

9.9 Improper Integral

1070. The definite integral $\int_a^b f(x)dx$ is called an **improper integral**

if

- a or b is infinite,
- $f(x)$ has one or more points of discontinuity in the interval $[a, b]$.

1071. If $f(x)$ is a continuous function on $[a, \infty)$, then

$$\int_a^{\infty} f(x)dx = \lim_{n \rightarrow \infty} \int_a^n f(x)dx.$$

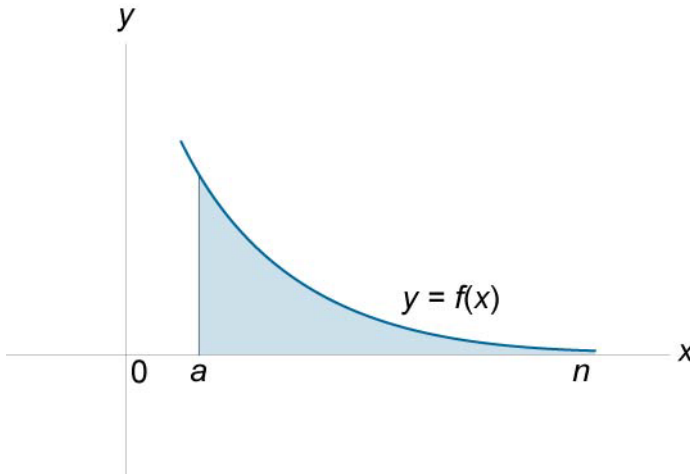


Figure 184.

1072. If $f(x)$ is a continuous function on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)dx = \lim_{n \rightarrow -\infty} \int_n^b f(x)dx.$$

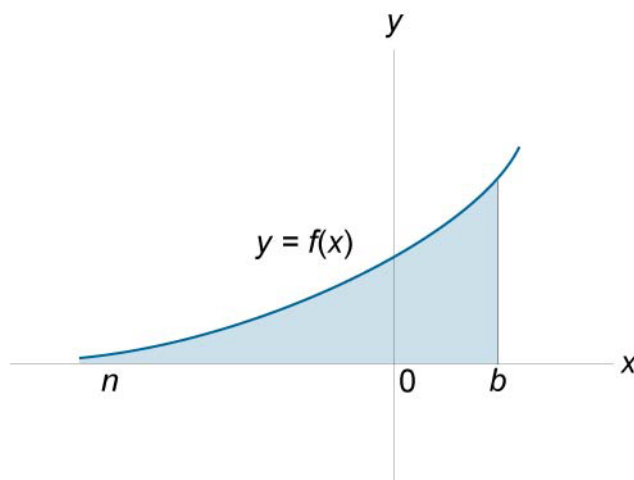


Figure 185.

Note : The improper integrals in 1071, 1072 are **convergent** if the limits exist and are finite; otherwise the integrals are **divergent**.

1073.
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^{\infty} f(x)dx$$

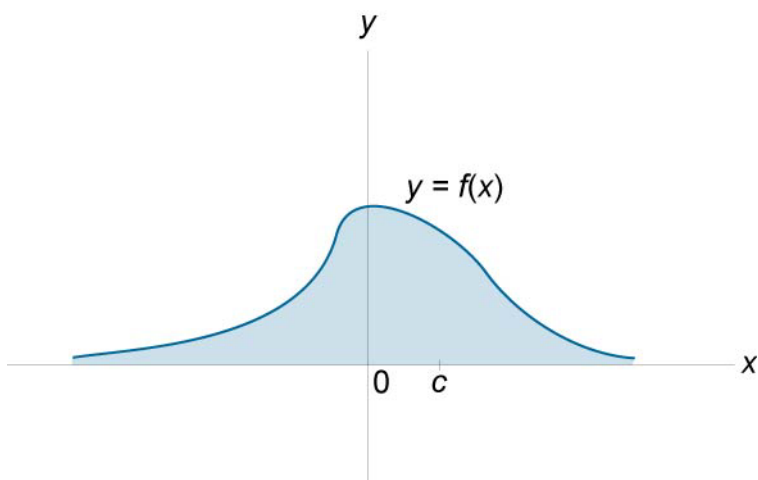


Figure 186.

If for some real number c , both of the integrals in the right side are convergent, then the integral $\int_{-\infty}^{\infty} f(x)dx$ is also convergent; otherwise it is divergent.

1074. Comparison Theorems

Let $f(x)$ and $g(x)$ be continuous functions on the closed interval $[a, \infty)$. Suppose that $0 \leq g(x) \leq f(x)$ for all x in $[a, \infty)$.

- If $\int_a^{\infty} f(x)dx$ is convergent, then $\int_a^{\infty} g(x)dx$ is also convergent,
- If $\int_a^{\infty} g(x)dx$ is divergent, then $\int_a^{\infty} f(x)dx$ is also divergent.

1075. Absolute Convergence

If $\int_a^{\infty} |f(x)|dx$ is convergent, then the integral $\int_a^{\infty} f(x)dx$ is absolutely convergent.

1076. Discontinuous Integrand

Let $f(x)$ be a function which is continuous on the interval $[a, b)$ but is discontinuous at $x = b$. Then

$$\int_a^b f(x)dx = \lim_{\varepsilon \rightarrow 0^+} \int_a^{b-\varepsilon} f(x)dx$$

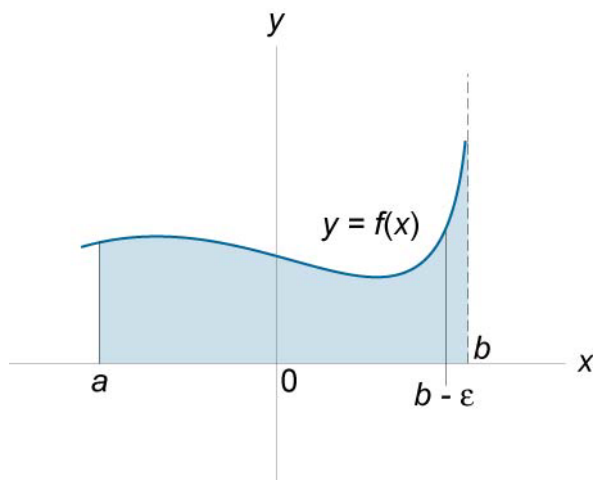


Figure 187.

1077. Let $f(x)$ be a continuous function for all real numbers x in the interval $[a, b]$ except for some point c in (a, b) . Then

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_a^{c-\varepsilon} f(x) dx + \lim_{\delta \rightarrow 0^+} \int_{c+\delta}^b f(x) dx.$$

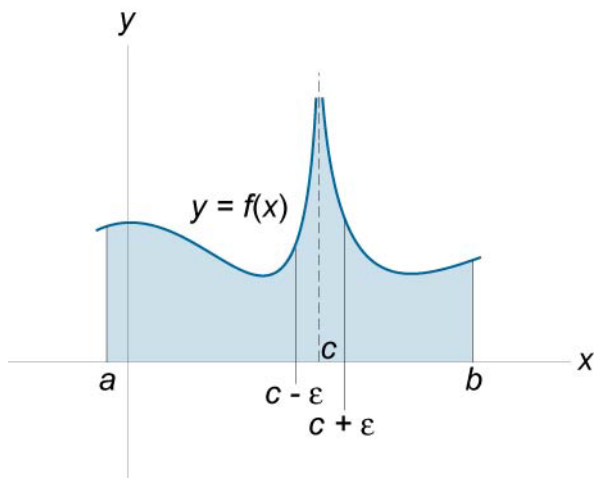


Figure 188.