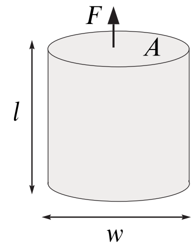
3.8 Elasticity

Elasticity definitions (simple)[a]

Stress	$\tau = F/A$	(3.228)	τ stress F applied force A cross-sectional area
Strain	$e = \delta l / l$	(3.229)	e strain δl change in length l length
Young modulus (Hooke's law)	$E = \tau / e = \text{constant}$	(3.230)	E Young modulus
Poisson ratio[b]	$\sigma = -\dfrac{\delta w / w}{\delta l / l}$	(3.231)	σ Poisson ratio δw change in width w width



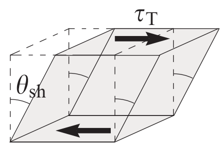
[a]These apply to a thin wire under longitudinal stress.
[b]Solids obeying Hooke's law are restricted by thermodynamics to $-1 \leq \sigma \leq 1/2$, but none are known with $\sigma < 0$. Non-Hookean materials can show $\sigma > 1/2$.

Elasticity definitions (general)

Stress tensor[a]	$\tau_{ij} = \dfrac{\text{force} \parallel i \text{ direction}}{\text{area} \perp j \text{ direction}}$	(3.232)	τ_{ij} stress tensor ($\tau_{ij} = \tau_{ji}$)
Strain tensor	$e_{kl} = \dfrac{1}{2}\left(\dfrac{\partial u_k}{\partial x_l} + \dfrac{\partial u_l}{\partial x_k}\right)$	(3.233)	e_{kl} strain tensor ($e_{kl} = e_{lk}$) u_k displacement $\parallel$ to x_k x_k coordinate system
Elastic modulus	$\tau_{ij} = \lambda_{ijkl} e_{kl}$	(3.234)	λ_{ijkl} elastic modulus
Elastic energy[b]	$U = \dfrac{1}{2}\lambda_{ijkl} e_{ij} e_{kl}$	(3.235)	U potential energy
Volume strain (dilatation)	$e_{\rm v} = \dfrac{\delta V}{V} = e_{11} + e_{22} + e_{33}$	(3.236)	$e_{\rm v}$ volume strain δV change in volume V volume
Shear strain	$e_{kl} = \underbrace{(e_{kl} - \frac{1}{3} e_{\rm v} \delta_{kl})}_{\text{pure shear}} + \underbrace{\frac{1}{3} e_{\rm v} \delta_{kl}}_{\text{dilatation}}$	(3.237)	δ_{kl} Kronecker delta
Hydrostatic compression	$\tau_{ij} = -p\delta_{ij}$	(3.238)	p hydrostatic pressure

[a]τ_{ii} are normal stresses, τ_{ij} $(i \neq j)$ are torsional stresses.
[b]As usual, products are implicitly summed over repeated indices.

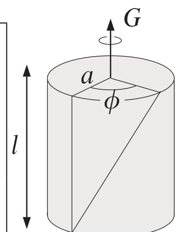
Isotropic elastic solids

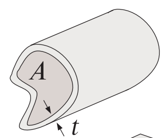
Lamé coefficients	$\mu = \frac{E}{2(1+\sigma)}$	(3.239)	μ,λ Lamé coefficients E Young modulus σ Poisson ratio
	$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$	(3.240)	
Longitudinal modulus[a]	$M_l = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} = \lambda + 2\mu$	(3.241)	M_l longitudinal elastic modulus
Diagonalised equations[b]	$e_{ii} = \frac{1}{E}[\tau_{ii} - \sigma(\tau_{jj} + \tau_{kk})]$	(3.242)	e_{ii} strain in i direction τ_{ii} stress in i direction $\mathbf{e}$ strain tensor $\mathbf{t}$ stress tensor $\mathbf{1}$ unit matrix $\mathrm{tr}(\cdot)$ trace
	$\tau_{ii} = M_l\left[e_{ii} + \frac{\sigma}{1-\sigma}(e_{jj} + e_{kk})\right]$	(3.243)	
	$\mathbf{t} = 2\mu\mathbf{e} + \lambda\mathbf{1}\,\mathrm{tr}(\mathbf{e})$	(3.244)	
Bulk modulus (compression modulus)	$K = \frac{E}{3(1-2\sigma)} = \lambda + \frac{2}{3}\mu$	(3.245)	K bulk modulus K_T isothermal bulk modulus V volume p pressure T temperature
	$\frac{1}{K_T} = -\frac{1}{V}\frac{\partial V}{\partial p}\Big\|_T$	(3.246)	
	$-p = K e_{\mathrm{v}}$	(3.247)	
Shear modulus (rigidity modulus)	$\mu = \frac{E}{2(1+\sigma)}$	(3.248)	e_{v} volume strain μ shear modulus τ_{T} transverse stress θ_{sh} shear strain
	$\tau_{\mathrm{T}} = \mu\theta_{\mathrm{sh}}$	(3.249)	
Young modulus	$E = \frac{9\mu K}{\mu + 3K}$	(3.250)	
Poisson ratio	$\sigma = \frac{3K - 2\mu}{2(3K + \mu)}$	(3.251)	

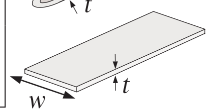
[a] In an extended medium.
[b] Axes aligned along eigenvectors of the stress and strain tensors.

Torsion

Torsional rigidity (for a homogeneous rod)	$G = C\frac{\phi}{l}$	(3.252)	G twisting couple C torsional rigidity l rod length ϕ twist angle in length l
Thin circular cylinder	$C = 2\pi a^3 \mu t$	(3.253)	a radius t wall thickness μ shear modulus
Thick circular cylinder	$C = \frac{1}{2}\mu\pi(a_2^4 - a_1^4)$	(3.254)	a_1 inner radius a_2 outer radius
Arbitrary thin-walled tube	$C = \frac{4A^2\mu t}{P}$	(3.255)	A cross-sectional area P perimeter
Long flat ribbon	$C = \frac{1}{3}\mu w t^3$	(3.256)	w cross-sectional width

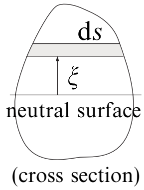


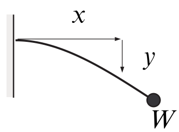


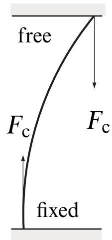


Bending beams[a]

Bending moment	$G_b = \frac{E}{R_c}\int \xi^2 \, ds$	(3.257)	G_b bending moment E Young modulus R_c radius of curvature ds area element ξ distance to neutral surface from ds I moment of area
	$= \frac{EI}{R_c}$	(3.258)	
Light beam, horizontal at $x=0$, weight at $x=l$	$y = \frac{W}{2EI}\left(l - \frac{x}{3}\right)x^2$	(3.259)	y displacement from horizontal W end-weight l beam length x distance along beam
Heavy beam	$EI\frac{d^4 y}{dx^4} = w(x)$	(3.260)	w beam weight per unit length
Euler strut failure	$F_c = \begin{cases} \pi^2 EI/l^2 & \text{(free ends)} \\ 4\pi^2 EI/l^2 & \text{(fixed ends)} \\ \pi^2 EI/(4l^2) & \text{(1 free end)} \end{cases}$	(3.261)	F_c critical compression force l strut length



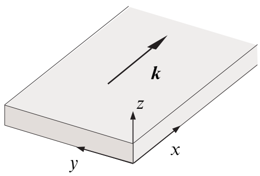




[a]The radius of curvature is approximated by $1/R_c \simeq d^2y/dx^2$.

Elastic wave velocities[a]

In an infinite isotropic solid[b]	$v_t = (\mu/\rho)^{1/2}$	(3.262)	v_t speed of transverse wave v_l speed of longitudinal wave μ shear modulus ρ density M_l longitudinal modulus $\left(= \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\right)$
	$v_l = (M_l/\rho)^{1/2}$	(3.263)	
	$\frac{v_l}{v_t} = \left(\frac{2-2\sigma}{1-2\sigma}\right)^{1/2}$	(3.264)	
In a fluid	$v_l = (K/\rho)^{1/2}$	(3.265)	K bulk modulus
On a thin plate (wave travelling along x, plate thin in z)	$v_l^{(x)} = \left[\frac{E}{\rho(1-\sigma^2)}\right]^{1/2}$	(3.266)	$v_l^{(i)}$ speed of longitudinal wave (displacement $\parallel i$) $v_t^{(i)}$ speed of transverse wave (displacement $\parallel i$) E Young modulus σ Poisson ratio k wavenumber ($=2\pi/\lambda$) t plate thickness (in z, $t \ll \lambda$)
	$v_t^{(y)} = (\mu/\rho)^{1/2}$	(3.267)	
	$v_t^{(z)} = k\left[\frac{Et^2}{12\rho(1-\sigma^2)}\right]^{1/2}$	(3.268)	
In a thin circular rod	$v_l = (E/\rho)^{1/2}$	(3.269)	
	$v_\phi = (\mu/\rho)^{1/2}$	(3.270)	v_ϕ torsional wave velocity a rod radius ($\ll \lambda$)
	$v_t = \frac{ka}{2}\left(\frac{E}{\rho}\right)^{1/2}$	(3.271)	



[a]Waves that produce "bending" are generally dispersive. Wave (phase) speeds are quoted throughout.
[b]Transverse waves are also known as shear waves, or S-waves. Longitudinal waves are also known as pressure waves, or P-waves.

Waves in strings and springs[a]

In a spring	$v_\mathrm{l} = (\kappa l/\rho_l)^{1/2}$	(3.272)	v_l speed of longitudinal wave κ spring constant[b] l spring length ρ_l mass per unit length[c]
On a stretched string	$v_\mathrm{t} = (T/\rho_l)^{1/2}$	(3.273)	v_t speed of transverse wave T tension
On a stretched sheet	$v_\mathrm{t} = (\tau/\rho_\mathrm{A})^{1/2}$	(3.274)	τ tension per unit width ρ_A mass per unit area

[a]Wave amplitude assumed $\ll$ wavelength.
[b]In the sense $\kappa = $ force/extension.
[c]Measured along the axis of the spring.

Propagation of elastic waves

Acoustic impedance	$Z = \frac{\text{force}}{\text{response velocity}} = \frac{F}{\dot{u}}$ $= (E'\rho)^{1/2}$	(3.275) (3.276)	Z impedance F stress force u strain displacement
Wave velocity/ impedance relation	if $v = \left(\frac{E'}{\rho}\right)^{1/2}$ then $Z = (E'\rho)^{1/2} = \rho v$	(3.277) (3.278)	E' elastic modulus ρ density v wave phase velocity
Mean energy density (nondispersive waves)	$\mathcal{U} = \frac{1}{2}E'k^2u_0^2$ $= \frac{1}{2}\rho\omega^2u_0^2$ $P = \mathcal{U}v$	(3.279) (3.280) (3.281)	$\mathcal{U}$ energy density k wavenumber ω angular frequency u_0 maximum displacement P mean energy flux
Normal coefficients[a]	$r = \frac{u_\mathrm{r}}{u_\mathrm{i}} = -\frac{\tau_\mathrm{r}}{\tau_\mathrm{i}} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$ $t = \frac{2Z_1}{Z_1 + Z_2}$	(3.282) (3.283)	r reflection coefficient t transmission coefficient τ stress
Snell's law[b]	$\frac{\sin\theta_\mathrm{i}}{v_\mathrm{i}} = \frac{\sin\theta_\mathrm{r}}{v_\mathrm{r}} = \frac{\sin\theta_\mathrm{t}}{v_\mathrm{t}}$	(3.284)	θ_i angle of incidence θ_r angle of reflection θ_t angle of refraction

[a]For stress and strain amplitudes. Because these reflection and transmission coefficients are usually defined in terms of displacement, u, rather than stress, there are differences between these coefficients and their equivalents defined in electromagnetism [see Equation (7.179) and page 154].
[b]Angles defined from the normal to the interface. An incident plane pressure wave will generally excite both shear and pressure waves in reflection and transmission. Use the velocity appropriate for the wave type.