

3.4 Particle motion

Dynamics definitions^a

Newtonian force	$\mathbf{F} = m\ddot{\mathbf{r}} = \dot{\mathbf{p}}$	(3.63)	$\mathbf{F}$ force m mass of particle $\mathbf{r}$ particle position vector
Momentum	$\mathbf{p} = m\dot{\mathbf{r}}$	(3.64)	$\mathbf{p}$ momentum
Kinetic energy	$T = \frac{1}{2}mv^2$	(3.65)	T kinetic energy v particle velocity
Angular momentum	$\mathbf{J} = \mathbf{r} \times \mathbf{p}$	(3.66)	$\mathbf{J}$ angular momentum
Couple (or torque)	$\mathbf{G} = \mathbf{r} \times \mathbf{F}$	(3.67)	$\mathbf{G}$ couple
Centre of mass (ensemble of N particles)	$\mathbf{R}_0 = \frac{\sum_{i=1}^N m_i \mathbf{r}_i}{\sum_{i=1}^N m_i}$	(3.68)	$\mathbf{R}_0$ position vector of centre of mass m_i mass of i th particle $\mathbf{r}_i$ position vector of i th particle

^aIn the Newtonian limit, $v \ll c$, assuming m is constant.

Relativistic dynamics^a

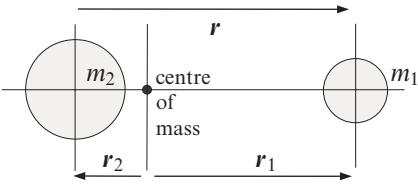
Lorentz factor	$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$	(3.69)	γ Lorentz factor $\mathbf{v}$ particle velocity c speed of light
Momentum	$\mathbf{p} = \gamma m_0 \mathbf{v}$	(3.70)	$\mathbf{p}$ relativistic momentum m_0 particle (rest) mass
Force	$\mathbf{F} = \frac{d\mathbf{p}}{dt}$	(3.71)	$\mathbf{F}$ force on particle t time
Rest energy	$E_r = m_0 c^2$	(3.72)	E_r particle rest energy
Kinetic energy	$T = m_0 c^2 (\gamma - 1)$	(3.73)	T relativistic kinetic energy
Total energy	$E = \gamma m_0 c^2$	(3.74)	E total energy ($= E_r + T$)
	$= (p^2 c^2 + m_0^2 c^4)^{1/2}$	(3.75)	

^aIt is now common to regard mass as a Lorentz invariant property and to drop the term “rest mass.” The symbol m_0 is used here to avoid confusion with the idea of “relativistic mass” ($= \gamma m_0$) used by some authors.

Constant acceleration

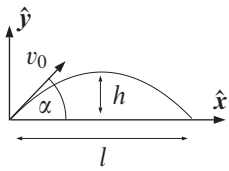
$v = u + at$	(3.76)	u initial velocity
$v^2 = u^2 + 2as$	(3.77)	v final velocity
$s = ut + \frac{1}{2}at^2$	(3.78)	t time
$s = \frac{u+v}{2}t$	(3.79)	s distance travelled
		a acceleration

Reduced mass (of two interacting bodies)

				
Reduced mass	$\mu = \frac{m_1 m_2}{m_1 + m_2}$	(3.80)	μ	reduced mass
Distances from centre of mass	$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2} \mathbf{r}$	(3.81)	m_i	interacting masses
	$\mathbf{r}_2 = \frac{-m_1}{m_1 + m_2} \mathbf{r}$	(3.82)	$\mathbf{r}_i$	position vectors from centre of mass
Moment of inertia	$I = \mu \mathbf{r} ^2$	(3.83)	$\mathbf{r}$	$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$
Total angular momentum	$\mathbf{J} = \mu \mathbf{r} \times \dot{\mathbf{r}}$	(3.84)	$ \mathbf{r} $	distance between masses
Lagrangian	$L = \frac{1}{2} \mu \dot{\mathbf{r}} ^2 - U(\mathbf{r})$	(3.85)	I	moment of inertia
			$\mathbf{J}$	angular momentum
			L	Lagrangian
			U	potential energy of interaction

Ballistics^a

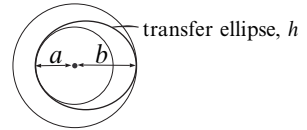
Velocity	$\mathbf{v} = v_0 \cos \alpha \hat{\mathbf{x}} + (v_0 \sin \alpha - gt) \hat{\mathbf{y}}$	(3.86)	v_0	initial velocity
	$v^2 = v_0^2 - 2gy$	(3.87)	$\mathbf{v}$	velocity at t
Trajectory	$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$	(3.88)	α	elevation angle
			g	gravitational acceleration
Maximum height	$h = \frac{v_0^2}{2g} \sin^2 \alpha$	(3.89)	$\hat{}$	unit vector
Horizontal range	$l = \frac{v_0^2}{g} \sin 2\alpha$	(3.90)	t	time
			h	maximum height
			l	range



^aIgnoring the curvature and rotation of the Earth and frictional losses, g is assumed constant.

Rocketry

Escape velocity ^a	$v_{\text{esc}} = \left(\frac{2GM}{r} \right)^{1/2}$	(3.91)	v_{esc} escape velocity G constant of gravitation M mass of central body r central body radius
Specific impulse	$I_{\text{sp}} = \frac{u}{g}$	(3.92)	I_{sp} specific impulse u effective exhaust velocity g acceleration due to gravity
Exhaust velocity (into a vacuum)	$u = \left[\frac{2\gamma RT_{\text{c}}}{(\gamma - 1)\mu} \right]^{1/2}$	(3.93)	R molar gas constant γ ratio of heat capacities T_{c} combustion temperature μ effective molecular mass of exhaust gas
Rocket equation ($g = 0$)	$\Delta v = u \ln \left(\frac{M_{\text{i}}}{M_{\text{f}}} \right) \equiv u \ln \mathcal{M}$	(3.94)	Δv rocket velocity increment M_{i} pre-burn rocket mass M_{f} post-burn rocket mass $\mathcal{M}$ mass ratio
Multistage rocket	$\Delta v = \sum_{i=1}^N u_i \ln \mathcal{M}_i$	(3.95)	N number of stages $\mathcal{M}_i$ mass ratio for i th burn u_i exhaust velocity of i th burn
In a constant gravitational field	$\Delta v = u \ln \mathcal{M} - gt \cos \theta$	(3.96)	t burn time θ rocket zenith angle
Hohmann cotangential transfer ^b	$\Delta v_{ah} = \left(\frac{GM}{r_a} \right)^{1/2} \left[\left(\frac{2r_b}{r_a + r_b} \right)^{1/2} - 1 \right]$	(3.97)	Δv_{ah} velocity increment, a to h Δv_{hb} velocity increment, h to b r_a radius of inner orbit r_b radius of outer orbit
	$\Delta v_{hb} = \left(\frac{GM}{r_b} \right)^{1/2} \left[1 - \left(\frac{2r_a}{r_a + r_b} \right)^{1/2} \right]$	(3.98)	

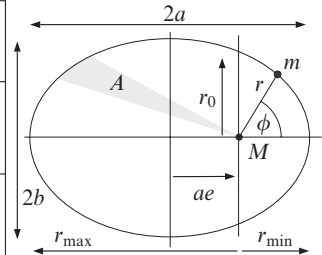


^aFrom the surface of a spherically symmetric, nonrotating body, mass M .

^bTransfer between coplanar, circular orbits a and b , via ellipse h with a minimal expenditure of energy.

Gravitationally bound orbital motion^a

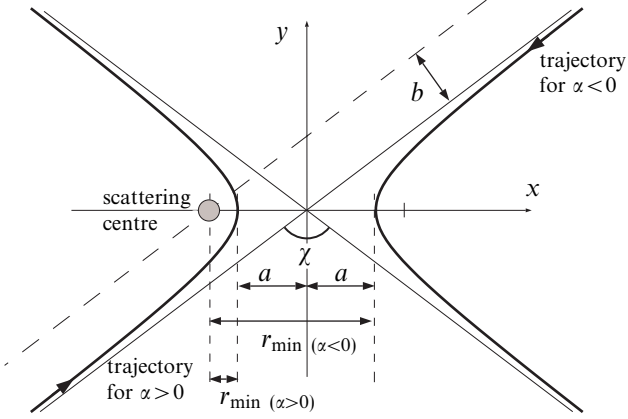
Potential energy of interaction	$U(r) = -\frac{GMm}{r} \equiv -\frac{\alpha}{r}$	(3.99)	$U(r)$ potential energy G constant of gravitation M central mass m orbiting mass ($\ll M$) α GMm (for gravitation) E total energy (constant) J total angular momentum (constant)
Total energy	$E = -\frac{\alpha}{r} + \frac{J^2}{2mr^2} = -\frac{\alpha}{2a}$	(3.100)	T kinetic energy $\langle \cdot \rangle$ mean value
Virial theorem (1/r potential)	$E = \langle U \rangle / 2 = -\langle T \rangle$	(3.101)	
	$\langle U \rangle = -2\langle T \rangle$	(3.102)	
Orbital equation (Kepler's 1st law)	$\frac{r_0}{r} = 1 + e \cos \phi$, or	(3.103)	r_0 semi-latus-rectum r distance of m from M e eccentricity ϕ phase (true anomaly)
	$r = \frac{a(1-e^2)}{1+e \cos \phi}$	(3.104)	
Rate of sweeping area (Kepler's 2nd law)	$\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$	(3.105)	A area swept out by radius vector (total area = πab)
Semi-major axis	$a = \frac{r_0}{1-e^2} = \frac{\alpha}{2 E }$	(3.106)	a semi-major axis b semi-minor axis
Semi-minor axis	$b = \frac{r_0}{(1-e^2)^{1/2}} = \frac{J}{(2m E)^{1/2}}$	(3.107)	
Eccentricity ^b	$e = \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2} = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$	(3.108)	
Semi-latus-rectum	$r_0 = \frac{J^2}{m\alpha} = \frac{b^2}{a} = a(1-e^2)$	(3.109)	
Pericentre	$r_{\min} = \frac{r_0}{1+e} = a(1-e)$	(3.110)	$r_{\min}$ pericentre distance
Apocentre	$r_{\max} = \frac{r_0}{1-e} = a(1+e)$	(3.111)	$r_{\max}$ apocentre distance
Speed	$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right)$	(3.112)	v orbital speed
Period (Kepler's 3rd law)	$P = \pi\alpha \left(\frac{m}{2 E ^3} \right)^{1/2} = 2\pi a^{3/2} \left(\frac{m}{\alpha} \right)^{1/2}$	(3.113)	P orbital period



^aFor an inverse-square law of attraction between two isolated bodies in the nonrelativistic limit. If m is not $\ll M$, then the equations are valid with the substitutions $m \rightarrow \mu = Mm/(M+m)$ and $M \rightarrow (M+m)$ and with r taken as the body separation. The distance of mass m from the centre of mass is then $r\mu/m$ (see earlier table on *Reduced mass*). Other orbital dimensions scale similarly, and the two orbits have the same eccentricity.

^bNote that if the total energy, E , is < 0 then $e < 1$ and the orbit is an ellipse (a circle if $e = 0$). If $E = 0$, then $e = 1$ and the orbit is a parabola. If $E > 0$ then $e > 1$ and the orbit becomes a hyperbola (see *Rutherford scattering* on next page).

Rutherford scattering^a

		
Scattering potential energy	$U(r) = -\frac{\alpha}{r} \quad (3.114)$ $\alpha \begin{cases} < 0 & \text{repulsive} \\ > 0 & \text{attractive} \end{cases} \quad (3.115)$	$U(r)$ potential energy r particle separation α constant
Scattering angle	$\tan \frac{\chi}{2} = \frac{ \alpha }{2Eb} \quad (3.116)$	χ scattering angle E total energy (> 0) b impact parameter
Closest approach	$r_{\min} = \frac{ \alpha }{2E} \left(\csc \frac{\chi}{2} - \frac{\alpha}{ \alpha } \right) \quad (3.117)$ $= a(e \pm 1) \quad (3.118)$	$r_{\min}$ closest approach a hyperbola semi-axis e eccentricity
Semi-axis	$a = \frac{ \alpha }{2E} \quad (3.119)$	
Eccentricity	$e = \left(\frac{4E^2 b^2}{\alpha^2} + 1 \right)^{1/2} = \csc \frac{\chi}{2} \quad (3.120)$	
Motion trajectory ^b	$\frac{4E^2}{\alpha^2} x^2 - \frac{y^2}{b^2} = 1 \quad (3.121)$	x, y position with respect to hyperbola centre
Scattering centre ^c	$x = \pm \left(\frac{\alpha^2}{4E^2} + b^2 \right)^{1/2} \quad (3.122)$	
Rutherford scattering formula ^d	$\frac{d\sigma}{d\Omega} = \frac{1}{n} \frac{dN}{d\Omega} \quad (3.123)$ $= \left(\frac{\alpha}{4E} \right)^2 \csc^4 \frac{\chi}{2} \quad (3.124)$	$\frac{d\sigma}{d\Omega}$ differential scattering cross section n beam flux density dN number of particles scattered into $d\Omega$ Ω solid angle

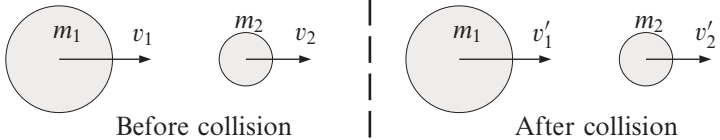
^aNonrelativistic treatment for an inverse-square force law and a fixed scattering centre. Similar scattering results from either an attractive or repulsive force. See also *Conic sections* on page 38.

^bThe correct branch can be chosen by inspection.

^cAlso the focal points of the hyperbola.

^d n is the number of particles per second passing through unit area perpendicular to the beam.

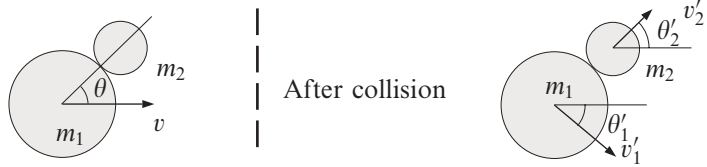
Inelastic collisions^a

			
Coefficient of restitution	$v'_2 - v'_1 = \epsilon(v_1 - v_2) \quad (3.125)$ $\epsilon = 1 \quad \text{if perfectly elastic} \quad (3.126)$ $\epsilon = 0 \quad \text{if perfectly inelastic} \quad (3.127)$	ϵ coefficient of restitution v_i pre-collision velocities v'_i post-collision velocities	T, T' total KE in zero momentum frame before and after collision m_i particle masses
Loss of kinetic energy ^b	$\frac{T - T'}{T} = 1 - \epsilon^2 \quad (3.128)$		
Final velocities	$v'_1 = \frac{m_1 - \epsilon m_2}{m_1 + m_2} v_1 + \frac{(1 + \epsilon)m_2}{m_1 + m_2} v_2 \quad (3.129)$ $v'_2 = \frac{m_2 - \epsilon m_1}{m_1 + m_2} v_2 + \frac{(1 + \epsilon)m_1}{m_1 + m_2} v_1 \quad (3.130)$		

^aAlong the line of centres, $v_1, v_2 \ll c$.

^bIn zero momentum frame.

Oblique elastic collisions^a

			
Directions of motion	$\tan \theta'_1 = \frac{m_2 \sin 2\theta}{m_1 - m_2 \cos 2\theta} \quad (3.131)$ $\theta'_2 = \theta \quad (3.132)$	θ angle between centre line and incident velocity θ'_i final trajectories m_i sphere masses	
Relative separation angle	$\theta'_1 + \theta'_2 \begin{cases} > \pi/2 & \text{if } m_1 < m_2 \\ = \pi/2 & \text{if } m_1 = m_2 \\ < \pi/2 & \text{if } m_1 > m_2 \end{cases} \quad (3.133)$		
Final velocities	$v'_1 = \frac{(m_1^2 + m_2^2 - 2m_1 m_2 \cos 2\theta)^{1/2}}{m_1 + m_2} v \quad (3.134)$ $v'_2 = \frac{2m_1 v}{m_1 + m_2} \cos \theta \quad (3.135)$	v incident velocity of m_1 v'_i final velocities	

^aCollision between two perfectly elastic spheres: m_2 initially at rest, velocities $\ll c$.