2.11 Fourier series and transforms

Fourier series

Real form	$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad (2.474)$$ $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \quad (2.475)$$ $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \quad (2.476)$$	$f(x)$ periodic function, period $2L$ a_n, b_n Fourier coefficients
Complex form	$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{\mathrm{i}n\pi x}{L}\right) \quad (2.477)$$ $$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(\frac{-\mathrm{i}n\pi x}{L}\right)\mathrm{d}x \quad (2.478)$$	c_n complex Fourier coefficient
Parseval's theorem	$$\frac{1}{2L}\int_{-L}^{L} \lvert f(x)\rvert^2\,\mathrm{d}x = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \quad (2.479)$$ $$= \sum_{n=-\infty}^{\infty} \lvert c_n\rvert^2 \quad (2.480)$$	$\lvert\,\rvert$ modulus

Fourier transform[a]

Definition 1	$$F(s) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi\mathrm{i}xs}\,\mathrm{d}x \quad (2.481)$$ $$f(x) = \int_{-\infty}^{\infty} F(s)\mathrm{e}^{2\pi\mathrm{i}xs}\,\mathrm{d}s \quad (2.482)$$	$f(x)$ function of x $F(s)$ Fourier transform of $f(x)$
Definition 2	$$F(s) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}x \quad (2.483)$$ $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s \quad (2.484)$$	
Definition 3	$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}x \quad (2.485)$$ $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s \quad (2.486)$$	

[a] All three (and more) definitions are used, but definition 1 is probably the best.

Fourier transform theorems[a]

Theorem	Relation	Eq.	Symbols
Convolution	$f(x)*g(x)=\int_{-\infty}^{\infty} f(u)g(x-u)\,\mathrm{d}u$	(2.487)	f,g general functions; $*$ convolution
Convolution rules	$f*g=g*f$	(2.488)	f $f(x)\rightleftharpoons F(s)$
	$f*(g*h)=(f*g)*h$	(2.489)	g $g(x)\rightleftharpoons G(s)$
Convolution theorem	$f(x)g(x)\rightleftharpoons F(s)*G(s)$	(2.490)	$\rightleftharpoons$ Fourier transform relation
Autocorrelation	$f^*(x)\star f(x)=\int_{-\infty}^{\infty} f^*(u-x)f(u)\,\mathrm{d}u$	(2.491)	$\star$ correlation; f^* complex conjugate of f
Wiener–Khintchine theorem	$f^*(x)\star f(x)\rightleftharpoons \lvert F(s)\rvert^2$	(2.492)	
Cross-correlation	$f^*(x)\star g(x)=\int_{-\infty}^{\infty} f^*(u-x)g(u)\,\mathrm{d}u$	(2.493)	
Correlation theorem	$h(x)\star j(x)\rightleftharpoons H(s)J^*(s)$	(2.494)	h,j real functions; H $H(s)\rightleftharpoons h(x)$; J $J(s)\rightleftharpoons j(x)$
Parseval's relation[b]	$\int_{-\infty}^{\infty} f(x)g^*(x)\,\mathrm{d}x=\int_{-\infty}^{\infty} F(s)G^*(s)\,\mathrm{d}s$	(2.495)	
Parseval's theorem[c]	$\int_{-\infty}^{\infty} \lvert f(x)\rvert^2\,\mathrm{d}x=\int_{-\infty}^{\infty} \lvert F(s)\rvert^2\,\mathrm{d}s$	(2.496)	
Derivatives	$\frac{\mathrm{d}f(x)}{\mathrm{d}x}\rightleftharpoons 2\pi\mathbf{i}sF(s)$	(2.497)	
	$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)*g(x)]=\frac{\mathrm{d}f(x)}{\mathrm{d}x}*g(x)=\frac{\mathrm{d}g(x)}{\mathrm{d}x}*f(x)$	(2.498)	

[a] Defining the Fourier transform as $F(s)=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi\mathbf{i}xs}\,\mathrm{d}x$.
[b] Also called the "power theorem."
[c] Also called "Rayleigh's theorem."

Fourier symmetry relationships

$f(x)$	$\rightleftharpoons$	$F(s)$	definitions
even	$\rightleftharpoons$	even	real: $f(x)=f^*(x)$
odd	$\rightleftharpoons$	odd	imaginary: $f(x)=-f^*(x)$
real, even	$\rightleftharpoons$	real, even	even: $f(x)=f(-x)$
real, odd	$\rightleftharpoons$	imaginary, odd	odd: $f(x)=-f(-x)$
imaginary, even	$\rightleftharpoons$	imaginary, even	Hermitian: $f(x)=f^*(-x)$
complex, even	$\rightleftharpoons$	complex, even	anti-Hermitian: $f(x)=-f^*(-x)$
complex, odd	$\rightleftharpoons$	complex, odd	
real, asymmetric	$\rightleftharpoons$	complex, Hermitian	
imaginary, asymmetric	$\rightleftharpoons$	complex, anti-Hermitian	

Fourier transform pairs[a]

$$f(x) \rightleftharpoons F(s)=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi \mathbf{i} sx}\,\mathrm{d}x \tag{2.499}$$

$$f(ax) \rightleftharpoons \frac{1}{|a|}F(s/a) \qquad (a\neq 0,\ \text{real}) \tag{2.500}$$

$$f(x-a) \rightleftharpoons \mathrm{e}^{-2\pi \mathbf{i} as}F(s) \qquad (a\ \text{real}) \tag{2.501}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x) \rightleftharpoons (2\pi \mathbf{i} s)^n F(s) \tag{2.502}$$

$$\delta(x) \rightleftharpoons 1 \tag{2.503}$$

$$\delta(x-a) \rightleftharpoons \mathrm{e}^{-2\pi \mathbf{i} as} \tag{2.504}$$

$$\mathrm{e}^{-a|x|} \rightleftharpoons \frac{2a}{a^2+4\pi^2 s^2} \qquad (a>0) \tag{2.505}$$

$$x\mathrm{e}^{-a|x|} \rightleftharpoons \frac{8\mathbf{i}\pi as}{(a^2+4\pi^2 s^2)^2} \qquad (a>0) \tag{2.506}$$

$$\mathrm{e}^{-x^2/a^2} \rightleftharpoons a\sqrt{\pi}\mathrm{e}^{-\pi^2 a^2 s^2} \tag{2.507}$$

$$\sin ax \rightleftharpoons \frac{1}{2\mathbf{i}}\left[\delta\left(s-\frac{a}{2\pi}\right)-\delta\left(s+\frac{a}{2\pi}\right)\right] \tag{2.508}$$

$$\cos ax \rightleftharpoons \frac{1}{2}\left[\delta\left(s-\frac{a}{2\pi}\right)+\delta\left(s+\frac{a}{2\pi}\right)\right] \tag{2.509}$$

$$\sum_{m=-\infty}^{\infty}\delta(x-ma) \rightleftharpoons \frac{1}{a}\sum_{n=-\infty}^{\infty}\delta\left(s-\frac{n}{a}\right) \tag{2.510}$$

$$f(x)=\begin{cases}0 & x<0\\ 1 & x>0\end{cases} \quad (\text{“step”}) \rightleftharpoons \frac{1}{2}\delta(s)-\frac{\mathbf{i}}{2\pi s} \tag{2.511}$$

$$f(x)=\begin{cases}1 & |x|\le a\\ 0 & |x|>a\end{cases} \quad (\text{“top hat”}) \rightleftharpoons \frac{\sin 2\pi as}{\pi s}=2a\,\mathrm{sinc}\,2as \tag{2.512}$$

$$f(x)=\begin{cases}\left(1-\frac{|x|}{a}\right) & |x|\le a\\ 0 & |x|>a\end{cases} \quad (\text{“triangle”}) \rightleftharpoons \frac{1}{2\pi^2 as^2}(1-\cos 2\pi as)=a\,\mathrm{sinc}^2\,as \tag{2.513}$$

[a] Equation (2.499) defines the Fourier transform used for these pairs. Note that $\mathrm{sinc}\,x\equiv(\sin\pi x)/(\pi x)$.