3.9 Fluid dynamics

Ideal fluids[a]

Continuity[b]	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$	(3.285)	ρ	density
			$\boldsymbol{v}$	fluid velocity field
			t	time
Kelvin circulation	$\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{l} = \text{constant}$	(3.286)	Γ	circulation
			$d\boldsymbol{l}$	loop element
	$= \int_S \boldsymbol{\omega} \cdot d\boldsymbol{s}$	(3.287)	$d\boldsymbol{s}$	element of surface bounded by loop
			$\boldsymbol{\omega}$	vorticity ($= \nabla \times \boldsymbol{v}$)
Euler's equation[c]	$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} + \boldsymbol{g}$	(3.288)	p	pressure
			$\boldsymbol{g}$	gravitational field strength
	or $\frac{\partial}{\partial t}(\nabla \times \boldsymbol{v}) = \nabla \times [\boldsymbol{v} \times (\nabla \times \boldsymbol{v})]$	(3.289)	$(\boldsymbol{v} \cdot \nabla)$	advective operator
Bernoulli's equation (incompressible flow)	$\frac{1}{2}\rho v^2 + p + \rho g z = \text{constant}$	(3.290)	z	altitude
Bernoulli's equation (compressible adiabatic flow)[d]	$\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + gz = \text{constant}$	(3.291)	γ	ratio of specific heat capacities (c_p/c_V)
	$= \frac{1}{2}v^2 + c_p T + gz$	(3.292)	c_p	specific heat capacity at constant pressure
			T	temperature
Hydrostatics	$\nabla p = \rho \boldsymbol{g}$	(3.293)		
Adiabatic lapse rate (ideal gas)	$\frac{dT}{dz} = -\frac{g}{c_p}$	(3.294)		

[a]No thermal conductivity or viscosity.
[b]True generally.
[c]The second form of Euler's equation applies to incompressible flow only.
[d]Equation (3.292) is true only for an ideal gas.

Potential flow[a]

Velocity potential	$\boldsymbol{v} = \nabla \phi$	(3.295)	$\boldsymbol{v}$	velocity
	$\nabla^2 \phi = 0$	(3.296)	ϕ	velocity potential
Vorticity condition	$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = 0$	(3.297)	$\boldsymbol{\omega}$	vorticity
			$\boldsymbol{F}$	drag force on moving sphere
Drag force on a sphere[b]	$\boldsymbol{F} = -\frac{2}{3}\pi \rho a^3 \dot{\boldsymbol{u}} = -\frac{1}{2}M_d \dot{\boldsymbol{u}}$	(3.298)	a	sphere radius
			$\dot{\boldsymbol{u}}$	sphere acceleration
			ρ	fluid density
			M_d	displaced fluid mass

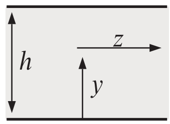
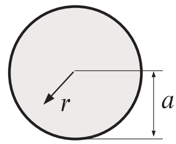
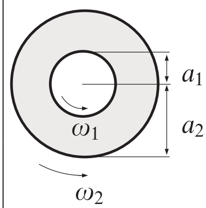
[a]For incompressible fluids.
[b]The effect of this drag force is to give the sphere an additional effective mass equal to half the mass of fluid displaced.

Viscous flow (incompressible)[a]

Fluid stress	$\tau_{ij}=-p\delta_{ij}+\eta\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)$	(3.299)	τ_{ij} fluid stress tensor p hydrostatic pressure η shear viscosity v_i velocity along i axis δ_{ij} Kronecker delta
Navier–Stokes equation[b]	$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\frac{\nabla p}{\rho}-\frac{\eta}{\rho}\nabla\times\boldsymbol{\omega}+\boldsymbol{g}$	(3.300)	$\boldsymbol{v}$ fluid velocity field $\boldsymbol{\omega}$ vorticity $\boldsymbol{g}$ gravitational acceleration ρ density
	$=-\frac{\nabla p}{\rho}+\frac{\eta}{\rho}\nabla^2\boldsymbol{v}+\boldsymbol{g}$	(3.301)	
Kinematic viscosity	$\nu=\eta/\rho$	(3.302)	ν kinematic viscosity

[a]I.e., $\nabla\cdot\boldsymbol{v}=0$, $\eta\neq 0$.
[b]Neglecting bulk (second) viscosity.

Laminar viscous flow

Between parallel plates	$v_z(y)=\frac{1}{2\eta}y(h-y)\frac{\partial p}{\partial z}$	(3.303)	v_z flow velocity z direction of flow y distance from plate η shear viscosity p pressure	
Along a circular pipe[a]	$v_z(r)=\frac{1}{4\eta}(a^2-r^2)\frac{\partial p}{\partial z}$	(3.304)	r distance from pipe axis a pipe radius V volume	
	$Q=\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\pi a^4}{8\eta}\frac{\partial p}{\partial z}$	(3.305)		
Circulating between concentric rotating cylinders[b]	$G_z=\frac{4\pi\eta a_1^2a_2^2}{a_2^2-a_1^2}(\omega_2-\omega_1)$	(3.306)	G_z axial couple between cylinders per unit length ω_i angular velocity of ith cylinder	
Along an annular pipe	$Q=\frac{\pi}{8\eta}\frac{\partial p}{\partial z}\left[a_2^4-a_1^4-\frac{(a_2^2-a_1^2)^2}{\ln(a_2/a_1)}\right]$	(3.307)	a_1 inner radius a_2 outer radius Q volume discharge rate	

[a]Poiseuille flow.
[b]Couette flow.

Drag[a]

On a sphere (Stokes's law)	$F=6\pi a\eta v$	(3.308)	F drag force a radius
On a disk, broadside to flow	$F=16a\eta v$	(3.309)	v velocity η shear viscosity
On a disk, edge on to flow	$F=32a\eta v/3$	(3.310)	

[a]For Reynolds numbers $\ll 1$.

Characteristic numbers

Reynolds number	$\mathsf{Re}=\frac{\rho UL}{\eta}=\frac{\text{inertial force}}{\text{viscous force}}$	(3.311)	Re ρ U L η	Reynolds number density characteristic velocity characteristic scale-length shear viscosity
Froude number[a]	$\mathsf{F}=\frac{U^2}{Lg}=\frac{\text{inertial force}}{\text{gravitational force}}$	(3.312)	F g	Froude number gravitational acceleration
Strouhal number[b]	$\mathsf{S}=\frac{U\tau}{L}=\frac{\text{evolution scale}}{\text{physical scale}}$	(3.313)	S τ	Strouhal number characteristic timescale
Prandtl number	$\mathsf{P}=\frac{\eta c_p}{\lambda}=\frac{\text{momentum transport}}{\text{heat transport}}$	(3.314)	P c_p λ	Prandtl number Specific heat capacity at constant pressure thermal conductivity
Mach number	$\mathsf{M}=\frac{U}{c}=\frac{\text{speed}}{\text{sound speed}}$	(3.315)	M c	Mach number sound speed
Rossby number	$\mathsf{Ro}=\frac{U}{\Omega L}=\frac{\text{inertial force}}{\text{Coriolis force}}$	(3.316)	Ro Ω	Rossby number angular velocity

[a]Sometimes the square root of this expression. L is usually the fluid depth.
[b]Sometimes the reciprocal of this expression.

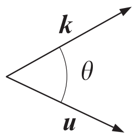
Fluid waves

Sound waves	$v_\mathrm{p}=\left(\frac{K}{\rho}\right)^{1/2}=\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)^{1/2}$	(3.317)	v_p K p ρ	wave (phase) speed bulk modulus pressure density
In an ideal gas (adiabatic conditions)[a]	$v_\mathrm{p}=\left(\frac{\gamma RT}{\mu}\right)^{1/2}=\left(\frac{\gamma p}{\rho}\right)^{1/2}$	(3.318)	γ R T μ	ratio of heat capacities molar gas constant (absolute) temperature mean molecular mass
Gravity waves on a liquid surface[b]	$\omega^2=gk\tanh kh$	(3.319)	v_g h λ	group speed of wave liquid depth wavelength
	$v_\mathrm{g}\simeq\begin{cases}\frac{1}{2}\left(\frac{g}{k}\right)^{1/2} & (h\gg\lambda)\\(gh)^{1/2} & (h\ll\lambda)\end{cases}$	(3.320)	k g ω	wavenumber gravitational acceleration angular frequency
Capillary waves (ripples)[c]	$\omega^2=\frac{\sigma k^3}{\rho}$	(3.321)	σ	surface tension
Capillary–gravity waves ($h\gg\lambda$)	$\omega^2=gk+\frac{\sigma k^3}{\rho}$	(3.322)		

[a]If the waves are isothermal rather than adiabatic then $v_\mathrm{p}=(p/\rho)^{1/2}$.
[b]Amplitude $\ll$ wavelength.
[c]In the limit $k^2\gg g\rho/\sigma$.

Doppler effect[a]

Source at rest, observer moving at u	$\frac{\nu'}{\nu} = 1 - \frac{\lvert\boldsymbol{u}\rvert}{v_{\mathrm{p}}}\cos\theta$	(3.323)	ν', ν'' observed frequency ν emitted frequency v_{p} wave (phase) speed in fluid
Observer at rest, source moving at u	$\frac{\nu''}{\nu} = \frac{1}{1 - \frac{\lvert\boldsymbol{u}\rvert}{v_{\mathrm{p}}}\cos\theta}$	(3.324)	$\boldsymbol{u}$ velocity θ angle between wavevector, $\boldsymbol{k}$, and $\boldsymbol{u}$



[a]For plane waves in a stationary fluid.

Wave speeds

Phase speed	$v_{\mathrm{p}} = \frac{\omega}{k} = \nu\lambda$	(3.325)	v_{p} phase speed ν frequency ω angular frequency $(=2\pi\nu)$ λ wavelength k wavenumber $(=2\pi/\lambda)$
Group speed	$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}$	(3.326)	v_{g} group speed
	$= v_{\mathrm{p}} - \lambda\frac{\mathrm{d}v_{\mathrm{p}}}{\mathrm{d}\lambda}$	(3.327)	

Shocks

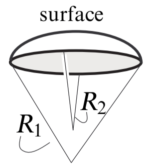
Mach wedge[a]	$\sin\theta_{\mathrm{w}} = \frac{v_{\mathrm{p}}}{v_{\mathrm{b}}}$	(3.328)	θ_{w} wedge semi-angle v_{p} wave (phase) speed v_{b} body speed
Kelvin wedge[b]	$\lambda_{\mathrm{K}} = \frac{4\pi v_{\mathrm{b}}^2}{3g}$	(3.329)	λ_{K} characteristic wavelength
	$\theta_{\mathrm{w}} = \arcsin(1/3) = 19^{\circ}.5$	(3.330)	g gravitational acceleration
Spherical adiabatic shock[c]	$r \simeq \left(\frac{Et^2}{\rho_0}\right)^{1/5}$	(3.331)	r shock radius E energy release t time ρ_0 density of undisturbed medium
Rankine–Hugoniot shock relations[d]	$\frac{p_2}{p_1} = \frac{2\gamma \mathsf{M}_1^2 - (\gamma - 1)}{\gamma + 1}$	(3.332)	$_1$ upstream values $_2$ downstream values p pressure
	$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{(\gamma - 1) + 2/\mathsf{M}_1^2}$	(3.333)	v velocity T temperature
	$\frac{T_2}{T_1} = \frac{[2\gamma\mathsf{M}_1^2 - (\gamma - 1)][2 + (\gamma - 1)\mathsf{M}_1^2]}{(\gamma + 1)^2\mathsf{M}_1^2}$	(3.334)	ρ density γ ratio of specific heats M Mach number

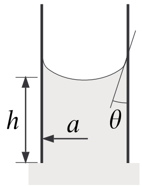
[a]Approximating the wake generated by supersonic motion of a body in a nondispersive medium.
[b]For gravity waves, e.g., in the wake of a boat. Note that the wedge semi-angle is independent of v_{b}.
[c]Sedov–Taylor relation.
[d]Solutions for a steady, normal shock, in the frame moving with the shock front. If $\gamma = 5/3$ then $v_1/v_2 \le 4$.

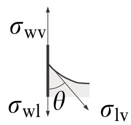
Surface tension

Definition	$\sigma_{lv} = \dfrac{\text{surface energy}}{\text{area}}$	(3.335)
	$= \dfrac{\text{surface tension}}{\text{length}}$	(3.336)
Laplace's formula[a]	$\Delta p = \sigma_{lv} \left(\dfrac{1}{R_1} + \dfrac{1}{R_2} \right)$	(3.337)
Capillary constant	$c_c = \left(\dfrac{2\sigma_{lv}}{g\rho} \right)^{1/2}$	(3.338)
Capillary rise (circular tube)	$h = \dfrac{2\sigma_{lv} \cos\theta}{\rho g a}$	(3.339)
Contact angle	$\cos\theta = \dfrac{\sigma_{wv} - \sigma_{wl}}{\sigma_{lv}}$	(3.340)

σ_{lv}	surface tension (liquid/vapour interface)
Δp	pressure difference over surface
R_i	principal radii of curvature
c_c	capillary constant
ρ	liquid density
g	gravitational acceleration
h	rise height
θ	contact angle
a	tube radius
σ_{wv}	wall/vapour surface tension
σ_{wl}	wall/liquid surface tension







[a]For a spherical bubble in a liquid $\Delta p = 2\sigma_{lv}/R$. For a soap bubble (two surfaces) $\Delta p = 4\sigma_{lv}/R$.