2.13 Probability and statistics

Discrete statistics

Mean	$\langle x\rangle = \frac{1}{N}\sum_{i=1}^{N} x_i$	(2.541)	x_i N $\langle\cdot\rangle$	data series series length mean value
Variance[a]	$\mathrm{var}[x] = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \langle x\rangle)^2$	(2.542)	var[·]	unbiased variance
Standard deviation	$\sigma[x] = (\mathrm{var}[x])^{1/2}$	(2.543)	σ	standard deviation
Skewness	$\mathrm{skew}[x] = \frac{N}{(N-1)(N-2)}\sum_{i=1}^{N}\left(\frac{x_i - \langle x\rangle}{\sigma}\right)^3$	(2.544)		
Kurtosis	$\mathrm{kurt}[x] \simeq \left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{x_i - \langle x\rangle}{\sigma}\right)^4\right] - 3$	(2.545)		
Correlation coefficient[b]	$r = \frac{\sum_{i=1}^{N}(x_i - \langle x\rangle)(y_i - \langle y\rangle)}{\sqrt{\sum_{i=1}^{N}(x_i - \langle x\rangle)^2}\sqrt{\sum_{i=1}^{N}(y_i - \langle y\rangle)^2}}$	(2.546)	x,y r	data series to correlate correlation coefficient

[a] If $\langle x\rangle$ is derived from the data, $\{x_i\}$, the relation is as shown. If $\langle x\rangle$ is known independently, then an unbiased estimate is obtained by dividing the right-hand side by N rather than $N-1$.
[b] Also known as "Pearson's r."

Discrete probability distributions

distribution	pr(x)	*mean*	*variance*	*domain*			
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$	$(x = 0,1,\ldots,n)$	(2.547)	$\binom{n}{x}$	binomial coefficient
Geometric	$(1-p)^{x-1}p$	$1/p$	$(1-p)/p^2$	$(x = 1,2,3,\ldots)$	(2.548)		
Poisson	$\lambda^x \exp(-\lambda)/x!$	λ	λ	$(x = 0,1,2,\ldots)$	(2.549)		

Continuous probability distributions

distribution	pr(x)	*mean*	*variance*	*domain*	
Uniform	$\dfrac{1}{b-a}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$	$(a \le x \le b)$	(2.550)
Exponential	$\lambda \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$	$(x \ge 0)$	(2.551)
Normal/ Gaussian	$\dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[\dfrac{-(x-\mu)^2}{2\sigma^2}\right]$	μ	σ^2	$(-\infty < x < \infty)$	(2.552)
Chi-squared[a]	$\dfrac{e^{-x/2} x^{(r/2)-1}}{2^{r/2}\Gamma(r/2)}$	r	$2r$	$(x \ge 0)$	(2.553)
Rayleigh	$\dfrac{x}{\sigma^2} \exp\left(\dfrac{-x^2}{2\sigma^2}\right)$	$\sigma\sqrt{\pi/2}$	$2\sigma^2\left(1-\dfrac{\pi}{4}\right)$	$(x \ge 0)$	(2.554)
Cauchy/ Lorentzian	$\dfrac{a}{\pi(a^2+x^2)}$	(none)	(none)	$(-\infty < x < \infty)$	(2.555)

[a]With r degrees of freedom. Γ is the gamma function.

Multivariate normal distribution

Density function	$\text{pr}(\boldsymbol{x}) = \dfrac{\exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})\mathsf{C}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})^T\right]}{(2\pi)^{k/2}[\det(\mathsf{C})]^{1/2}}$	(2.556)
Mean	$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$	(2.557)
Covariance	$\mathsf{C} = \sigma_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$	(2.558)
Correlation coefficient	$r = \dfrac{\sigma_{ij}}{\sigma_i \sigma_j}$	(2.559)
Box–Muller transformation	$x_1 = (-2\ln y_1)^{1/2} \cos 2\pi y_2$	(2.560)
	$x_2 = (-2\ln y_1)^{1/2} \sin 2\pi y_2$	(2.561)

pr	probability density
k	number of dimensions
C	covariance matrix
$\boldsymbol{x}$	variable (k dimensional)
$\boldsymbol{\mu}$	vector of means
^{T}	transpose
det	determinant
μ_i	mean of ith variable
σ_{ij}	components of C
r	correlation coefficient
x_i	normally distributed deviates
y_i	deviates distributed uniformly between 0 and 1

Random walk

One-dimensional	$$\mathrm{pr}(x)=\frac{1}{(2\pi Nl^2)^{1/2}}\exp\left(\frac{-x^2}{2Nl^2}\right)$$ (2.562)	x displacement after N steps (can be positive or negative) $\mathrm{pr}(x)$ probability density of x ($\int_{-\infty}^{\infty}\mathrm{pr}(x)\,\mathrm{d}x=1$) N number of steps l step length (all equal)
rms displacement	$$x_{\mathrm{rms}}=N^{1/2}l$$ (2.563)	x_{rms} root-mean-squared displacement from start point
Three-dimensional	$$\mathrm{pr}(r)=\left(\frac{a}{\pi^{1/2}}\right)^3\exp(-a^2r^2)$$ (2.564) where $$a=\left(\frac{3}{2Nl^2}\right)^{1/2}$$	r radial distance from start point $\mathrm{pr}(r)$ probability density of r ($\int_0^{\infty}4\pi r^2\,\mathrm{pr}(r)\,\mathrm{d}r=1$) a (most probable distance)$^{-1}$
Mean distance	$$\langle r\rangle=\left(\frac{8}{3\pi}\right)^{1/2}N^{1/2}l$$ (2.565)	$\langle r\rangle$ mean distance from start point
rms distance	$$r_{\mathrm{rms}}=N^{1/2}l$$ (2.566)	r_{rms} root-mean-squared distance from start point

Bayesian inference

Conditional probability	$$\mathrm{pr}(x)=\int\mathrm{pr}(x\|y')\,\mathrm{pr}(y')\,\mathrm{d}y'$$ (2.567)	$\mathrm{pr}(x)$ probability (density) of x $\mathrm{pr}(x\|y')$ conditional probability of x given y'
Joint probability	$$\mathrm{pr}(x,y)=\mathrm{pr}(x)\,\mathrm{pr}(y\|x)$$ (2.568)	$\mathrm{pr}(x,y)$ joint probability of x and y
Bayes' theorem[a]	$$\mathrm{pr}(y\|x)=\frac{\mathrm{pr}(x\|y)\,\mathrm{pr}(y)}{\mathrm{pr}(x)}$$ (2.569)	

[a]In this expression, $\mathrm{pr}(y|x)$ is known as the posterior probability, $\mathrm{pr}(x|y)$ the likelihood, and $\mathrm{pr}(y)$ the prior probability.