3.5 Rigid body dynamics

Moment of inertia tensor

Moment of inertia tensor[a]	$I_{ij}=\int (r^2\delta_{ij}-x_ix_j)\,\mathrm{d}m$	(3.136)
	$\mathsf{I}=\begin{pmatrix}\int(y^2+z^2)\,\mathrm{d}m & -\int xy\,\mathrm{d}m & -\int xz\,\mathrm{d}m\\ -\int xy\,\mathrm{d}m & \int(x^2+z^2)\,\mathrm{d}m & -\int yz\,\mathrm{d}m\\ -\int xz\,\mathrm{d}m & -\int yz\,\mathrm{d}m & \int(x^2+y^2)\,\mathrm{d}m\end{pmatrix}$	(3.137)
Parallel axis theorem	$I_{12}=I^{\star}_{12}-ma_1a_2$	(3.138)
	$I_{11}=I^{\star}_{11}+m(a_2^2+a_3^2)$	(3.139)
	$I_{ij}=I^{\star}_{ij}+m(\lvert\boldsymbol{a}\rvert^2\delta_{ij}-a_ia_j)$	(3.140)
Angular momentum	$\boldsymbol{J}=\mathsf{I}\boldsymbol{\omega}$	(3.141)
Rotational kinetic energy	$T=\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{J}=\frac{1}{2}I_{ij}\omega_i\omega_j$	(3.142)

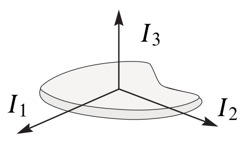
r	$r^2=x^2+y^2+z^2$
δ_{ij}	Kronecker delta
I	moment of inertia tensor
$\mathrm{d}m$	mass element
x_i	position vector of $\mathrm{d}m$
I_{ij}	components of I
$I^{\star}_{ij}$	tensor with respect to centre of mass
$a_i,\boldsymbol{a}$	position vector of centre of mass
m	mass of body
$\boldsymbol{J}$	angular momentum
$\boldsymbol{\omega}$	angular velocity
T	kinetic energy

[a] I_{ii} are the moments of inertia of the body. I_{ij} $(i\neq j)$ are its products of inertia. The integrals are over the body volume.

Principal axes

Principal moment of inertia tensor	$\mathsf{I}'=\begin{pmatrix}I_1 & 0 & 0\\ 0 & I_2 & 0\\ 0 & 0 & I_3\end{pmatrix}$	(3.143)
Angular momentum	$\boldsymbol{J}=(I_1\omega_1,I_2\omega_2,I_3\omega_3)$	(3.144)
Rotational kinetic energy	$T=\frac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)$	(3.145)
Moment of inertia ellipsoid[a]	$T=T(\omega_1,\omega_2,\omega_3)$	(3.146)
	$J_i=\frac{\partial T}{\partial\omega_i}$ ($\boldsymbol{J}$ is $\perp$ ellipsoid surface)	(3.147)
Perpendicular axis theorem	$I_1+I_2\begin{cases}\geq I_3 & \text{generally}\\ =I_3 & \text{flat lamina}\perp\text{to 3-axis}\end{cases}$	(3.148)
Symmetries	$I_1\neq I_2\neq I_3$ asymmetric top $I_1=I_2\neq I_3$ symmetric top $I_1=I_2=I_3$ spherical top	(3.149)

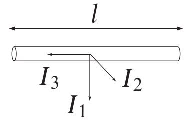
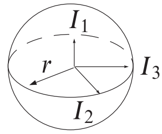
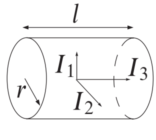
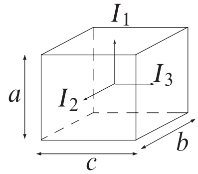
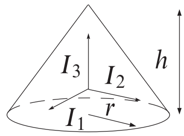
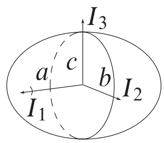
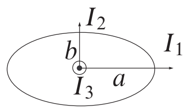
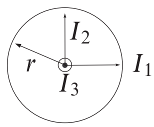
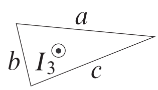
I'	principal moment of inertia tensor
I_i	principal moments of inertia
$\boldsymbol{J}$	angular momentum
ω_i	components of $\boldsymbol{\omega}$ along principal axes
T	kinetic energy



lamina

[a] The ellipsoid is defined by the surface of constant T.

Moments of inertia[a]

Thin rod, length l	$I_1 = I_2 = \dfrac{ml^2}{12}$ $I_3 \simeq 0$	(3.150) (3.151)	
Solid sphere, radius r	$I_1 = I_2 = I_3 = \dfrac{2}{5}mr^2$	(3.152)	
Spherical shell, radius r	$I_1 = I_2 = I_3 = \dfrac{2}{3}mr^2$	(3.153)	
Solid cylinder, radius r, length l	$I_1 = I_2 = \dfrac{m}{4}\left(r^2 + \dfrac{l^2}{3}\right)$ $I_3 = \dfrac{1}{2}mr^2$	(3.154) (3.155)	
Solid cuboid, sides a,b,c	$I_1 = m(b^2 + c^2)/12$ $I_2 = m(c^2 + a^2)/12$ $I_3 = m(a^2 + b^2)/12$	(3.156) (3.157) (3.158)	
Solid circular cone, base radius r, height h[b]	$I_1 = I_2 = \dfrac{3}{20}m\left(r^2 + \dfrac{h^2}{4}\right)$ $I_3 = \dfrac{3}{10}mr^2$	(3.159) (3.160)	
Solid ellipsoid, semi-axes a,b,c	$I_1 = m(b^2 + c^2)/5$ $I_2 = m(c^2 + a^2)/5$ $I_3 = m(a^2 + b^2)/5$	(3.161) (3.162) (3.163)	
Elliptical lamina, semi-axes a,b	$I_1 = mb^2/4$ $I_2 = ma^2/4$ $I_3 = m(a^2 + b^2)/4$	(3.164) (3.165) (3.166)	
Disk, radius r	$I_1 = I_2 = mr^2/4$ $I_3 = mr^2/2$	(3.167) (3.168)	
Triangular plate[c]	$I_3 = \dfrac{m}{36}(a^2 + b^2 + c^2)$	(3.169)	

[a]With respect to principal axes for bodies of mass m and uniform density. The radius of gyration is defined as $k = (I/m)^{1/2}$.
[b]Origin of axes is at the centre of mass ($h/4$ above the base).
[c]Around an axis through the centre of mass and perpendicular to the plane of the plate.

Centres of mass

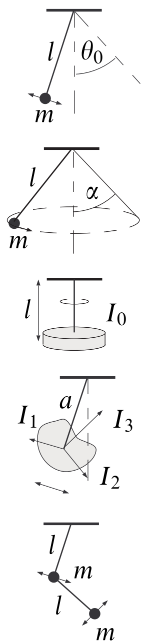
Solid hemisphere, radius r	$d=3r/8$ from sphere centre	(3.170)
Hemispherical shell, radius r	$d=r/2$ from sphere centre	(3.171)
Sector of disk, radius r, angle 2θ	$d=\frac{2}{3}r\frac{\sin\theta}{\theta}$ from disk centre	(3.172)
Arc of circle, radius r, angle 2θ	$d=r\frac{\sin\theta}{\theta}$ from circle centre	(3.173)
Arbitrary triangular lamina, height h[a]	$d=h/3$ perpendicular from base	(3.174)
Solid cone or pyramid, height h	$d=h/4$ perpendicular from base	(3.175)
Spherical cap, height h, sphere radius r	solid: $d=\frac{3}{4}\frac{(2r-h)^2}{3r-h}$ from sphere centre	(3.176)
	shell: $d=r-h/2$ from sphere centre	(3.177)
Semi-elliptical lamina, height h	$d=\frac{4h}{3\pi}$ from base	(3.178)

[a] h is the perpendicular distance between the base and apex of the triangle.

Pendulums

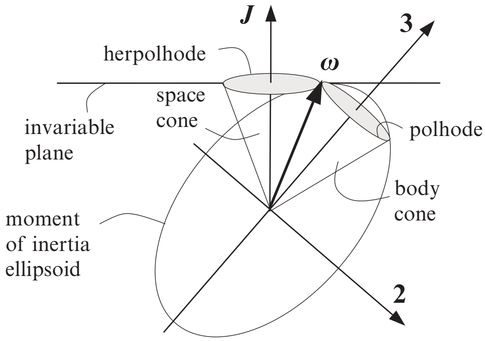
Simple pendulum	$P=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{16}+\cdots\right)$	(3.179)
Conical pendulum	$P=2\pi\left(\frac{l\cos\alpha}{g}\right)^{1/2}$	(3.180)
Torsional pendulum[a]	$P=2\pi\left(\frac{lI_0}{C}\right)^{1/2}$	(3.181)
Compound pendulum[b]	$P\simeq 2\pi\left[\frac{1}{mga}(ma^2+I_1\cos^2\gamma_1+I_2\cos^2\gamma_2+I_3\cos^2\gamma_3)\right]^{1/2}$	(3.182)
Equal double pendulum[c]	$P\simeq 2\pi\left[\frac{l}{(2\pm\sqrt{2})g}\right]^{1/2}$	(3.183)

P period
g gravitational acceleration
l length
θ_0 maximum angular displacement
α cone half-angle
I_0 moment of inertia of bob
C torsional rigidity of wire (see page 81)
a distance of rotation axis from centre of mass
m mass of body
I_i principal moments of inertia
γ_i angles between rotation axis and principal axes

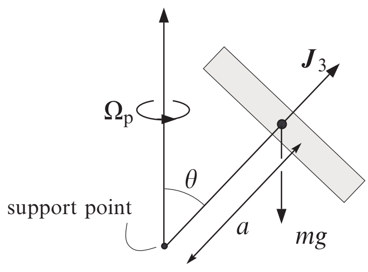


[a] Assuming the bob is supported parallel to a principal rotation axis.
[b] I.e., an arbitrary triaxial rigid body.
[c] For very small oscillations (two eigenmodes).

Tops and gyroscopes



prolate symmetric top



gyroscope

Euler's equations[a]	$G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$ (3.184) $G_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$ (3.185) $G_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$ (3.186)	G_i	external couple ($=0$ for free rotation)
		I_i	principal moments of inertia
		ω_i	angular velocity of rotation
Free symmetric top[b] ($I_3 < I_2 = I_1$)	$\Omega_b = \frac{I_1 - I_3}{I_1}\omega_3$ (3.187) $\Omega_s = \frac{J}{I_1}$ (3.188)	Ω_b	body frequency
		Ω_s	space frequency
		J	total angular momentum
Free asymmetric top[c]	$\Omega_b^2 = \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2}\omega_3^2$ (3.189)		
Steady gyroscopic precession	$\Omega_p^2 I_1' \cos\theta - \Omega_p J_3 + mga = 0$ (3.190) $\Omega_p \simeq \begin{cases} Mga/J_3 & \text{(slow)} \\ J_3/(I_1' \cos\theta) & \text{(fast)} \end{cases}$ (3.191)	Ω_p	precession angular velocity
		θ	angle from vertical
		J_3	angular momentum around symmetry axis
		m	mass
		g	gravitational acceleration
Gyroscopic stability	$J_3^2 \geq 4I_1' mga\cos\theta$ (3.192)	a	distance of centre of mass from support point
		I_1'	moment of inertia about support point
Gyroscopic limit ("sleeping top")	$J_3^2 \gg I_1' mga$ (3.193)		
Nutation rate	$\Omega_n = J_3/I_1'$ (3.194)	Ω_n	nutation angular velocity
Gyroscope released from rest	$\Omega_p = \frac{mga}{J_3}(1 - \cos\Omega_n t)$ (3.195)	t	time

[a]Components are with respect to the principal axes, rotating with the body.
[b]The body frequency is the angular velocity (with respect to principal axes) of $\boldsymbol{\omega}$ around the **3**-axis. The space frequency is the angular velocity of the **3**-axis around $\boldsymbol{J}$, i.e., the angular velocity at which the body cone moves around the space cone.
[c]$\boldsymbol{J}$ close to **3**-axis. If $\Omega_b^2 < 0$, the body tumbles.