8.8 Multivariable Functions

Functions of two variables: $z(x,y)$, $f(x,y)$, $g(x,y)$, $h(x,y)$
Arguments: x, y, t
Small changes in x, y, z, respectively: Δx, Δy, Δz.

846. First Order Partial Derivatives
The partial derivative with respect to x

$$\frac{\partial f}{\partial x} = f_x \text{ (also } \frac{\partial z}{\partial x} = z_x \text{)},$$

The partial derivative with respect to y

$$\frac{\partial f}{\partial y} = f_y \text{ (also } \frac{\partial z}{\partial y} = z_y \text{)}.$$

847. Second Order Partial Derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

If the derivatives are continuous, then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

848. Chain Rules

If $f(x,y)=g(h(x,y))$ (g is a function of one variable h), then

$$\frac{\partial f}{\partial x}=g'(h(x,y))\frac{\partial h}{\partial x},\ \frac{\partial f}{\partial y}=g'(h(x,y))\frac{\partial h}{\partial y}.$$

If $h(t)=f(x(t),y(t))$, then $h'(t)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$.

If $z=f(x(u,v),y(u,v))$, then

$$\frac{\partial z}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u},\ \frac{\partial z}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

849. Small Changes

$$\Delta z\approx\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y$$

850. Local Maxima and Minima

$f(x,y)$ has a local maximum at (x_0,y_0) if $f(x,y)\le f(x_0,y_0)$ for all (x,y) sufficiently close to (x_0,y_0).

$f(x,y)$ has a local minimum at (x_0,y_0) if $f(x,y)\ge f(x_0,y_0)$ for all (x,y) sufficiently close to (x_0,y_0).

851. Stationary Points

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0.$$

Local maxima and local minima occur at stationary points.

852. Saddle Point

A stationary point which is neither a local maximum nor a local minimum

853. Second Derivative Test for Stationary Points

Let (x_0, y_0) be a stationary point ($\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$).

$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}.$$

If $D > 0$, $f_{xx}(x_0, y_0) > 0$, (x_0, y_0) is a point of local minima.
If $D > 0$, $f_{xx}(x_0, y_0) < 0$, (x_0, y_0) is a point of local maxima.
If $D < 0$, (x_0, y_0) is a saddle point.
If $D = 0$, the test fails.

854. Tangent Plane

The equation of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

855. Normal to Surface

The equation of the normal to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}.$$