

4.2 Quantum definitions

Quantum uncertainty relations

De Broglie relation	$p = \frac{h}{\lambda}$	(4.1)	$p, \mathbf{p}$	particle momentum
	$\mathbf{p} = \hbar \mathbf{k}$	(4.2)	h	Planck constant
Planck–Einstein relation	$E = h\nu = \hbar\omega$	(4.3)	$\hbar$	$h/(2\pi)$
			λ	de Broglie wavelength
Dispersion ^a	$(\Delta a)^2 = \langle (a - \langle a \rangle)^2 \rangle$ $= \langle a^2 \rangle - \langle a \rangle^2$	(4.4)	$\mathbf{k}$	de Broglie wavevector
		(4.5)	E	energy
General uncertainty relation	$(\Delta a)^2(\Delta b)^2 \geq \frac{1}{4} \langle [\hat{a}, \hat{b}] \rangle^2$	(4.6)	ν	frequency
			ω	angular frequency ($= 2\pi\nu$)
Momentum–position uncertainty relation ^c	$\Delta p \Delta x \geq \frac{\hbar}{2}$	(4.7)	a, b	observables ^b
Energy–time uncertainty relation	$\Delta E \Delta t \geq \frac{\hbar}{2}$	(4.8)	$\langle \cdot \rangle$	expectation value
Number–phase uncertainty relation	$\Delta n \Delta \phi \geq \frac{1}{2}$	(4.9)	$(\Delta a)^2$	dispersion of a
			$\hat{a}$	operator for observable a
			$[\cdot, \cdot]$	commutator (see page 26)
			x	particle position
			t	time
			n	number of photons
			ϕ	wave phase

^aDispersion in quantum physics corresponds to variance in statistics.

^bAn observable is a directly measurable parameter of a system.

^cAlso known as the “Heisenberg uncertainty relation.”

Wavefunctions

Probability density	$\text{pr}(x, t) \, dx = \psi(x, t) ^2 \, dx$	(4.10)	pr	probability density
Probability density current ^a	$j(x) = \frac{\hbar}{2im} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right)$	(4.11)	ψ	wavefunction
	$\mathbf{j} = \frac{\hbar}{2im} [\psi^*(\mathbf{r}) \nabla \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla \psi^*(\mathbf{r})]$	(4.12)	$\mathbf{j}, j$	probability density current
	$= \frac{1}{m} \Re(\psi^* \hat{\mathbf{p}} \psi)$	(4.13)	$\hbar$	(Planck constant)/(2π)
			x	position coordinate
Continuity equation	$\nabla \cdot \mathbf{j} = -\frac{\partial}{\partial t}(\psi \psi^*)$	(4.14)	$\hat{\mathbf{p}}$	momentum operator
Schrödinger equation	$\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$	(4.15)	m	particle mass
Particle stationary states ^b	$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$	(4.16)	$\Re$	real part of
			t	time
			H	Hamiltonian
			V	potential energy
			E	total energy

^aFor particles. In three dimensions, suitable units would be particles $\text{m}^{-2}\text{s}^{-1}$.

^bTime-independent Schrödinger equation for a particle, in one dimension.

Operators

Hermitian conjugate operator	$\int (\hat{a}\phi)^* \psi \, dx = \int \phi^* \hat{a}\psi \, dx \quad (4.17)$	$\hat{a}$ Hermitian conjugate operator ψ, ϕ normalisable functions
Position operator	$\hat{x}^n = x^n \quad (4.18)$	$*$ complex conjugate x, y position coordinates
Momentum operator	$\hat{p}_x^n = \frac{\hbar^n}{i^n} \frac{\partial^n}{\partial x^n} \quad (4.19)$	n arbitrary integer ≥ 1 p_x momentum coordinate
Kinetic energy operator	$\hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \quad (4.20)$	T kinetic energy $\hbar$ (Planck constant)/(2 π) m particle mass
Hamiltonian operator	$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \quad (4.21)$	H Hamiltonian V potential energy
Angular momentum operators	$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \quad (4.22)$ $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad (4.23)$	L_z angular momentum along z axis (sim. x and y) L total angular momentum
Parity operator	$\hat{P}\psi(\mathbf{r}) = \psi(-\mathbf{r}) \quad (4.24)$	$\hat{P}$ parity operator $\mathbf{r}$ position vector

Expectation value

Expectation value ^a	$\langle a \rangle = \langle \hat{a} \rangle = \int \Psi^* \hat{a} \Psi \, dx \quad (4.25)$ $= \langle \Psi \hat{a} \Psi \rangle \quad (4.26)$	$\langle a \rangle$ expectation value of a $\hat{a}$ operator for a Ψ (spatial) wavefunction x (spatial) coordinate
Time dependence	$\frac{d}{dt} \langle \hat{a} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{a}] \rangle + \left\langle \frac{\partial \hat{a}}{\partial t} \right\rangle \quad (4.27)$	t time $\hbar$ (Planck constant)/(2 π)
Relation to eigenfunctions	if $\hat{a}\psi_n = a_n\psi_n$ and $\Psi = \sum c_n\psi_n$ then $\langle a \rangle = \sum c_n ^2 a_n \quad (4.28)$	ψ_n eigenfunctions of $\hat{a}$ a_n eigenvalues n dummy index c_n probability amplitudes
Ehrenfest's theorem	$m \frac{d}{dt} \langle \mathbf{r} \rangle = \langle \mathbf{p} \rangle \quad (4.29)$ $\frac{d}{dt} \langle \mathbf{p} \rangle = -\langle \nabla V \rangle \quad (4.30)$	m particle mass $\mathbf{r}$ position vector $\mathbf{p}$ momentum V potential energy

^aEquation (4.26) uses the Dirac “bra-ket” notation for integrals involving operators. The presence of vertical bars distinguishes this use of angled brackets from that on the left-hand side of the equations. Note that $\langle a \rangle$ and $\langle \hat{a} \rangle$ are taken as equivalent.

Dirac notation

Matrix element ^a	$a_{nm} = \int \psi_n^* \hat{a} \psi_m \, dx \quad (4.31)$ $= \langle n \hat{a} m \rangle \quad (4.32)$	n, m eigenvector indices a_{nm} matrix element ψ_n basis states $\hat{a}$ operator x spatial coordinate
Bra vector	bra state vector = $\langle n $ (4.33)	$\langle \cdot $ bra
Ket vector	ket state vector = $ m \rangle$ (4.34)	$ \cdot\rangle$ ket
Scalar product	$\langle n m \rangle = \int \psi_n^* \psi_m \, dx \quad (4.35)$	
Expectation	if $\Psi = \sum_n c_n \psi_n \quad (4.36)$ then $\langle a \rangle = \sum_m \sum_n c_n^* c_m a_{nm} \quad (4.37)$	Ψ wavefunction c_n probability amplitudes

^aThe Dirac bracket, $\langle n | \hat{a} | m \rangle$, can also be written $\langle \psi_n | \hat{a} | \psi_m \rangle$.