

## 9.12 Line Integral

Scalar functions:  $F(x,y,z)$ ,  $F(x,y)$ ,  $f(x)$

Scalar potential:  $u(x,y,z)$

Curves:  $C$ ,  $C_1$ ,  $C_2$

Limits of integrations:  $a$ ,  $b$ ,  $\alpha$ ,  $\beta$

Parameters:  $t$ ,  $s$

Polar coordinates:  $r$ ,  $\theta$

Vector field:  $\vec{F}(P,Q,R)$

Position vector:  $\vec{r}(s)$

Unit vectors:  $\vec{i}$ ,  $\vec{j}$ ,  $\vec{k}$ ,  $\vec{\tau}$

Area of region:  $S$

Length of a curve:  $L$

Mass of a wire:  $m$

Density:  $\rho(x,y,z)$ ,  $\rho(x,y)$

Coordinates of center of mass:  $\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$

First moments:  $M_{xy}$ ,  $M_{yz}$ ,  $M_{xz}$

Moments of inertia:  $I_x$ ,  $I_y$ ,  $I_z$

Volume of a solid:  $V$

Work:  $W$

Magnetic field:  $\vec{B}$

Current:  $I$

Electromotive force:  $\varepsilon$

Magnetic flux:  $\psi$

### 1117. Line Integral of a Scalar Function

Let a curve  $C$  be given by the vector function  $\vec{r} = \vec{r}(s)$ ,

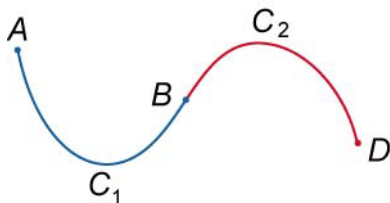
$0 \leq s \leq S$ , and a **scalar function**  $F$  is defined over the curve  $C$ .

Then

$$\int_0^S F(\vec{r}(s))ds = \int_C F(x,y,z)ds = \int_C Fds,$$

where  $ds$  is the arc length differential.

$$1118. \int_{C_1 \cup C_2} F \, ds = \int_{C_1} F \, ds + \int_{C_2} F \, ds$$



**Figure 203.**

- 1119.** If the smooth curve  $C$  is parametrized by  $\vec{r} = \vec{r}(t)$ ,  $\alpha \leq t \leq \beta$ , then

$$\int_C F(x, y, z) \, ds = \int_{\alpha}^{\beta} F(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt.$$

- 1120.** If  $C$  is a smooth curve in the  $xy$ -plane given by the equation  $y = f(x)$ ,  $a \leq x \leq b$ , then

$$\int_C F(x, y) \, ds = \int_a^b F(x, f(x)) \sqrt{1 + (f'(x))^2} \, dx.$$

- 1121.** Line Integral of Scalar Function in Polar Coordinates

$$\int_C F(x, y) \, ds = \int_{\alpha}^{\beta} F(r \cos \theta, r \sin \theta) \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta,$$

where the curve  $C$  is defined by the polar function  $r(\theta)$ .

- 1122.** Line Integral of Vector Field

Let a curve  $C$  be defined by the vector function  $\vec{r} = \vec{r}(s)$ ,  $0 \leq s \leq S$ . Then

$$\frac{d\vec{r}}{ds} = \vec{\tau} = (\cos \alpha, \cos \beta, \cos \gamma)$$

is the unit vector of the tangent line to this curve.

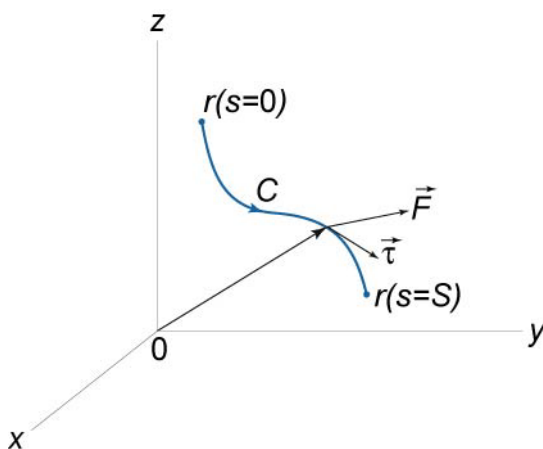


Figure 204.

Let a **vector field**  $\vec{F}(P, Q, R)$  is defined over the curve  $C$ . Then the line integral of the vector field  $\vec{F}$  along the curve  $C$  is

$$\int_C Pdx + Qdy + Rdz = \int_0^s (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds.$$

**1123.** Properties of Line Integrals of Vector Fields

$$\int_{-C} (\vec{F} \cdot d\vec{r}) = - \int_C (\vec{F} \cdot d\vec{r}),$$

where  $-C$  denote the curve with the opposite orientation.

$$\int_C (\vec{F} \cdot d\vec{r}) = \int_{C_1 \cup C_2} (\vec{F} \cdot d\vec{r}) = \int_{C_1} (\vec{F} \cdot d\vec{r}) + \int_{C_2} (\vec{F} \cdot d\vec{r}),$$

where  $C$  is the union of the curves  $C_1$  and  $C_2$ .

**1124.** If the curve  $C$  is parameterized by  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ ,  $\alpha \leq t \leq \beta$ , then

$$\int Pdx + Qdy + Rdz =$$

$$= \int_{\alpha}^{\beta} \left( P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt$$

**1125.** If  $C$  lies in the  $xy$ -plane and given by the equation  $y = f(x)$ , then

$$\int_C P dx + Q dy = \int_a^b \left( P(x, f(x)) + Q(x, f(x)) \frac{df}{dx} \right) dx.$$

**1126.** Green's Theorem

$$\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_C P dx + Q dy,$$

where  $\vec{F} = P(x, y)\vec{i} + Q(x, y)\vec{j}$  is a continuous vector function with continuous first partial derivatives  $\frac{\partial P}{\partial y}$ ,  $\frac{\partial Q}{\partial x}$  in a some domain  $R$ , which is bounded by a closed, piecewise smooth curve  $C$ .

**1127.** Area of a Region  $R$  Bounded by the Curve  $C$

$$S = \iint_R dx dy = \frac{1}{2} \oint_C x dy - y dx$$

**1128.** Path Independence of Line Integrals

The line integral of a vector function  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$  is said to be **path independent**, if and only if  $P$ ,  $Q$ , and  $R$  are continuous in a domain  $D$ , and if there exists some scalar function  $u = u(x, y, z)$  (a **scalar potential**) in  $D$  such that

$$\vec{F} = \text{grad } u, \text{ or } \frac{\partial u}{\partial x} = P, \frac{\partial u}{\partial y} = Q, \frac{\partial u}{\partial z} = R.$$

Then

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C P dx + Q dy + R dz = u(B) - u(A).$$

**1129. Test for a Conservative Field**

A vector field of the form  $\vec{F} = \text{grad } u$  is called a **conservative field**. The line integral of a vector function  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$  is path independent if and only if

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{0}.$$

If the line integral is taken in xy-plane so that

$$\int_C Pdx + Qdy = u(B) - u(A),$$

then the test for determining if a vector field is conservative can be written in the form

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

**1130. Length of a Curve**

$$L = \int_C ds = \int_{\alpha}^{\beta} \left| \frac{d\vec{r}}{dt}(t) \right| dt = \int_{\alpha}^{\beta} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} dt,$$

where  $C$  is a piecewise smooth curve described by the position vector  $\vec{r}(t)$ ,  $\alpha \leq t \leq \beta$ .

If the curve  $C$  is two-dimensional, then

$$L = \int_C ds = \int_{\alpha}^{\beta} \left| \frac{d\vec{r}}{dt}(t) \right| dt = \int_{\alpha}^{\beta} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt.$$

If the curve  $C$  is the graph of a function  $y = f(x)$  in the xy-plane ( $a \leq x \leq b$ ), then

$$L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx.$$

**1131. Length of a Curve in Polar Coordinates**

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta,$$

where the curve  $C$  is given by the equation  $r = r(\theta)$ ,  
 $\alpha \leq \theta \leq \beta$  in polar coordinates.

**1132. Mass of a Wire**

$$m = \int_C \rho(x, y, z) ds,$$

where  $\rho(x, y, z)$  is the mass per unit length of the wire.

If  $C$  is a curve parametrized by the vector function  
 $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ , then the mass can be computed by  
the formula

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$

If  $C$  is a curve in  $xy$ -plane, then the mass of the wire is given  
by

$$m = \int_C \rho(x, y) ds,$$

or

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \text{ (in parametric form).}$$

**1133. Center of Mass of a Wire**

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \int_C x \rho(x, y, z) ds,$$

$$M_{xz} = \int_C y\rho(x, y, z)ds,$$

$$M_{xy} = \int_C z\rho(x, y, z)ds.$$

### 1134. Moments of Inertia

The moments of inertia about the x-axis, y-axis, and z-axis are given by the formulas

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)ds,$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)ds,$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)ds.$$

### 1135. Area of a Region Bounded by a Closed Curve

$$S = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx.$$

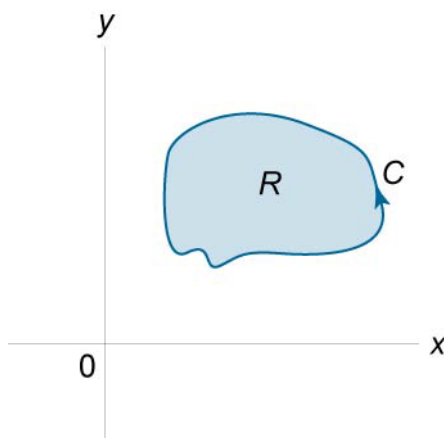


Figure 205.

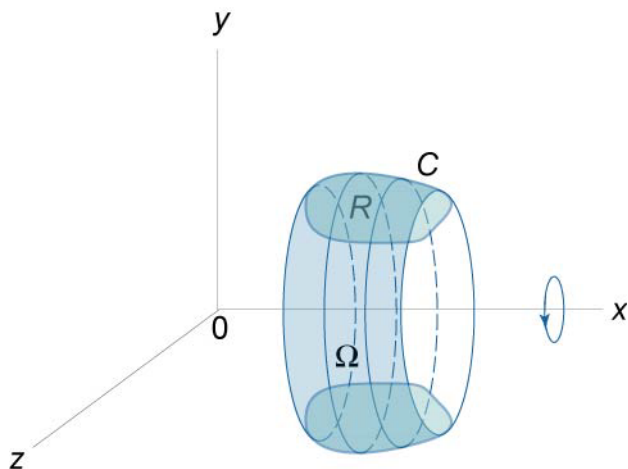
If the closed curve C is given in parametric form

$\vec{r}(t) = \langle x(t), y(t) \rangle$ , then the area can be calculated by the formula

$$S = \int_{\alpha}^{\beta} x(t) \frac{dy}{dt} dt = - \int_{\alpha}^{\beta} y(t) \frac{dx}{dt} dt = \frac{1}{2} \int_{\alpha}^{\beta} \left( x(t) \frac{dy}{dt} - y(t) \frac{dx}{dt} \right) dt.$$

**1136.** Volume of a Solid Formed by Rotating a Closed Curve about the x-axis

$$V = -\pi \oint_C y^2 dx = -2\pi \oint_C xy dy = -\frac{\pi}{2} \oint_C 2xy dy + y^2 dx$$



**Figure 206.**

**1137.** Work

Work done by a force  $\vec{F}$  on an object moving along a curve  $C$  is given by the line integral

$$W = \int_C \vec{F} \cdot d\vec{r},$$

where  $\vec{F}$  is the vector force field acting on the object,  $d\vec{r}$  is the unit tangent vector.

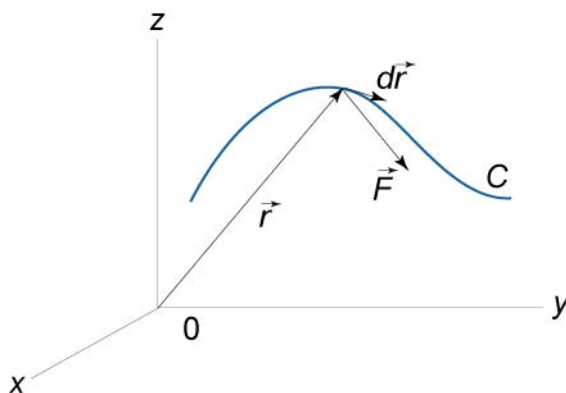


Figure 207.

If the object is moved along a curve  $C$  in the  $xy$ -plane, then

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy,$$

If a path  $C$  is specified by a parameter  $t$  ( $t$  often means time), the formula for calculating work becomes

$$W = \int_{\alpha}^{\beta} \left[ P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right] dt,$$

where  $t$  goes from  $\alpha$  to  $\beta$ .

If a vector field  $\vec{F}$  is conservative and  $u(x, y, z)$  is a scalar potential of the field, then the work on an object moving from  $A$  to  $B$  can be found by the formula

$$W = u(B) - u(A).$$

### 1138. Ampere's Law

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I.$$

The line integral of a magnetic field  $\vec{B}$  around a closed path  $C$  is equal to the total current  $I$  flowing through the area bounded by the path.

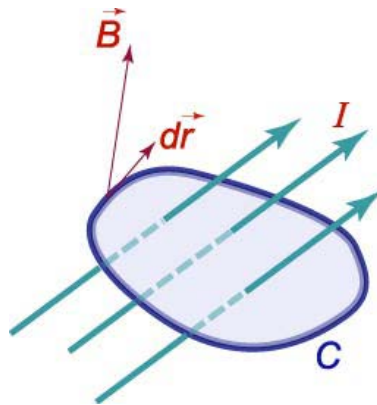


Figure 208.

### 1139. Faraday's Law

$$\varepsilon = \oint_C \vec{E} \cdot d\vec{r} = -\frac{d\psi}{dt}$$

The electromotive force (emf)  $\varepsilon$  induced around a closed loop C is equal to the rate of the change of magnetic flux  $\psi$  passing through the loop.

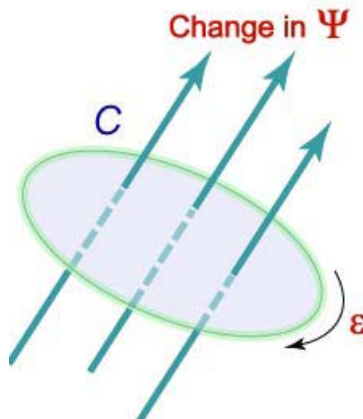


Figure 209.