Chapter – 10

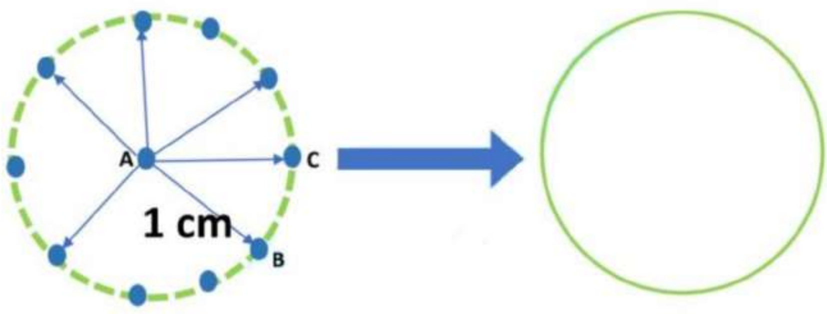
Circles

Introduction to Circles

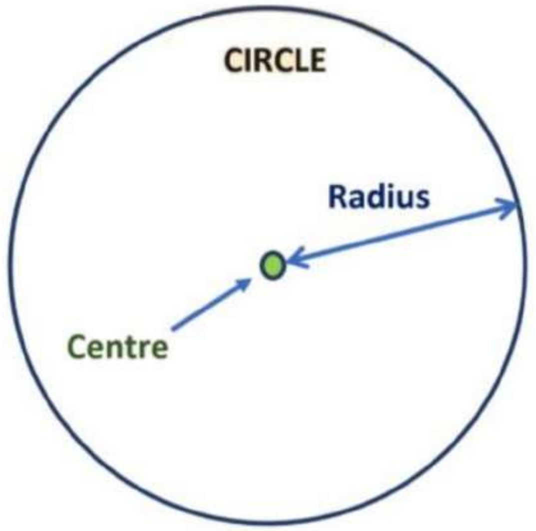
A circle is a unique figure; it is everywhere around us. We see the dials of clocks, buttons of shirts, coins, wheels of a vehicle, etc. All these are in the shape of a circle.

How to Draw a Circle

Suppose, we have a point 'A', let us take another point 'B' which is 1 cm away from point A. Let us take a third point 'C' which is also 1 cm away from point A. Similarly, the fourth point, fifth point and so on. All these points are 1 cm away from point A. When we join all these points together, they form a circle.

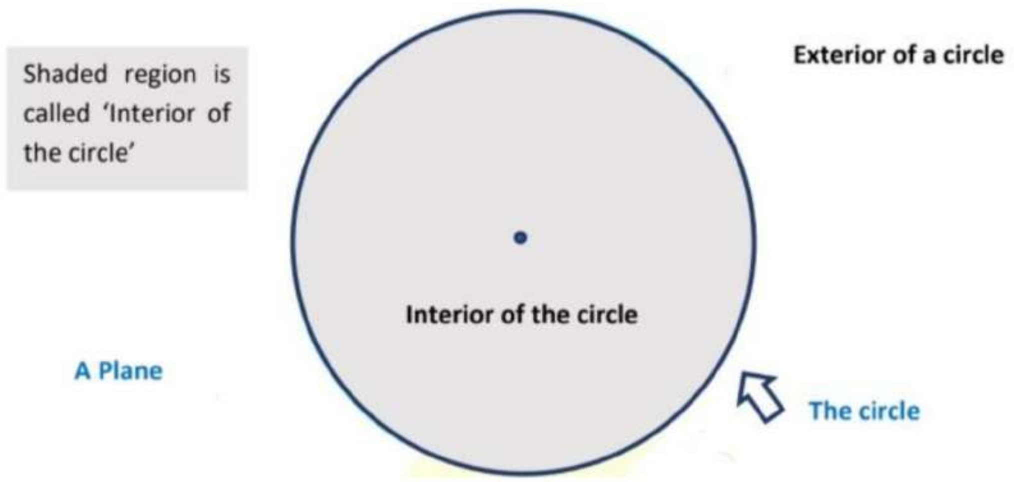


So, a circle is the set of all points in a plane that are equally distant from a fixed point. The fixed point is called the Centre of the circle and the distance from the Centre to any point on the circle is called the radius of the circle.



A circle divides the plane on which it lies into three parts:

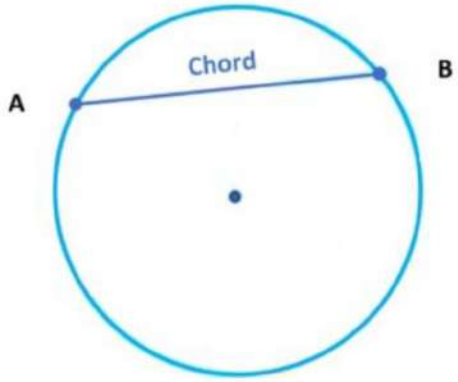
I. The circle
II. Inside the circle, which is also called interior of the circle.
III. Outside the circle, which is also called exterior of the circle.



Note: The interior of the circle together with its circumference is called the circular region.

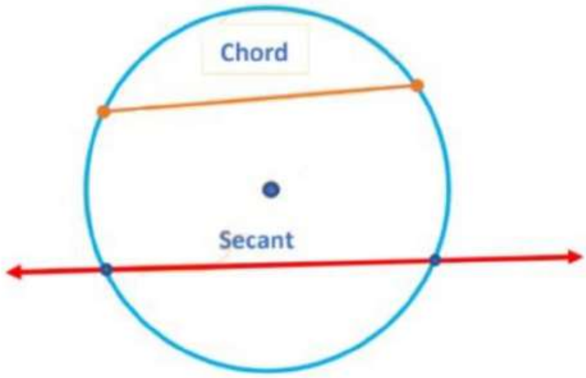
Terms related to circles

Chord: The chord of a circle is a straight line segment whose endpoints lie on the circle.

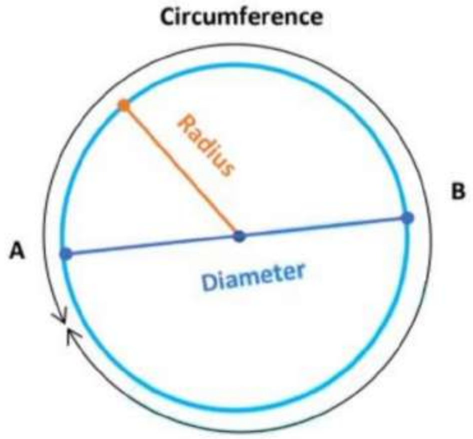


Secant: The secant of a circle is a straight line that intersects the circle at two distinct points.

So, a line that touches the circle at two points is called a chord and a line that intersects the circle at two distinct points is called a Secant.



Diameter: The chord, which passes through the center of the circle is called a diameter of the circle. Diameter is the longest chord and all diameters have the same length, which is equal to two times the radius of the circle.

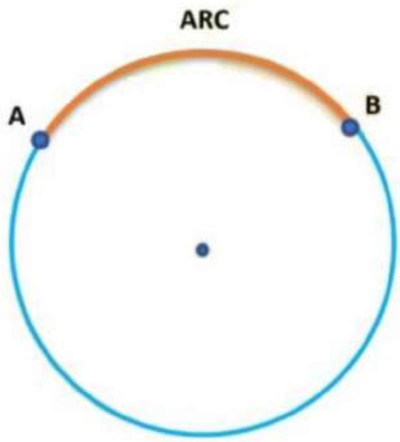


The length of the complete circle is called its circumference.

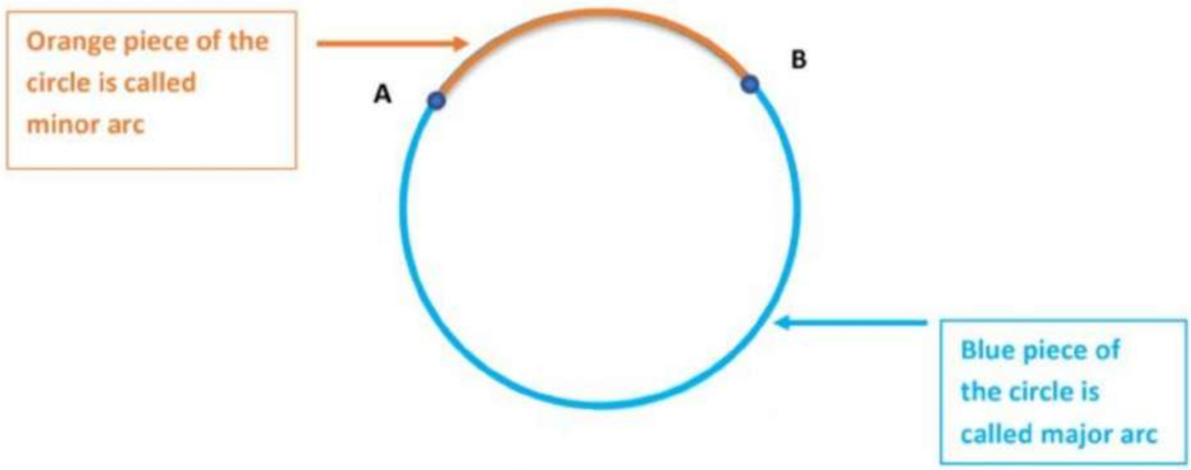
Arc: The arc of a circle is a portion of the circumference of a circle.

Or

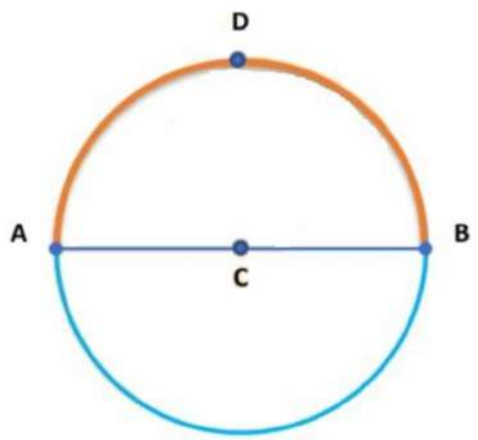
A piece of a circle between two points is also called an arc.



Two points lying on the circle define two arcs: The shorter one is called a minor arc and the longer one is called a major arc.

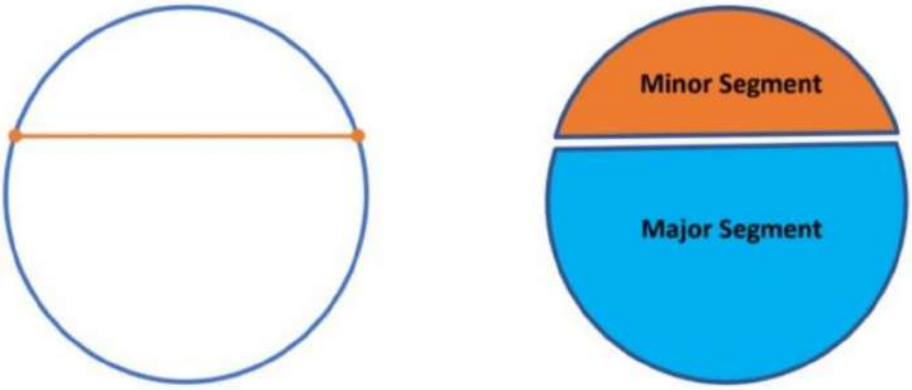


The minor arc AB is also denoted by $\widehat{AB}$ and the major arc AB by $\widehat{ADB}$ where D is some point on the arc between A and B. When A and B are ends of a diameter, then both arcs are equal and each is called a semicircle.

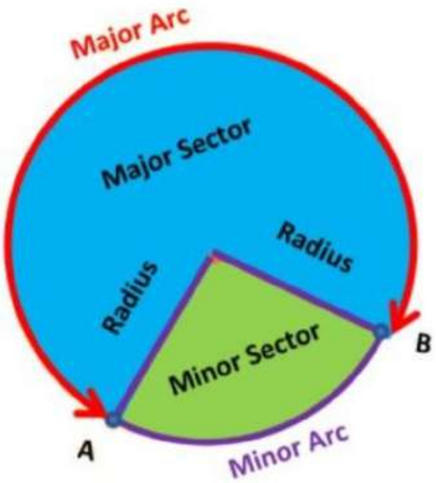


Arc ADB is a semi-circle.

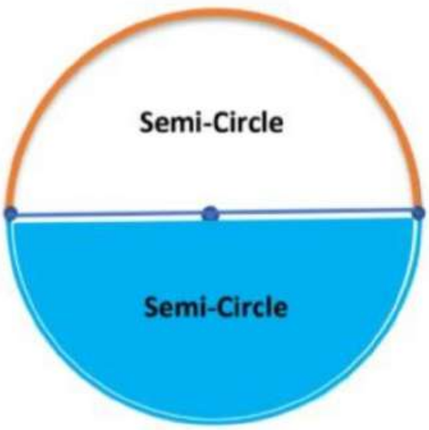
Segment: The region between a chord and either of its arc is called a segment of the circle. There are two types of segments also: which are the major segment and the minor segment.



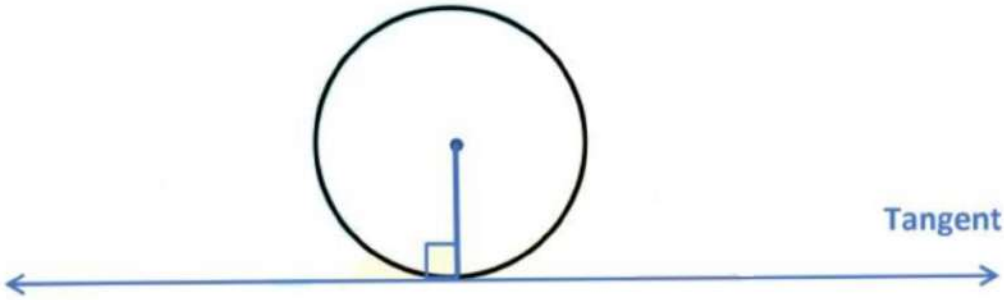
Sector: The region between the arc and the two radii, joining the center to the ends points of the arc is called a sector. The minor arc corresponds to the minor sector and the major arc corresponds to the major sector.



When the two arcs are equal, then both segments and both sectors become equal and each is known as a semicircle.



Tangent: It is a line that touches a circle at one point; without cutting across the circle. The angle between a tangent and the radius of a circle is 90°.

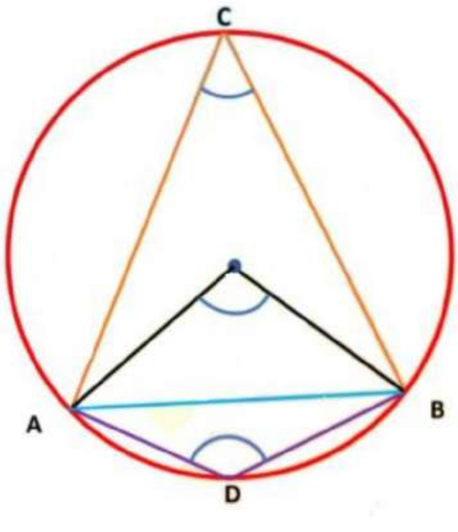


Equal chords of a circle subtend equal angles at the centre

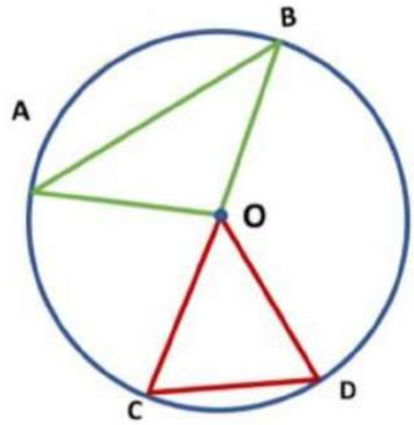
Angle Subtended by a Chord at a Point

Take a line segment AB and a point C on the circle not on the line containing AB. Then ∠ACB is called the angle subtended by the line segment AB at point C.

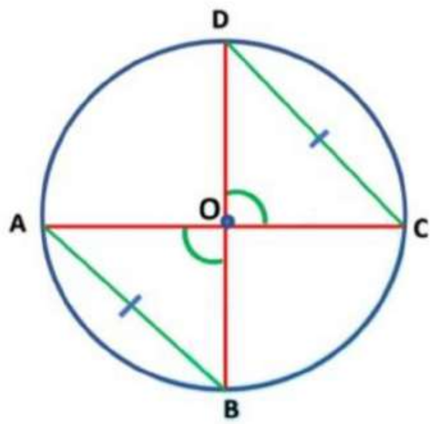
Also, ∠AOB is the angle subtended by the chord AB at the center O, ∠ADB and ∠ACB are respectively the angles subtended by the chord AB at points D and C on the minor and major arcs AB.



We may see by drawing different chords of a circle and angles subtended by them at the center that the longer is the chord AB, the bigger will be the angle ∠AOB subtended by it at the center.



Theorem 1: Equal chords of a circle subtend equal angles at the center.



Given: AB and CD are two equal chords of a circle with a center O.

To Prove: $\angle AOB = \angle COD$

Proof: In Δ AOB and Δ COD,

$\Rightarrow$ $OA = OC$ (Radii of a circle)

$\Rightarrow$ $OB = OD$ (Radii of a circle)

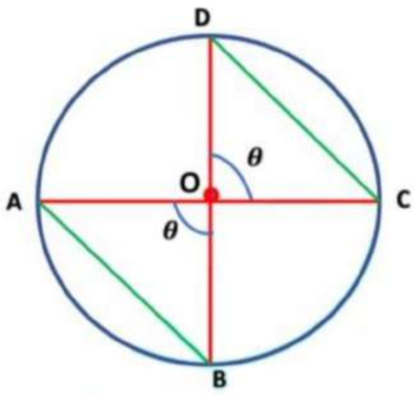
$\Rightarrow$ $AB = CD$ (Given)

Therefore, $\Delta AOB \cong \Delta COD$ (By SSS-criterion of congruence)

By using corresponding parts of congruent triangles

$\Rightarrow \angle AOB = \angle COD$. Proved.

Theorem 2: If the angles subtended by the chords of a circle at the center are equal, then chords are equal.



Given: In a circle, $\angle AOB$ and $\angle COD$ are two equal angles at the center.

To Prove: AB and CD are two equal chords of a circle with a center O.

Proof: In Δ AOB and Δ COD,

$\Rightarrow$ $OA = OC$ (Radii of a circle)

$\Rightarrow$ $\angle AOB = \angle COD$ (Given)

$\Rightarrow$ $OB = OD$ (Radii of a circle)

Therefore, $\Delta AOB \cong \Delta COD$ (By SAS-criterion of congruence)

$\Rightarrow AB = CD$ (By using corresponding parts of congruent triangles)

Hence, AB and CD are two equal chords of a circle with a center O.
Proved.

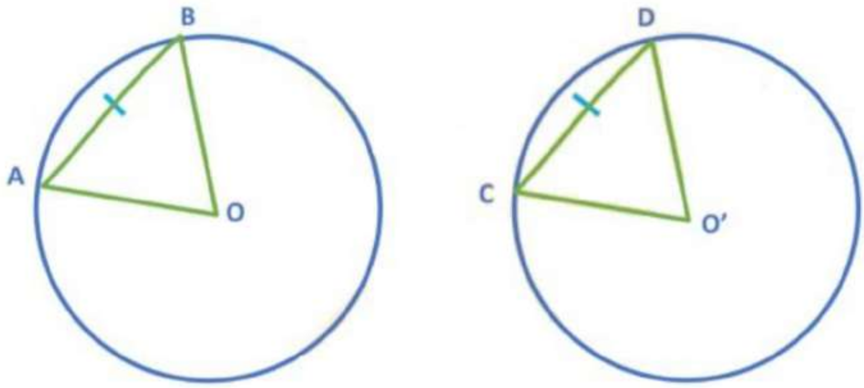
Theorem 3: Equal chords of congruent circles subtended equal angles at their centers.

Given: Two chords of congruent circles are equal.

To Prove: The angles subtended at the centers are equal.

Proof: A circle is the collection of points that are equidistant from a fixed point. This fixed point is called the centre of the circle and this equal distance is called a radius of the circle. And thus, size of a circle depends on its radius. Therefore, it can be observed that if we try to superimpose two circles of equal radius, then both circles will cover each other. Therefore, two circles are congruent if they have an equal radius.

Consider two congruent circles having center O and O' and two chords AB and CD of equal length.



In Δ AOB and Δ CO'D,

⇒ OA = O'C (Radii of congruent circles)

⇒ OB = O'D (Radii of congruent circles)

⇒ AB = CD (Given)

Therefore, Δ AOB ≅ Δ CO'D (By SSS-criterion of congruence)

By using corresponding parts of congruent triangles

This gives, ∠AOB = ∠CO'D.

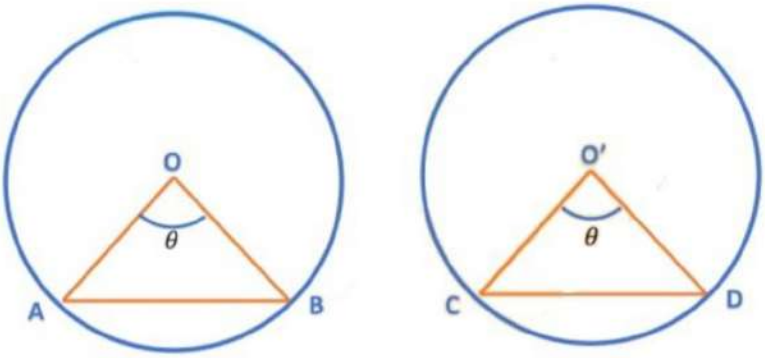
Hence, equal chords of congruent circles subtended equal angles at their centers. Proved.

Theorem 4: If chords of congruent circles subtend equal angles at their centers, then the chords are equal.

Given: The angles subtended at the centers are equal.

To Prove: Two chords of congruent circles are also equal.

Proof: Let us consider two congruent circles having center O and O'.



In Δ AOB and Δ CO'D,

⇒ OA = O'C (Radii of congruent circles)

⇒ ∠AOB = ∠CO'D (Given)

⇒ OB = O'D (Radii of congruent circles)

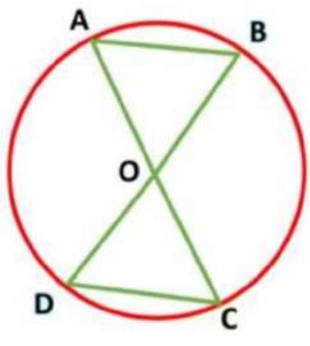
Therefore, Δ AOB ≅ Δ CO'D (By SAS-criterion of congruence)

By using corresponding parts of congruent triangles

This gives AB = CD.

Hence, if chords of congruent circles subtend equal angles at their centres, then the chords are equal. Proved.

Examples: AB and CD are two chords of a circle whose diameters are BD and AC. To Prove: AB = CD.



Solution: AC and BD are diameters of a circle which intersect at the center O.

Therefore, $\angle AOB = \angle COD$

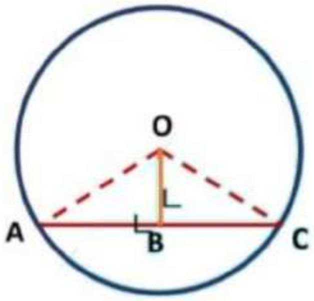
($\because$ $\angle AOB$ and $\angle COD$ are vertically opposite angles)

We know that the angles subtended by the chords of a circle at the center are equal, then chords are equal.

Thus, $AB = CD$. Proved.

Perpendicular from the centre of a circle to the chord

Theorem 5: The perpendicular from the center of a circle to its chord, bisects the chord.



Given: From center O, OB is perpendicular to the chord AB.

To Prove: OB is the bisector of chord AB.

Construction: Join OA and OC.

Proof: In Δ AOB and Δ COD,

$\Rightarrow$ $OA = OC$ (Radii of the circle)

$\Rightarrow$ $\angle ABO = \angle CBO$ (Both equal to $90°$)

$\Rightarrow$ $OB = OB$ (common)

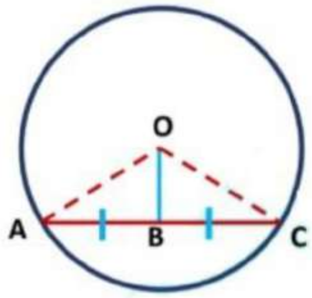
Therefore, $\Delta AOB \cong \Delta BOC$ (By RHS-criterion of congruence)

By using corresponding parts of congruent triangles

This gives, AB = BC.

Hence, the perpendicular from the center of a circle to a chord bisects the chord. Proved.

Theorem 6: The line joining the center of a circle to the mid-point of a chord is perpendicular to the chord.



Given: OB is the bisector of chord AC.

To Prove: From center O, OB is perpendicular to the chord AC.

Construction: Join OA and OC.

Proof : In Δ AOB and Δ COB,

⇒ OA = OC (Radii of a circle)

⇒ AB = BC (B is the mid-point of AC)

⇒ OB = OB (common)

Therefore, Δ AOB ≅ Δ BOC (By SSS-criterion of congruence)

By using corresponding parts of congruent triangles

This gives, ∠ABO = ∠CBO

⇒∠ABO +∠CBO = 180° (Linear pair of angles)

⇒∠ABO+∠ABO = 180° (∠ABO=∠CBO proved above)

⇒ 2∠ABO = 180°

$\Rightarrow \angle ABO = 90°$

Therefore, $\angle ABO = \angle CBO$.

$\Rightarrow \angle CBO = 90°$.

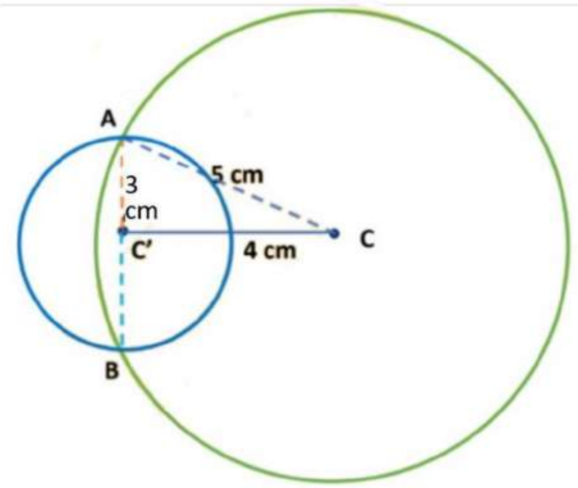
Hence, OB is perpendicular to the chord AC. Proved.

Example 1: Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centers is 4 cm. Find the common chord.

(REFERENCE: NCERT)

Given: The circle with center C has a radius of 5 cm and the circle with center C' has a radius of 3 cm, and the distance between these two circles is 4 cm.

Construction: Join AC and AB.



Solution: Suppose AB be the common chord of the two circles having their centers at C and C'.

Then, CA = 5 cm, C'A = 3 cm and CC' = 4 cm.

We have, $5^2 = 4^2 + 3^2$ (By Pythagoras Theorem)

$\Rightarrow (AC)^2 = (C'C)^2 + (AC')^2$

$\Rightarrow \angle AC'C = 90°$

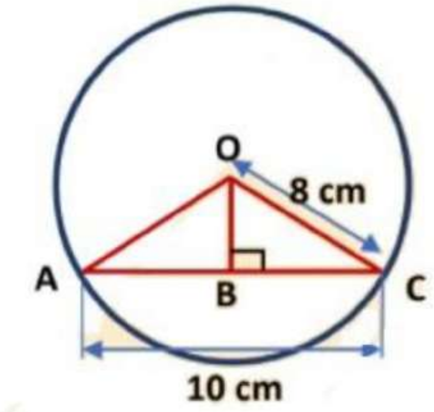
⇒ C' lies on the common chord AB.

By theorem, the perpendicular from the center of a circle to a chord bisects the chord. So, C'C bisects AB.

Since CC' ⊥ AB and line segment joining the centers of two circles bisects the common chord. Therefore, C' is the mid-point of AB.

Hence, AB = 2 C'A = 2 x 3 cm = 6 cm.

Examples 2: A circle has a chord of the length of 10 cm. If the radius of the circle is 8 cm, find the shortest distance from the center of the circle to the chord.



Solution: The shortest distance is the perpendicular distance. The line drawn from the center of the circle, perpendicular to a chord, bisects the chord.

Therefore, AB = $\frac{1}{2}$ AC.

⇒ AB = $\frac{1}{2}$ x 10 cm

⇒ AB = 5 cm

In triangle AOB,

$(AB)^2 + (OB)^2 = (OA)^2$ (By Pythagoras theorem)

⇒ $(5\text{ cm})^2 + (OB)^2 = (8\text{ cm})^2$

$\Rightarrow 25\ cm^2 + (OB)^2 = 64\ cm^2$

$\Rightarrow (OB)^2 = 64\ cm^2 - 25\ cm^2 \Rightarrow (OB)^2 = 39\ cm^2$

$\Rightarrow OB = \sqrt{39}\ cm$

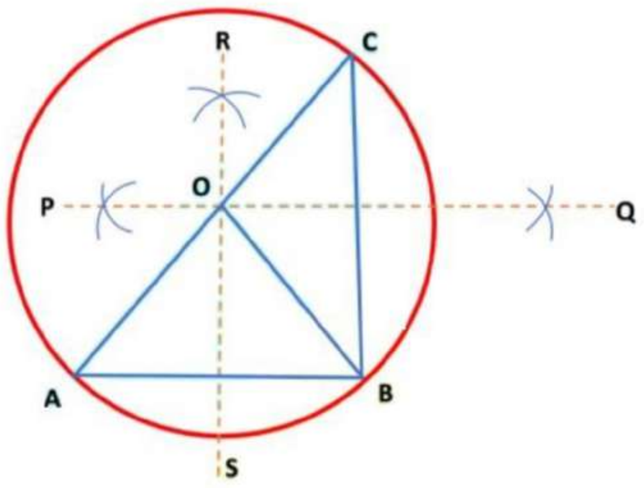
$\Rightarrow OB \approx 6.24\ cm$

Hence, the shortest distance is approx 6.24 cm, because OB is the shortest distance.

Circle passing through three points

Circle passing through three points

Theorem 7: There is one and only one circle passing through three non-collinear points.



Given: Three non-collinear points are A, B, and C.

To Prove: There is one and only one circle passing through points A, B, and C.

Construction: Join AB and BC. Draw perpendicular bisectors RS and PQ of AB and BC respectively. A, B and C are not collinear. Therefore, the perpendicular bisectors RS and PQ are not parallel. Let RS and PQ intersect at O. Join OA, OB, and OC.

Proof: Since O lies on the perpendicular bisector of AB and also by construction OQ and OS are perpendicular to AB and BC respectively.

Therefore, OA = OB

Again, O lies on the perpendicular bisector of BC.

Therefore, OB = OC

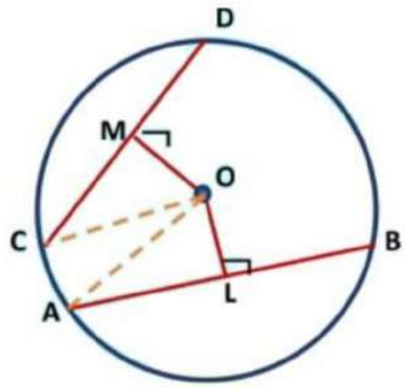
Thus, OA = OB = OC = radius = r ....(I)

From (I), O as the center, draw a circle of radius r. Clearly, circle passes through A, B and C. This proves that there is a circle passing through points A, B, and C.

Equal Chords and their Distance from Centre

Equal Chords and their Distance from Centre

Theorem 8: Equal chords of a circle (or congruent circles) are equidistant from the center (or centers).



Given: Two chords AB and CD of a circle with center O are equal and OL ⊥ AB and OM⊥CD.

To Prove: AB = CD

Construction: Join OA and OC.

Proof: Since the perpendicular from the center of a circle to a chord bisects the chord.

Therefore, OL ⊥ AB ⇒ AL = $\frac{1}{2}$ AB ....(1)

And, OM ⊥ CD ⇒ CM = $\frac{1}{2}$ CD ....(2)

But, AB = CD

⇒ $\frac{1}{2}$ AB = $\frac{1}{2}$ CD

From equations (1) and (2)

we have, AL = CM ....(3)

Now, In right angled triangles Δ OAL and Δ OCM,

We have, OA = OC [Radii of the same circle]

⇒ AL = CM [From equation (3)]

And, ∠ALO = ∠CMO [Each equal to 90°]

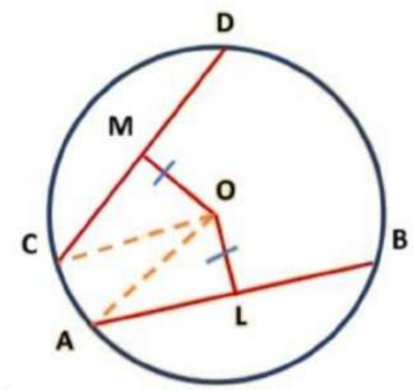
Therefore, Δ OAL ≅ Δ OCM [By RHS-criterion of congruence]

By using corresponding parts of congruent triangles

∴ OL = OM.

Hence, chords AB and CD are equidistant from the center O.

Theorem 9: Chords equidistant from the center of a circle are equal in length.



Given: Two chords AB and CD of the circle which are equidistant from its center i.e. OL = OM where OL ⊥ AB and OM⊥CD.

To Prove: AB = CD

Construction: Join OA and OC.

Proof: Since the perpendicular from the center of a circle to the chord bisects the chord.

Therefore, $OL \perp AB \Rightarrow AL = \frac{1}{2} AB$ ....(1)

And, $OM \perp CD \Rightarrow CM = \frac{1}{2} CD$ ....(2)

Now, In right-angled triangles OAL and OCM,

We have, OA = OC [Radii of the same circle]

OL = OM [Given]

And, $\angle ALO = \angle CMO$ [Each equal to 90°]

Therefore, Δ OAL $\cong$ Δ OCM (By RHS-criterion of congruence)

By using corresponding parts of congruent triangles.

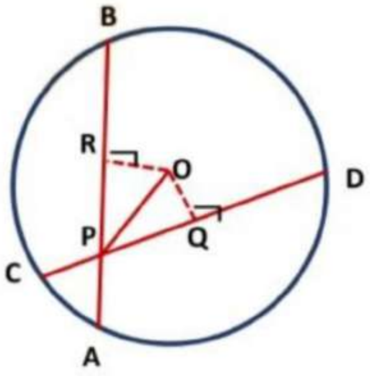
$\therefore$ AL =CM

From equation (1) and (2).

we can write, AB = CD

Hence, Chords AB and CD are equal.

Example: In the figure, O is the center of the circle and PO bisects $\angle BPD$. To Prove: AB = CD. (REFERENCE: NCERT)



Given: O is the center of the circle and ∠BPO = ∠OPD.

To Prove: AB = CD.

Construction: Draw OR ⊥ AB and OQ ⊥ CD.

Solution: In Δ ORP and Δ OQP, we have

⇒ ∠ORP = ∠OQP [Each equal to 90°]

⇒ OP = OP [Common]

And, ∠OPR = ∠OPQ [∵ OP bisects ∠BPD]

Thus, Δ ORP ≅ Δ OQP [By AAS-criterion of congruence]

(By Corresponding parts of congruent triangles)

∴ OR =OQ
Here, chords AB and CD are equidistant from the center O of the circle.

We know that Chords equidistant from the center of a circle are equal in length.

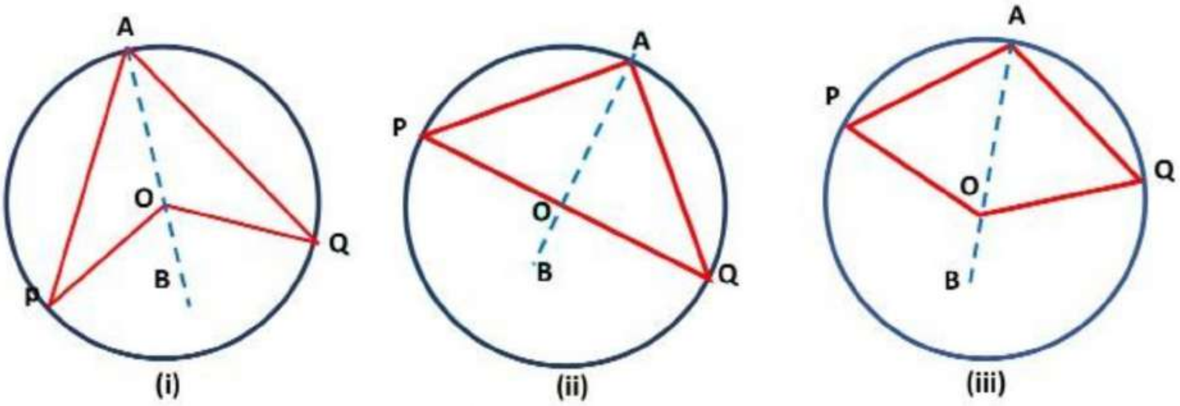
Hence,

AB = CD. Proved.

Angle Subtended by an Arc of a Circle

Angle Subtended by an Arc of a Circle

Theorem 10: The angle subtended by an arc of a circle at the center is double the angle subtended by it at any point on the circle or at a point on its circumference.



Given: An arc PQ of a circle and a point A on the remaining part of the circle i.e. arc QP.

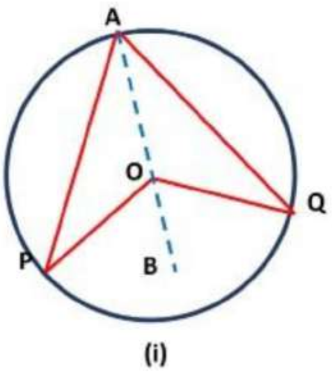
To Prove: $\angle POQ = 2\angle PAQ$

Construction: Join OA and produce it to a point B.

Proof: CASE I: when PQ is a minor arc in the figure (i).

We know that an exterior angle of a triangle is equal to the sum of the interior opposite angles.

In Δ POA, $\angle$POB is the exterior angle.



(i)

$\Rightarrow \angle POB = \angle OPA + \angle OAP$

$\Rightarrow \angle POB = \angle OAP + \angle OAP$

[$\because OP = OA = r, \therefore \angle OPA = \angle OAP$]

$\Rightarrow \angle POB = 2\angle OAP$ .... (i)

In Δ QOB, ∠QOB is the exterior angle.

$\therefore \angle QOB = \angle OQA + \angle OAQ$

$\Rightarrow \angle QOB = \angle OAQ + \angle OAQ$ [$\because OQ = OA = r, \therefore \angle OQA = \angle OAQ$]

$\Rightarrow \angle QOB = 2\angle OAQ$ .... (ii)

Adding equations (i) and (ii), we get

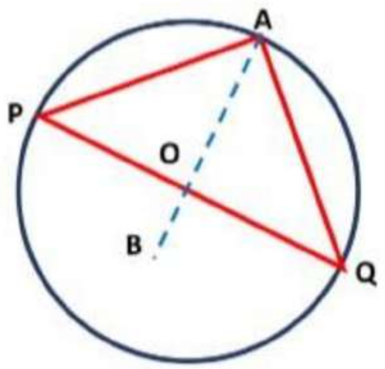
$\angle POB + \angle QOB = 2\angle OAP + 2\angle OAQ$

$\Rightarrow \angle POQ = 2(\angle OAP + \angle OAQ)$

$\Rightarrow \angle POQ = 2\ \angle PAQ$

CASE II: when PQ is a semi-circle in the figure (ii).

We know that an exterior angle of a triangle is equal to the sum of the interior opposite angles.



(ii)

In Δ POA, ∠POB is the exterior angle.

$\therefore \angle POB = \angle OPA + \angle OAP$

$\Rightarrow \angle POB = \angle OAP + \angle OAP$

$[\because OP = OA = r, \therefore \angle OPA = \angle OAP]$

$\Rightarrow \angle POB = 2\angle OAP$ .... (iii)

In Δ QOB, $\angle QOB$ is the exterior angle.

$\therefore \angle QOB = \angle OQA + \angle OAQ$

$\Rightarrow \angle QOB = \angle OAQ + \angle OAQ$ $[\because OQ = OA = r, \therefore \angle OQA = \angle OAQ]$

$\Rightarrow \angle QOB = 2\angle OAQ$ .... (iv)

Adding equations (iii) and (iv), we get

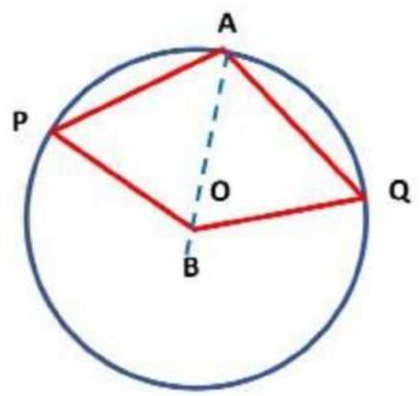
$\angle POB + \angle QOB = 2\angle OAP + 2\angle OAQ$

$\Rightarrow \angle POQ = 2(\angle OAP + \angle OAQ)$

$\Rightarrow \angle POQ = 2\ \angle PAQ$

CASE III : when PQ is a major arc in the figure (iii).

We know that an exterior angle of a triangle is equal to the sum of the interior opposite angles.



(iii)

In Δ POA, $\angle POB$ is the exterior angle.

$\therefore \angle POB = \angle OPA + \angle OAP$

$\Rightarrow \angle POB = \angle OAP + \angle OAP$

[$\because OP = OA = r, \therefore \angle OPA = \angle OAP$]

$\Rightarrow \angle POB = 2\angle OAP$ .... (v)

In Δ QOB, $\angle QOB$ is the exterior angle.

$\therefore \angle QOB = \angle OQA + \angle OAQ$

$\Rightarrow \angle QOB = \angle OAQ + \angle OAQ$ [$\because OQ = OA = r \therefore \angle OQA = \angle OAQ$]

$\Rightarrow \angle QOB = 2\angle OAQ$ .... (vi)

Adding equations (v) and (vi), we get

$\angle POB + \angle QOB = 2\angle OAP + 2\angle OAQ$

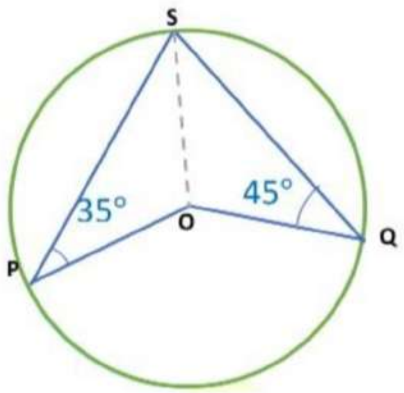
$\Rightarrow$ Reflex $\angle POQ = 2(\angle OAP + \angle OAQ)$

$\Rightarrow$ Reflex $\angle POQ = 2\ \angle PAQ$

Hence, the angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the circle.

Proved.

Examples 1: In the figure, calculate the measure of $\angle POQ$.



Given: $\angle SPO = 35°$ and $\angle SQO = 45°$

Construction: Join OS.

Solution: In Δ SPO, we have

$OP = OS$ (Each equal to the radius)

$\Rightarrow \angle SPO = \angle OSP$

($\because$ Angle opposite to equal sides of a triangle are equal)

$\Rightarrow \angle OSP = 35°$ ($\because \angle SPO = 35°$) ......(i)

In Δ SQO, we have

$OQ = OS$ (Each equal to the radius)

$\Rightarrow \angle SQO = \angle OSQ$

($\because$ Angle opposite to equal sides of a triangle are equal)

$\Rightarrow \angle OSQ = 45°$ ($\because \angle SQO = 45°$) ......(ii)

$\therefore \angle PSQ = \angle OSP + \angle OSQ$.

$\Rightarrow \angle PSQ = 35° + 45°$

$\Rightarrow \angle PSQ = 80°$

Since the angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the circle.

Therefore,

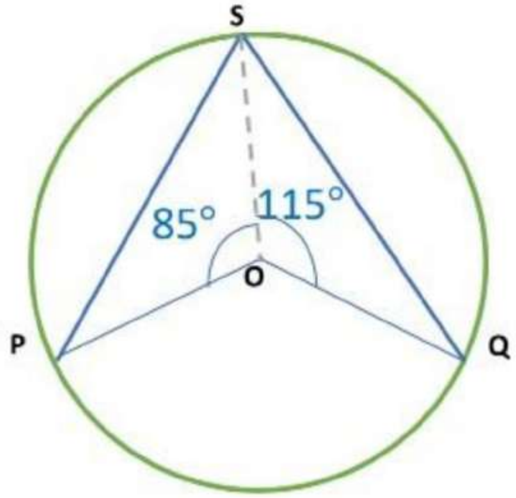
$\angle POQ = 2 \angle PSQ$.

$\Rightarrow \angle POQ = 2 \text{ x } 80°$.

$\Rightarrow \angle POQ = 160°$.

Examples 2: In the figure, P, S, and Q are three points on a circle such that the angles subtended by the chords PS and SQ at the center O are 85° and 115°, respectively.

Determine $\angle PSQ$.



Solution : We have, ∠POS = 85° and ∠QOS = 115°

∴∠POQ = 360° - (∠POS + ∠QOS).

⇒∠POQ = 360° - (85° + 115°).

⇒∠POQ = 360° - (200°).

⇒∠POQ = 160°.

Since the angle subtended by an arc of a circle at the center is double the angle subtended by it at any point on the circle.

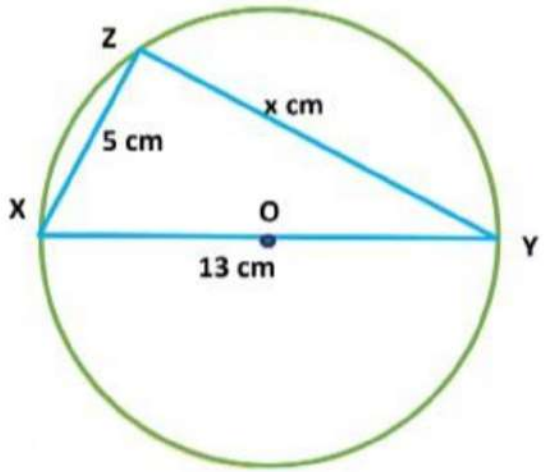
So, 2 ∠PSQ = ∠POQ.

⇒ 2 ∠PSQ = 160°

⇒ $\angle PSQ = \frac{160}{2}^{\circ}$

⇒ ∠PSQ = 80°.

Example 3: A circle has a diameter XY of length 13 cm. Z is a point on the circle such that XZ is 5 cm. Find the length YZ.



Solution: We know that the angle subtended by an arc of a circle at the center is double the angle subtended by it at any point on the circle.

So, Δ XYZ is a right-angled triangle.

Let the length YZ is x cm.

$(xz)^2 + (zy)^2 = (xy)^2$ (By Pythagoras theorem)

$\Rightarrow (5\text{ cm})^2 + (x)^2 = (13\text{ cm})^2$

$\Rightarrow 25\text{ cm}^2 + (x)^2 = 169\text{ cm}^2$

$\Rightarrow (x)^2 = 169\text{ cm}^2 - 25\text{ cm}^2$

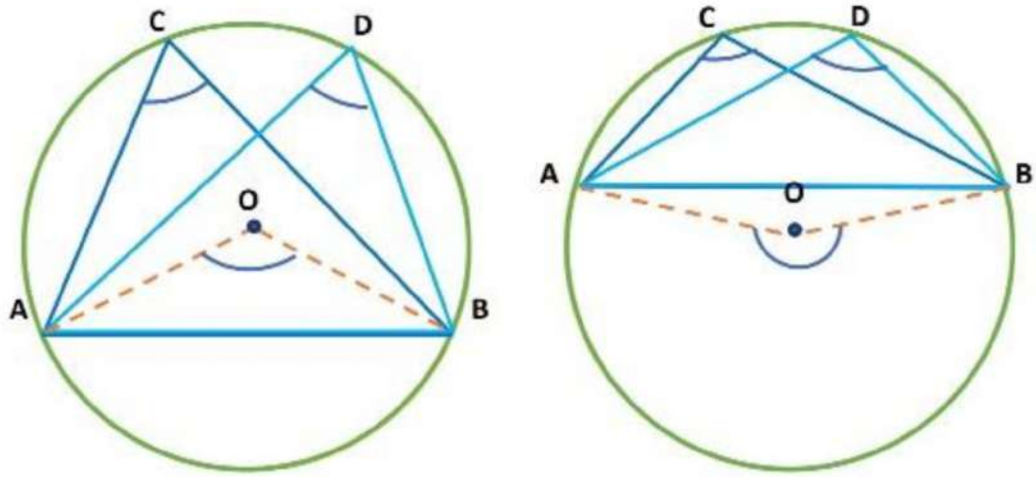
$\Rightarrow (x)^2 = 144\text{ cm}^2$

$\Rightarrow \quad x = \sqrt{144}\text{ cm}$

$\Rightarrow \quad x = 12\text{ cm}$

Hence, YZ has length 12 cm.

Theorem 11: Angles in the same segment of a circle are equal.



Given: An arc AB and two angles ∠ACB and ∠ADB in the same segment of the circle.

To Prove: ∠ACB = ∠ADB

Construction: Join OA and OB.

Proof: We know that the angle subtended by an arc at the center is double the angle subtended by the arc at any point in the remaining part of the circle.

CASE I : In the figure (i), we have

∠AOB = 2∠ACB ....(i)

And, ∠AOB = 2∠ADB ....(ii)

From equation (i) and (ii), we ge

2∠ACB = 2∠ADB

∴∠ACB = ∠ADB

CASE II : In the figure (ii), we have

Reflex ∠AOB = 2∠ACB ....(iii)

And, Reflex ∠AOB = 2∠ADB ....(iv)

From equation (iii) and (iv), we get , 2∠ACB = 2∠ADB

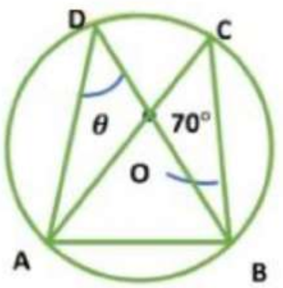
∴∠ACB = ∠ADB

Thus, in both case, we have

∠ACB = ∠ADB

Hence, angles in the same segment of a circle are equal.

Example 4: In the figure, ∠ACB = 70° then find ∠ADB.

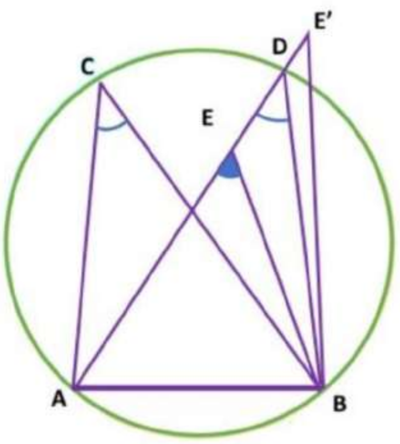


Solution: We know that angles in the same segment of a circle are equal.

Therefore, ∠ADB = ∠ACB.
⇒ θ = 70°.

Theorem 12: If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie

on a circle (i.e. they are concyclic).



Given: AB is a line segment, which subtends equal angles at two

points C and D. That is, ∠ACB = ∠ADB

To Prove: The points A, B, C, and D lie on a circle (i.e they are concyclic).

Construction: Draw a circle through points A, C, and B.

Proof: If possible, suppose D does not lie on this circle. Then, this circle will intersect AD (or extended AD) at a point, say E (or E').

If points A, C, E and B lie on a circle,

∠ACB = ∠AEB.

(Angles in the same segment of a circle are equal.)

But it is given that ∠ACB = ∠ADB.

Therefore, ∠AEB = ∠ADB.

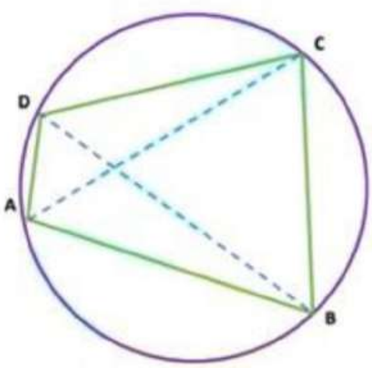
It is not possible unless E coincides with D.

Similarly, E' should also coincide with D.

Hence, the points A, B, C, and D lie on a circle (i.e they are concyclic). Proved.

Cyclic Quadrilateral

Cyclic Quadrilateral
Theorem 13: The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.



Given: ABCD is a cyclic quadrilateral.

To Prove: The opposite angles of a cyclic quadrilateral are 180°.

Construction: Join AC and BD.

Proof: Consider side AB of quadrilateral ABCD as the chord of the circle. Clearly, ∠ACB and ∠ADB are angles in the same segment determined by chord AB of the circles.

$$\therefore \angle ACB = \angle ADB. \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

(Since angles in the same segment of a circle are equal.)

Now, consider the side BC of quadrilateral ABCD as the chord of the circle. We find that ∠BAC and ∠BDC are angles in the same segment.

$$\therefore \angle BAC = \angle BDC. \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

(Since angles in the same segment of a circle are equal.)

Adding equations (i) and (ii), we get

$$\Rightarrow \quad \angle ACB + \angle BAC = \angle ADB + \angle BDC.$$

$$\Rightarrow \quad \angle ACB + \angle BAC = \angle ADC.$$

$$\Rightarrow \quad \angle ABC + \angle ACB + \angle BAC = \angle ABC + \angle ADC.$$

$$\Rightarrow \quad 180^\circ = \angle ABC + \angle ADC.$$

(∵ The sum of the angles of a triangle is 180°)

$$\Rightarrow \quad \angle ABC + \angle ADC = 180^\circ.$$

$$\Rightarrow \quad \angle B + \angle D = 180^\circ.$$

But, $\angle A + \angle B + \angle C + \angle D = 360^\circ.$

$$\therefore \quad \angle A + \angle C = 360^\circ - (\angle B + \angle D)$$

$$\Rightarrow \quad \angle A + \angle C = 360^\circ - 180^\circ$$

$$\Rightarrow \quad \angle A + \angle C = 180^\circ$$

Hence, the opposite angles of a cyclic quadrilateral are 180°.

Theorem 14: If the sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.

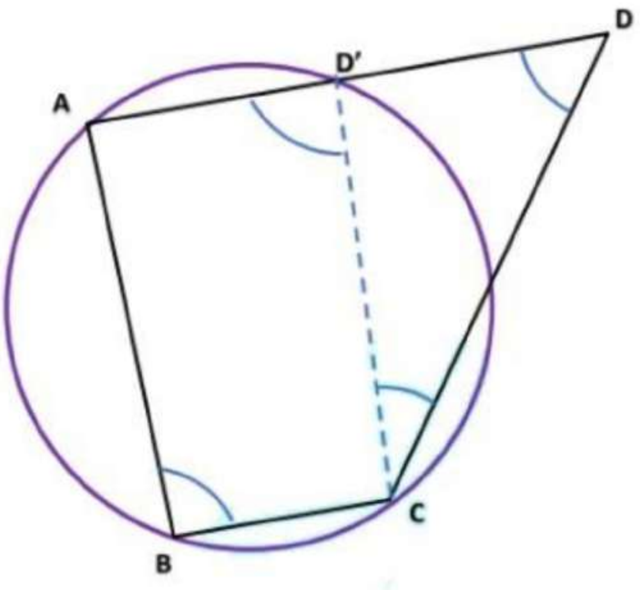
Given: ABCD is a quadrilateral such that

$$\angle B + \angle D = 180^\circ$$

$$\angle A + \angle C = 180^\circ$$

To Prove: ABCD is a cyclic quadrilateral.

Construction : Join D'C.



Proof: If possible, let ABCD be not a cyclic quadrilateral. Draw a circle passing through three non-collinear points A, B and C.

Suppose the circle meets AD or AD produced at D'.

Now, ABCD' is a cyclic quadrilateral.

⇒ $\angle ABC + \angle AD'C = 180^\circ$. ....(i)

(∵ ∠ABC and ∠AD'C are opposite angles of the cyclic quad ABCD')

But, $\angle B + \angle D = 180°$ (Given)

i.e. $\angle ABC + \angle ADC = 180°$. ....(ii)

From (i) and (ii), we get

$$\angle ABC + \angle AD'C = \angle ABC + \angle ADC$$

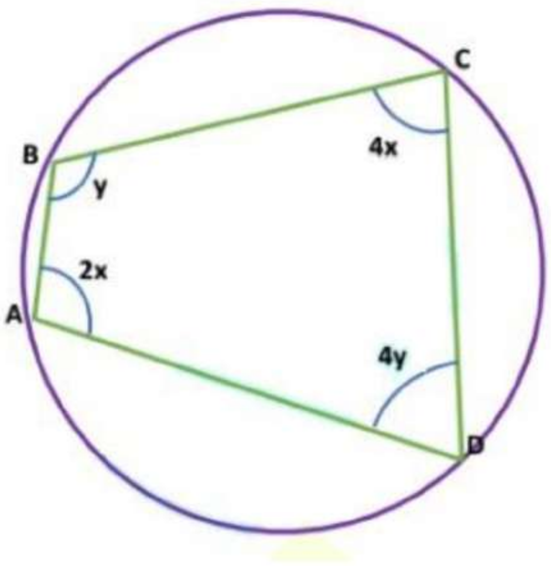
$\Rightarrow$ $\angle AD'C = \angle ADC$

$\Rightarrow$ An exterior angle of Δ CDD' is equal to an interior opposite angle.

But, this is not possible unless D' coincides with D. Thus, the circle passing through A, B, C also passes through D.

Hence, ABCD is a cyclic quadrilateral. Proved.

Examples 1: In the figure, ABCD is a cyclic quadrilateral. Find the measure of each of its angles.



Solution: Since the opposite angles of a cyclic quadrilateral are 180°.

$\therefore$ $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

$\Rightarrow$ $2x + 4x = 180°$ and $y + 4y = 180°$.

$\Rightarrow \qquad 6x = 180°$ and $5y = 180°$.

$\Rightarrow \qquad x = \frac{180}{6}°$ and $y = \frac{180}{5}°$

$\Rightarrow \qquad x = 30°$ and $y = 36°$.

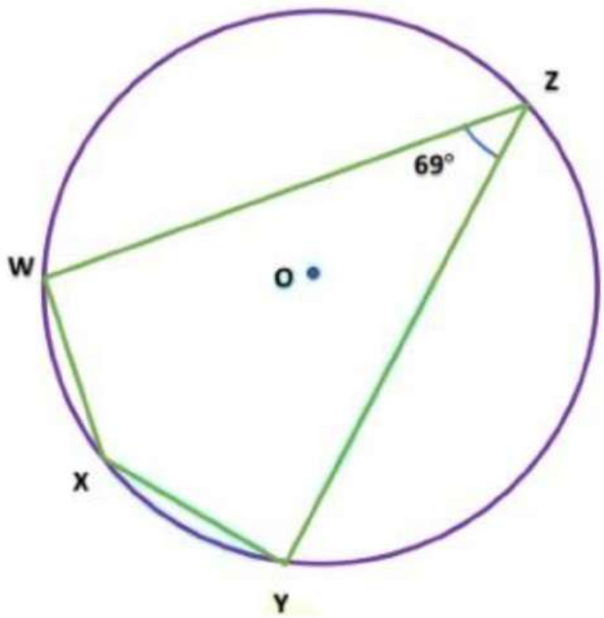
Therefore, $\angle A = 2x = 2\times 30° = 60°$.

$\Rightarrow \qquad \angle B = y = 36°$.

$\Rightarrow \qquad \angle C = 4x = 4\times 30° = 120°$.

$\Rightarrow \qquad \angle D = 4y = 4\times 36° = 144°$.

Example 2: What is the measure of angle WXY?



Solution: We know that the opposite angles of a cyclic quadrilateral are 180°.

Therefore, $\angle WXY + \angle WZY = 180°$

$\Rightarrow \qquad \angle WXY + 69° = 180°$

$\Rightarrow \qquad \angle WXY = 180° - 69°$

$\Rightarrow \qquad \angle WXY = 111°$