9.13 Surface Integral

Scalar functions: $f(x,y,z)$, $z(x,y)$
Position vectors: $\vec{r}(u,v)$, $\vec{r}(x,y,z)$
Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$
Surface: S
Vector field: $\vec{F}(P,Q,R)$
Divergence of a vector field: $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$
Curl of a vector field: $\operatorname{curl} \vec{F} = \nabla \times \vec{F}$
Vector element of a surface: $d\vec{S}$
Normal to surface: $\vec{n}$
Surface area: A
Mass of a surface: m
Density: $\mu(x,y,z)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$
First moments: M_{xy}, M_{yz}, M_{xz}
Moments of inertia: I_{xy}, I_{yz}, I_{xz}, I_x, I_y, I_z
Volume of a solid: V
Force: $\vec{F}$
Gravitational constant: G
Fluid velocity: $\vec{v}(\vec{r})$
Fluid density: ρ
Pressure: $p(\vec{r})$
Mass flux, electric flux: Φ
Surface charge: Q
Charge density: $\sigma(x,y)$
Magnitude of the electric field: $\vec{E}$

1140. Surface Integral of a Scalar Function
Let a surface S be given by the position vector
$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$,
where (u,v) ranges over some domain $D(u,v)$ of the uv-

plane.
The surface integral of a scalar function $f(x,y,z)$ over the surface S is defined as

$$\iint_S f(x,y,z)dS = \iint_{D(u,v)} f(x(u,v),y(u,v),z(u,v))\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|dudv,$$

where the partial derivatives $\frac{\partial\vec{r}}{\partial u}$ and $\frac{\partial\vec{r}}{\partial v}$ are given by

$$\frac{\partial\vec{r}}{\partial u} = \frac{\partial x}{\partial u}(u,v)\vec{i} + \frac{\partial y}{\partial u}(u,v)\vec{j} + \frac{\partial z}{\partial u}(u,v)\vec{k},$$

$$\frac{\partial\vec{r}}{\partial v} = \frac{\partial x}{\partial v}(u,v)\vec{i} + \frac{\partial y}{\partial v}(u,v)\vec{j} + \frac{\partial z}{\partial v}(u,v)\vec{k}$$

and $\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}$ is the cross product.

1141. If the surface S is given by the equation $z = z(x,y)$ where $z(x,y)$ is a differentiable function in the domain $D(x,y)$, then

$$\iint_S f(x,y,z)dS = \iint_{D(x,y)} f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy.$$

1142. Surface Integral of the Vector Field $\vec{F}$ over the Surface S

- If S is oriented outward, then

$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot\vec{n}dS$$

$$= \iint_{D(u,v)} \vec{F}(x(u,v),y(u,v),z(u,v))\cdot\left[\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right]dudv.$$

- If S is oriented inward, then

$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot\vec{n}dS$$

F x u,v ,y u,v ,z u,v

$$= \iint\limits_{D(u,v)} \vec{\ }\big(\ (\quad)\ (\quad)\ (\quad)\big)\cdot\left[\frac{\partial r}{\partial v}\times\frac{\partial r}{\partial u}\right]dudv\,.$$

$d\vec{S}=\vec{n}dS$ is called the vector element of the surface. Dot means the scalar product of the appropriate vectors. The partial derivatives $\frac{\partial\vec{r}}{\partial u}$ and $\frac{\partial\vec{r}}{\partial v}$ are given by

$$\frac{\partial\vec{r}}{\partial u}=\frac{\partial x}{\partial u}(u,v)\cdot\vec{i}+\frac{\partial y}{\partial u}(u,v)\cdot\vec{j}+\frac{\partial z}{\partial u}(u,v)\cdot\vec{k}\,,$$
$$\frac{\partial\vec{r}}{\partial v}=\frac{\partial x}{\partial v}(u,v)\cdot\vec{i}+\frac{\partial y}{\partial v}(u,v)\cdot\vec{j}+\frac{\partial z}{\partial v}(u,v)\cdot\vec{k}\,.$$

1143. If the surface S is given by the equation $z=z(x,y)$, where $z(x,y)$ is a differentiable function in the domain $D(x,y)$, then

- If S is oriented upward, i.e. the k-th component of the normal vector is positive, then
$$\iint\limits_{S}\vec{F}(x,y,z)\cdot d\vec{S}=\iint\limits_{S}\vec{F}(x,y,z)\cdot\vec{n}dS$$
$$=\iint\limits_{D(x,y)}\vec{F}(x,y,z)\cdot\left(-\frac{\partial z}{\partial x}\vec{i}-\frac{\partial z}{\partial y}\vec{j}+\vec{k}\right)dxdy\,,$$

- If S is oriented downward, i.e. the k-th component of the normal vector is negative, then
$$\iint\limits_{S}\vec{F}(x,y,z)\cdot d\vec{S}=\iint\limits_{S}\vec{F}(x,y,z)\cdot\vec{n}dS$$
$$=\iint\limits_{D(x,y)}\vec{F}(x,y,z)\cdot\left(\frac{\partial z}{\partial x}\vec{i}+\frac{\partial z}{\partial y}\vec{j}-\vec{k}\right)dxdy\,.$$

1144. $$\iint\limits_{S}(\vec{F}\cdot\vec{n})dS=\iint\limits_{S}Pdydz+Qdzdx+Rdxdy$$
$$=\iint\limits_{S}(P\cos\alpha+Q\cos\beta+R\cos\gamma)dS\,,$$

where $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ are the components of the vector field $\vec{F}$.
$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the angles between the outer unit normal vector $\vec{n}$ and the x-axis, y-axis, and z-axis, respectively.

1145. If the surface S is given in parametric form by the vector $\vec{r}(x(u,v),y(u,v),z(u,v))$, then the latter formula can be written as

$$\iint_S (\vec{F}\cdot\vec{n})dS = \iint_S Pdydz + Qdzdx + Rdxdy = \iint_{D(u,v)} \begin{vmatrix} P & Q & R \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv,$$

where (u,v) ranges over some domain $D(u,v)$ of the uv-plane.

1146. Divergence Theorem

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_G (\nabla\cdot\vec{F})dV,$$

where

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

is a vector field whose components P, Q, and R have continuous partial derivatives,

$$\nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is the divergence of $\vec{F}$, also denoted $\operatorname{div}\vec{F}$. The symbol $\oiint$ indicates that the surface integral is taken over a closed surface.

1147. Divergence Theorem in Coordinate Form

$$\oiint_S Pdydz + Qdxdz + Rdxdy = \iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dxdydz.$$

1148. Stoke's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \left(\nabla\times\vec{F}\right)\cdot d\vec{S},$$

where

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

is a vector field whose components P, Q, and R have continuous partial derivatives,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

is the curl of $\vec{F}$, also denoted $\operatorname{curl}\vec{F}$.

The symbol $\oint$ indicates that the line integral is taken over a closed curve.

1149. Stoke's Theorem in Coordinate Form

$$\oint_C Pdx + Qdy + Rdz$$

$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

1150. Surface Area

$$A = \iint_S dS$$

1151. If the surface S is parameterized by the vector

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k},$$

then the surface area is

$$A = \iint_{D(u,v)} \left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right| dudv,$$

where $D(u,v)$ is the domain where the surface $\vec{r}(u,v)$ is defined.

1152. If S is given explicitly by the function $z(x,y)$, then the surface area is

$$A = \iint\limits_{D(x,y)} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy,$$

where $D(x,y)$ is the projection of the surface S onto the xy-plane.

1153. Mass of a Surface

$$m = \iint\limits_{S} \mu(x,y,z)dS,$$

where $\mu(x,y,z)$ is the mass per unit area (density function).

1154. Center of Mass of a Shell

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iint\limits_{S} x\mu(x,y,z)dS,$$

$$M_{xz} = \iint\limits_{S} y\mu(x,y,z)dS,$$

$$M_{xy} = \iint\limits_{S} z\mu(x,y,z)dS$$

are the first moments about the coordinate planes $x=0$, $y=0$, $z=0$, respectively. $\mu(x,y,z)$ is the density function.

1155. Moments of Inertia about the xy-plane (or $z=0$), yz-plane ($x=0$), and xz-plane ($y=0$)

$$I_{xy} = \iint\limits_{S} z^2\mu(x,y,z)dS,$$

$$I_{yz} = \iint\limits_{S} x^2\mu(x,y,z)dS,$$

$$I_{xz} = \iint_S y^2 \mu(x,y,z)\,dS.$$

1156. Moments of Inertia about the x-axis, y-axis, and z-axis

$$I_x = \iint_S \left(y^2 + z^2\right)\mu(x,y,z)\,dS,$$

$$I_y = \iint_S \left(x^2 + z^2\right)\mu(x,y,z)\,dS,$$

$$I_z = \iint_S \left(x^2 + y^2\right)\mu(x,y,z)\,dS.$$

1157. Volume of a Solid Bounded by a Closed Surface

$$V = \frac{1}{3}\left|\oiint_S x\,dydz + y\,dxdz + z\,dxdy\right|$$

1158. Gravitational Force

$$\vec{F} = Gm\iint_S \mu(x,y,z)\frac{\vec{r}}{r^3}\,dS,$$

where m is a mass at a point $\langle x_0, y_0, z_0 \rangle$ outside the surface,

$\vec{r} = \langle x - x_0, y - y_0, z - z_0 \rangle$,

$\mu(x,y,z)$ is the density function,

and G is gravitational constant.

1159. Pressure Force

$$\vec{F} = \iint_S p(\vec{r})\,d\vec{S},$$

where the pressure $p(\vec{r})$ acts on the surface S given by the position vector $\vec{r}$.

1160. Fluid Flux (across the surface S)

$$\Phi = \oiint_S \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{v}(\vec{r})$ is the fluid velocity.

1161. Mass Flux (across the surface S)

$$\Phi = \oiint_S \rho \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{F} = \rho \vec{v}$ is the vector field, ρ is the fluid density.

1162. Surface Charge

$$Q = \iint_S \sigma(x, y) dS,$$

where $\sigma(x, y)$ is the surface charge density.

1163. Gauss' Law

The electric flux through any closed surface is proportional to the charge Q enclosed by the surface

$$\Phi = \oiint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0},$$

where

Φ is the electric flux,

$\vec{E}$ is the magnitude of the electric field strength,

$\varepsilon_0 = 8{,}85 \times 10^{-12} \frac{F}{m}$ is permittivity of free space.