

9.11 Triple Integral

Functions of three variables: $f(x, y, z)$, $g(x, y, z)$, ...

Triple integrals: $\iiint_G f(x, y, z) dV$, $\iiint_G g(x, y, z) dV$, ...

Riemann sum: $\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(u_i, v_j, w_k) \Delta x_i \Delta y_j \Delta z_k$

Small changes: Δx_i , Δy_j , Δz_k

Limits of integration: a , b , c , d , r , s

Regions of integration: G , T , S

Cylindrical coordinates: r , θ , z

Spherical coordinates: r , θ , φ

Volume of a solid: V

Mass of a solid: m

Density: $\mu(x, y, z)$

Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$

First moments: M_{xy} , M_{yz} , M_{xz}

Moments of inertia: I_{xy} , I_{yz} , I_{xz} , I_x , I_y , I_z , I_0

1099. Definition of Triple Integral

The triple integral over a parallelepiped $[a, b] \times [c, d] \times [r, s]$ is defined to be

$$\iiint_{[a, b] \times [c, d] \times [r, s]} f(x, y, z) dV = \lim_{\substack{\max \Delta x_i \rightarrow 0 \\ \max \Delta y_j \rightarrow 0 \\ \max \Delta z_k \rightarrow 0}} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(u_i, v_j, w_k) \Delta x_i \Delta y_j \Delta z_k,$$

where (u_i, v_j, w_k) is some point in the parallelepiped

$(x_{i-1}, x_i) \times (y_{j-1}, y_j) \times (z_{k-1}, z_k)$, and $\Delta x_i = x_i - x_{i-1}$,

$\Delta y_j = y_j - y_{j-1}$, $\Delta z_k = z_k - z_{k-1}$.

$$1100. \iiint_G [f(x, y, z) + g(x, y, z)] dV = \iiint_G f(x, y, z) dV + \iiint_G g(x, y, z) dV$$

$$1101. \iiint_G [f(x, y, z) - g(x, y, z)] dV = \iiint_G f(x, y, z) dV - \iiint_G g(x, y, z) dV$$

$$1102. \iiint_G kf(x, y, z) dV = k \iiint_G f(x, y, z) dV,$$

where k is a constant.

1103. If $f(x, y, z) \geq 0$ and G and T are nonoverlapping basic regions, then

$$\iiint_{G \cup T} f(x, y, z) dV = \iiint_G f(x, y, z) dV + \iiint_T f(x, y, z) dV.$$

Here $G \cup T$ is the union of the regions G and T .

1104. Evaluation of Triple Integrals by Repeated Integrals

If the solid G is the set of points (x, y, z) such that $(x, y) \in R$, $\chi_1(x, y) \leq z \leq \chi_2(x, y)$, then

$$\iiint_G f(x, y, z) dx dy dz = \iint_R \left[\int_{\chi_1(x, y)}^{\chi_2(x, y)} f(x, y, z) dz \right] dx dy,$$

where R is projection of G onto the xy -plane.

If the solid G is the set of points (x, y, z) such that $a \leq x \leq b$, $\varphi_1(x) \leq y \leq \varphi_2(x)$, $\chi_1(x, y) \leq z \leq \chi_2(x, y)$, then

$$\iiint_G f(x, y, z) dx dy dz = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} \left(\int_{\chi_1(x, y)}^{\chi_2(x, y)} f(x, y, z) dz \right) dy \right] dx$$

1105. Triple Integrals over Parallelepiped

If G is a parallelepiped $[a, b] \times [c, d] \times [r, s]$, then

$$\iiint_G f(x, y, z) dx dy dz = \int_a^b \left[\int_c^d \left(\int_r^s f(x, y, z) dz \right) dy \right] dx.$$

In the special case where the integrand $f(x, y, z)$ can be written as $g(x)h(y)k(z)$ we have

$$\iiint_G f(x, y, z) dx dy dz = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right) \left(\int_r^s k(z) dz \right).$$

1106. Change of Variables

$$\begin{aligned} \iiint_G f(x, y, z) dx dy dz &= \\ &= \iiint_S f[x(u, v, w), y(u, v, w), z(u, v, w)] \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dx dy dz, \end{aligned}$$

$$\text{where } \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0 \text{ is the jacobian of}$$

the transformations $(x, y, z) \rightarrow (u, v, w)$, and S is the pull-back of G which can be computed by $x = x(u, v, w)$,
 $y = y(u, v, w)$
 $z = z(u, v, w)$ into the definition of G .

1107. Triple Integrals in Cylindrical Coordinates

The differential $dx dy dz$ for cylindrical coordinates is

$$dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| dr d\theta dz = r dr d\theta dz.$$

Let the solid G is determined as follows:

$$(x, y) \in R, \chi_1(x, y) \leq z \leq \chi_2(x, y),$$

where R is projection of G onto the xy -plane. Then

$$\iiint_G f(x, y, z) dx dy dz = \iiint_S f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

$$= \iint_{R(r,\theta)} \left[\int_{\chi_1(r \cos \theta, r \sin \theta)}^{\chi_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) dz \right] r dr d\theta.$$

Here S is the pullback of G in cylindrical coordinates.

1108. Triple Integrals in Spherical Coordinates

The Differential $dx dy dz$ for Spherical Coordinates is

$$dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} \right| dr d\theta d\varphi = r^2 \sin \theta dr d\theta d\varphi$$

$$\iiint_G f(x, y, z) dx dy dz =$$

$$= \iiint_S f(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) r^2 \sin \theta dr d\theta d\varphi,$$

where the solid S is the pullback of G in spherical coordinates. The angle θ ranges from 0 to 2π , the angle φ ranges from 0 to π .

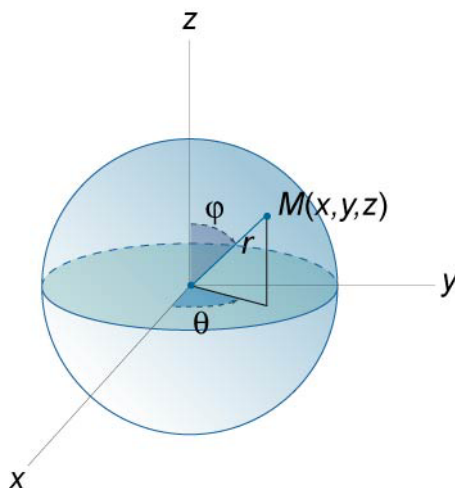


Figure 202.

1109. Volume of a Solid

$$V = \iiint_G dx dy dz$$

1110. Volume in Cylindrical Coordinates

$$V = \iiint_{S(r,\theta,z)} r dr d\theta dz$$

1111. Volume in Spherical Coordinates

$$V = \iiint_{S(r,\theta,\phi)} r^2 \sin \theta dr d\theta d\phi$$

1112. Mass of a Solid

$$m = \iiint_G \mu(x,y,z) dV,$$

where the solid occupies a region G and its density at a point (x,y,z) is $\mu(x,y,z)$.

1113. Center of Mass of a Solid

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iiint_G x \mu(x,y,z) dV,$$

$$M_{xz} = \iiint_G y \mu(x,y,z) dV,$$

$$M_{xy} = \iiint_G z \mu(x,y,z) dV$$

are the first moments about the coordinate planes $x=0$, $y=0$, $z=0$, respectively, $\mu(x,y,z)$ is the density function.

1114. Moments of Inertia about the xy -plane (or $z=0$), yz -plane ($x=0$), and xz -plane ($y=0$)

$$I_{xy} = \iiint_G z^2 \mu(x, y, z) dV ,$$

$$I_{yz} = \iiint_G x^2 \mu(x, y, z) dV ,$$

$$I_{xz} = \iiint_G y^2 \mu(x, y, z) dV .$$

1115. Moments of Inertia about the x-axis, y-axis, and z-axis

$$I_x = I_{xy} + I_{xz} = \iiint_G (z^2 + y^2) \mu(x, y, z) dV ,$$

$$I_y = I_{xy} + I_{yz} = \iiint_G (z^2 + x^2) \mu(x, y, z) dV ,$$

$$I_z = I_{xz} + I_{yz} = \iiint_G (y^2 + x^2) \mu(x, y, z) dV .$$

1116. Polar Moment of Inertia

$$I_0 = I_{xy} + I_{yz} + I_{xz} = \iiint_G (x^2 + y^2 + z^2) \mu(x, y, z) dV$$