

XII - ISC Board

Mathematics - Question Paper Solutions

Date: 26.02.2018

Max. Marks : 100

SECTION - A (80 Marks)

Question 1

(i) The binary operation $*$: $R \times R \rightarrow R$ is defined as $a*b = 2a + b$. Find $(2*3)*4$.

Ans. Given $a*b = 2a + b$

$$\begin{aligned}(2*3)*4 &= (4+3)*4 \\ &= 7*4 \\ &= 14 + 4 = 18\end{aligned}$$

(ii) If $A = \begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix}$ and A is symmetric matrix, show that $a = b$

Ans. $A = \begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix}$ given A is symmetric matrix

$$A = A^T$$

$$\begin{bmatrix} 5 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} 5 & a \\ b & 0 \end{bmatrix}^T = \begin{bmatrix} 5 & b \\ a & 0 \end{bmatrix}$$

$$\therefore a = b$$

(iii) Solve: $3 \tan^{-1} x + \cot^{-1} x = \pi$

Ans. $3 \tan^{-1} x + \cot^{-1} x = \pi$

$$\therefore 2 \tan^{-1} x + \tan^{-1} x + \cot^{-1} x = \pi$$

$$\therefore 2 \tan^{-1} x + \frac{\pi}{2} = \pi$$

$$2 \tan^{-1} x = \pi - \frac{\pi}{2}$$

$$\tan^{-1}(x) = \frac{\pi}{2} - \frac{\pi}{4}$$

$$x = \tan\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$x = 1$$

(iv) Without expanding at any stage, find the value of:

$$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

Ans. $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

$$R_1 / R_1 + R_3, \quad R_2 / R_2 - R_3$$

$$= \begin{vmatrix} a+x & b+y & c+z \\ a+x & b+y & c+z \\ x & y & z \end{vmatrix} = 0 \quad (\because R_1 = R_2)$$

(v) Find the value of constant 'k' so that the function $f(x)$ defined as:

$$f(x) = \begin{cases} \frac{x^2 - 2x - 3}{x+1}, & x \neq -1 \\ k, & x = -1 \end{cases}$$

is continuous at $x = -1$.

Ans. Given $f(x)$ is continuous at $x = -1$

$$\therefore f(-1) = \lim_{x \rightarrow -1} f(x)$$

$$\therefore k = \lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{x+1}$$

$$= \lim_{x \rightarrow -1} \frac{(x-3)(x+1)}{(x+1)} \quad [\because x \rightarrow -1 \Rightarrow x+1 \neq 0]$$

$$= -1 - 3 = -4$$

$$\therefore K = -4$$

- (vi) Find the approximate change in the volume 'V' of a cube of side x metres caused by decreasing the side by 1%.

Ans. Volume of a cube

$$V = x^3$$

$$\frac{dV}{dx} = 3x^2$$

$$\therefore \delta x = x \cdot \frac{1}{100} = \frac{-x}{100}$$

$\therefore$ Change in volume

$$\delta V = \left(\frac{dV}{dx} \right) \delta x = (3x^2) \cdot \left(\frac{-x}{100} \right) = - \left(\frac{3}{100} \right) x^3$$

$$= - \frac{3}{100} V = -V \frac{3}{100}$$

$\therefore$ change in volume decrease by 3%

- (vii) Evaluate : $\int \frac{x^3 + 5x^2 + 4x + 1}{x^2} dx$.

Ans. $I = \int \frac{x^3 + 5x^2 + 4x + 1}{x^2} dx$

$$= \int \left[\frac{x^3}{x^2} + \frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2} \right] dx$$

$$= \int \left[x + 5 + \frac{4}{x} + \frac{1}{x^2} \right] dx$$

$$= \frac{x^2}{2} + 5x + 4 \log|x| - \frac{1}{x} + c$$

- (viii) Find the differential equation of the family of concentric circles $x^2 + y^2 = a^2$

Ans. Family of concentric circles is $x^2 + y^2 = a^2$

$\therefore$ Differential w.r.t. x

$$2x + 2y \frac{dy}{dx} = 0$$

$$\therefore y \frac{dy}{dx} + x = 0$$

(ix) If A and B are events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$, then find:

(a) $P(A/B)$

(b) $P(B/A)$

Ans. $P(A) = \frac{1}{2}$ $P(B) = \frac{1}{3}$ $P(A \cap B) = \frac{1}{4}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

(x) In a race, the probabilities of A and B winning the race are $\frac{1}{3}$ and $\frac{1}{6}$ respectively. Find the probability of neither of them winning the race.

Ans. Let A win the race be E_1
 B win the race be E_2

$$P(E_1) = \frac{1}{3}, \quad P(E_2) = \frac{1}{6}$$

$$\begin{aligned} P(E'_1 \cap E'_2) &= P(E'_1) \cdot P(E'_2) \\ &= [1 - P(E_1)][1 - P(E_2)] \\ &= \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{6}\right) \\ &= \frac{2}{3} \times \frac{5}{6} = \frac{5}{9} \end{aligned}$$

Question 2

If the function $f(x) = \sqrt{2x-3}$ is invertible then find its inverse. Hence prove that $(f \circ f^{-1})(x) = x$

Ans. Let $y = \sqrt{2x-3}$

$$\therefore y^2 = 2x - 3$$

$$x = \frac{y^2 + 3}{2}$$

$$\therefore f^{-1}(x) = \frac{x^2 + 3}{2}$$

Now,

$$L.H.S = f \circ f^{-1}(x) = f[f^{-1}(x)]$$

$$= \sqrt{2f^{-1}(x) - 3}$$

$$= \sqrt{2\left(\frac{x^2 + 3}{2}\right) - 3} = x$$

$$\therefore f \circ f^{-1}(x) = x$$

Question 3

If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$, prove that $a + b + c = abc$.

Ans. $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$

$$\tan^{-1} b + \tan^{-1} c = \pi - \tan^{-1}(a)$$

$$\tan^{-1}\left(\frac{b+c}{1-bc}\right) = \pi - \tan^{-1} a$$

$$\frac{b+c}{1-bc} = \tan(\pi - \tan^{-1} a)$$

$$\frac{b+c}{1-bc} = -\tan(\tan^{-1} a)$$

$$\frac{b+c}{1-bc} = -a$$

$$b+c = -a + abc$$

$$\therefore a+b+c = abc$$

Question 4

Use properties of determinants to solve for x:

$$\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0 \text{ and } x \neq 0.$$

Ans. $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0 \text{ and } x \neq 0.$

$$C_1 / C_1 + (C_2 + C_3)$$

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix} = 0$$

$$(x+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix} = 0$$

$$R_1 | R_1 - R_3$$

$$(x+a+b+c) \begin{vmatrix} 0 & 0 & -x \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix} = 0$$

$$\therefore (x+a+b+c)[0-0-x(b-x-b)]=0$$

$$(x+a+b+c)(x^2)=0$$

$$\therefore x^2=0 \quad \text{or} \quad x+a+b+c=0$$

but $x \neq 0$

$$\therefore x = -(a+b+c)$$

Question 5

- (a) Show that the function $f(x) = \begin{cases} x^2, & x \leq 1 \\ \frac{1}{2}, & x > 1 \end{cases}$ is continuous at $x=1$ but not differentiable.

Ans. Continuity at $x=1$

$$f(x=1) = x^2 = (1)^2 = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1$$

$$\therefore f(x=1) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 1$$

$\therefore f(x)$ is continuous at $x=1$

Now differentiable at $x=1$

$$(R.H.D. \text{ at } x=1) = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x-1}$$

$$= \lim_{x \rightarrow 1} \frac{-(x-1) \frac{1}{x}}{(x-1)}$$

$$= -\frac{1}{1} = -1$$

$$(L.H.D. \text{ at } x=1) = \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{x - 1} = 2$$

$\therefore L.H.D. \neq R.H.D.$

$\therefore f(x)$ is not differentiable at $x=1$

OR

(b) Verify Rolle's theorem for the following function:

$$f(x) = e^{-x} \sin x \text{ on } [0, \pi]$$

Ans. $f(x) = e^{-x} \cdot \sin x$

(i) $f(x)$ is continuous on $[0, \pi]$ because e^{-x} & $\sin x$ are continuous function on its domain.

(ii) e^{-x} and $\sin x$ is differentiable on $(0, \pi)$

(iii) $f(0) = e^{-0} \cdot \sin 0 = 0$

$$f(\pi) = e^{-\pi} \cdot \sin \pi = 0$$

(iv) Let c be number such that $f'(c) = 0$

$$\therefore f'(x) = e^{-x} \cdot \cos x + \sin x \cdot e^{-x}(-1)$$

$$\therefore f'(c) = e^{-c} (\cos c - \sin c)$$

$$\therefore f'(c) = 0$$

$$e^{-c} (\cos c - \sin c) = 0$$

$$\therefore e^{-c} \neq 0 \therefore \cos C - \sin C = 0$$

$$\tan C = 1$$

$$\therefore C = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$$

$$\therefore \frac{\pi}{4} \in [0, \pi]$$

$\therefore$ Rolle's theorem verify

Question 6

If $x = \tan\left(\frac{1}{a} \log y\right)$, prove that $(1+x^2) \frac{d^2y}{dx^2} + (2x-a) \frac{dy}{dx} = 0$

Ans. $x = \tan\left(\frac{1}{a} \log y\right)$

$$\therefore \frac{1}{a} \log y = \tan^{-1} x$$

differentiating both sides w.r.t. x

$$y = e^{a \tan^{-1} x}$$

$$\frac{dy}{dx} = e^{a \tan^{-1} x} \cdot \left(\frac{a}{1+x^2} \right)$$

$$(1+x^2) \frac{dy}{dx} = ay$$

Again differentiating both sides w.r.t. x

$$(1+x^2) \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 2x = a \frac{dy}{dx}$$

$$(1+x^2) \frac{d^2 y}{dx^2} + \frac{dy}{dx} (2x - a) = 0$$

Question 7

Evaluate : $\int \tan^{-1} \sqrt{x} \, dx$

Ans. $I = \int \tan^{-1} \sqrt{x} \, dx$

Put $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$dx = 2\sqrt{x} dt \rightarrow dx = 2t \, dt$$

$$I = \int 2t \tan^{-1} t \, dt$$

$$I = 2 \left[\frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \frac{t^2}{1+t^2} dt \right]$$

$$I = 2 \left[\frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \left[\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2} \right] dt \right]$$

$$I = \left[t^2 \tan^{-1} t - t + \tan^{-1} t \right] + c$$

$$I = t^2 \tan^{-1} t - t + \tan^{-1} t + c$$

$$I = (x+1) \tan^{-1} \sqrt{x} - \sqrt{x} + c$$

Question 8

- (a) Find the points on the curve $y = 4x^3 - 3x + 5$ at which the equation of the tangent is parallel to the x-axis.

Ans. $y = 4x^3 - 3x + 5 \quad \dots(i)$

$$\frac{dy}{dx} = 12x^2 - 3$$

Given that lines is parallel to x-a xis

$$\therefore \frac{dy}{dx} = 0$$

$$12x^2 - 3 = 0$$

$$12x^2 = 3$$

$$x^2 = \frac{1}{4}$$

$$x = \pm \frac{1}{2}$$

Put $x = \pm \frac{1}{2}$ in equation (i)

$$x = \frac{1}{2} \text{ then, } y = 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right) + 5 = 4$$

$$\therefore \text{Point} \left(\frac{1}{2}, 4 \right)$$

$$\text{when } x = -\frac{1}{2} \text{ then } y = 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right) + 5 = 6$$

$$\therefore \text{Points } (x, y) = \left(\frac{1}{2}, 4 \right) \text{ amd } \left(-\frac{1}{2}, 6 \right)$$

OR

- (b) Water is dripping out from a conial funnel of semi-verticle angle $\frac{\pi}{4}$ at the uniform rate of $2 \text{ cm}^2/\text{sec}$ in the surface, through a tiny hole at the vertex of the bottom. When the slant height of the water level is 4 cm, find the rate of decrease of the slant height of the water.

Ans. Let r be the radius, h be the height and V be the volume of the funnel at any time t .

$$V = \frac{1}{3} \pi r^2 h$$

...(i)

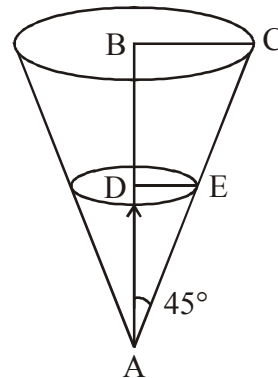
Let l be the slant height of the funnel

Given : Semi-verticle angle = 45°

in the triangle ADE :

$$\sin 45^\circ = \frac{DE}{AE} \Rightarrow \frac{1}{\sqrt{2}} = \frac{r}{l}$$

$$\cos 45^\circ = \frac{AD}{AE} \Rightarrow \frac{1}{\sqrt{2}} = \frac{h}{l}$$



$$r = \frac{1}{\sqrt{2}} \text{ and } h = \frac{1}{\sqrt{2}} \quad \dots(\text{ii})$$

therefore the equation (i) can be rewritten as :

$$V = \frac{1}{3} \pi \times \left(\frac{I}{\sqrt{2}} \right)^2 \times \frac{I}{\sqrt{2}} = \frac{\pi}{3 \times 2 \times \sqrt{2}} \times I^3$$

$$V = \frac{\pi}{6\sqrt{2}} I^3 \quad \dots(\text{iii})$$

Differentiate w.r.t. t :

$$\frac{dV}{dt} = \frac{\pi}{6\sqrt{2}} \times 3I^2 \times \frac{dI}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{2\sqrt{2}} \times I^2 \times \frac{dI}{dt}$$

$$\frac{dI}{dt} = \frac{2\sqrt{2}}{\pi I^2} \times \frac{dV}{dt} \quad \dots(\text{iv})$$

Since it is given that rate of change (decrease) of volume of water w.r.t. t is

$$\frac{dV}{dt} = -2 \text{ cm}^3 / \text{sec}$$

therefore

$$\frac{dI}{dt} = \frac{2\sqrt{2}}{\pi I^2} \times (-2) = -\frac{4\sqrt{2}}{\pi I^2}$$

$$\left. \frac{dI}{dt} \right|_{\text{at } I=4} = -\frac{4\sqrt{2}}{\pi \times (4)^2} = -\frac{\sqrt{2}}{4\pi} \text{ cm / sec}$$

Question 9

(a) Solve : $\sin x \frac{dy}{dx} - y = \sin x \cdot \tan \frac{x}{2}$

Ans. $\frac{dy}{dx} - y \cdot \operatorname{cosec} x = \tan \frac{x}{2}$

$$\frac{dy}{dx} - y \cdot \operatorname{cosec} x = \tan \frac{x}{2} \quad \dots(\text{i})$$

Compare $\frac{dy}{dx} + Py = Q$

$$P = -\operatorname{cosec} x, Q = \tan \frac{x}{2}$$

$$\text{I.F.} = e^{\int P dx}$$

$$\text{I.F.} = e^{\int -\operatorname{cosec} x dx}$$

$$I.F. = e^{-\log_e(\operatorname{cosec} x - \cot x)}$$

$$I.F. = e^{\log_e(\operatorname{cosec} x - \cot x)^{-1}}$$

$$I.F. = (\operatorname{cosec} x - \cot x)^{-1} = (\operatorname{cosec} x + \cot x)$$

$\therefore$ solution of the linear differential equation

$$y \cdot IF = \int Q \cdot IF \cdot dx$$

$$y \cdot (\operatorname{cosec} x + \cot x) = \int \tan \frac{x}{2} \cdot \left(\frac{1 + \cos x}{\sin x} \right) dx$$

$$y \cdot (\operatorname{cosec} x - \cot x) = \int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \frac{2 \cdot \cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}} dx = \int 1 dx$$

$$y \cdot (\operatorname{cosec} x + \cot x) = x + c$$

OR

- (b) The population of a town grows at the rate of 10% per year. Using differential equation, find how long will it take for the population to grow 4 times.

Ans. Here $\frac{dx}{dt} \propto x$ [Since increase in population speeds up with increase in population] and let x be the population at anytime t .

$$\therefore \frac{dx}{dt} = rx \quad (\text{where } r \text{ is proportionality constant})$$

$$\therefore \frac{dx}{x} = r \cdot dt$$

integrating both sides

$$\ln x = rt + c, \quad (\text{where } c \text{ is the integration constant})$$

$$\therefore x = e^{rt+c}$$

$$x = K e^{rt}, \text{ where } K = e^c$$

Here r is the rate of increase and K is the initial population let x_0 then $t = 0$

$$x_0 = K e^0 \Rightarrow K = x_0$$

Given to find the time t taken to attain 4 times population, so $x = 4x_0$

$$\text{So, } x = K e^{rt}$$

$$\Rightarrow 4x_0 = x_0 e^{0.10t}$$

$$2 = e^{0.05t}$$

Taking log on both sides

$$\ln 2 = \ln e^{0.1t}$$

$$0.69314 = 0.1t \quad t = 6.9314$$

Question 10

(a) Using matrices, solve the following system of equations :

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

Ans.

(a) Using this let us solve the system of given equation

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

This can be written in the form $AX = B$

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\text{we know } A^{-1} = \frac{1}{|A|} (\text{adj } A)$$

$$|A| = 2(2 \times -2 - 1 \times -4) - (-3)(3 \times -2 - 1 \times -4) + 5(3 \times 1 - 2 \times 1)$$

$$= 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$$

$$= 0 - 6 + 5 = -1 \neq 0$$

Hence it is a non-singular matrix

Therefore A^{-1} exists

Let us find the (adj A) by finding the minors and cofactors

$$M_{11} = \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -4 + 4 = 0$$

$$M_{12} = \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -6 + 4 = -2$$

$$M_{13} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = 6 - 5 = 1$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -4 - 5 = -9$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 2 + 3 = 5$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 12 - 10 = 2$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -8 - 15 = -23$$

$$M_{33} = \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 4 + 9 = 13$$

$$A_{11} = 0 \quad A_{12} = 2 \quad A_{13} = 1$$

$$A_{21} = -1 \quad A_{22} = -9 \quad A_{23} = -5$$

$$A_{31} = 2 \quad A_{32} = 23 \quad A_{33} = 13$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

We know $AX = B$, then $X = A^{-1} B$

$$\text{Therefore } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

Matrix multiplication can be done by multiplying the rows of matrix A with the column of matrix B.

$$\text{Therefore, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -5 & +6 \\ -22 & -45 & +69 \\ -11 & -25 & +39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence $x=1$, $y=2$ and $z=3$

OR

(b) Using elementary transformation, find the inverse of the matrix :

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\text{Ans. Let } A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$|A| = 0[2-3] - 1[1-9] + 2[1-6]$$

$$= 8 + 2(-5) = 8 - 10 = -2 \neq 0$$

$$\Rightarrow A^{-1} \text{ exists.}$$

$$\therefore A = IA$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_3 / R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$$R_3 / R_3 + 5R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$$R_2 / R_2 - R_3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 3 & -1 \\ 5 & -3 & 1 \end{bmatrix} A$$

$$R_1 / R_1 - 2R_2 \text{ and } R_3 \times \frac{1}{2}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -5 & 2 \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

$$R_1 / R_1 - 3R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

$$\therefore A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Question 11

A speaks truth in 60% of the cases, while B is 40% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact ?

Ans. A speaks truth $P(A) = \frac{60}{100}$, $P(A') = \frac{40}{100}$

B speaks truth $P(B) = \frac{40}{100}$, $P(B') = \frac{60}{100}$

they contradict each other $= P(A) \cdot (B') + P(A') \cdot P(B)$

$$= \frac{60}{100} \times \frac{60}{100} + \frac{40}{100} \times \frac{40}{100}$$

$$= \frac{3600 + 1600}{10000}$$

$$= \frac{5200}{10000}$$

$$= \frac{52}{100}$$

% of cases they likely to contradict each other $= \frac{52}{100} \times 100 = 52\%$

Question 12

A cone is inscribed in a sphere of radius 12 cm. If the volume of the cone is maximum, find its height.

Ans. Let VAB be a cone of greatest volume inscribed in a sphere of radius 12. It is obvious that for maximum volume the axis of the cone must be along a diameter of the sphere. Let VC be the axis of the cone and O be the centre of the sphere such that $OC = x$.

Then,

$$VC = VO + OC = R + x = (12 + x) = \text{height of cone}$$

Applying Pythagoras theorem,

$$OA^2 = AC^2 + OC^2$$

$$AC^2 = 12^2 - x^2$$

$$AC^2 = 144 - x^2$$

Let V be the volume of the cone, then

$$\begin{aligned} V &= \frac{1}{3} \pi (AC)^2 (VC) \\ &= \frac{1}{3} \pi (144 - x^2)(12 + x) \\ &= \frac{1}{3} \pi [1728 + 144x - 12x^2 - x^3] \quad \dots(i) \end{aligned}$$

$$\frac{dV}{dx} = \frac{1}{3} \pi [144 - 24x - 3x^2]$$

$$\frac{d^2V}{dx^2} = \frac{1}{3} \pi [-24 - 6x] = \frac{1}{3} \pi (-6)^2 [4 + x] = -2\pi (4 + x)$$

$$\text{Now, } \frac{dV}{dx} = 0 \text{ gives } \frac{1}{3} \pi [144 - 24x - 3x^2] = 0$$

$$\text{i.e., } 144 - 24x - 3x^2 = 0$$

$$\text{i.e., } x^2 + 8x - 48 = 0$$

$$\text{i.e., } (x + 12)(x - 4) = 0$$

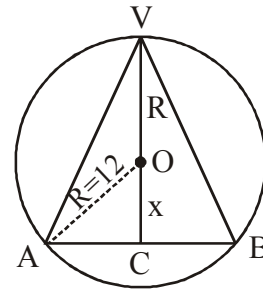
$$\text{i.e., } x = -12 \text{ or } x = 4$$

$$\left[\frac{d^2V}{dx^2} \right]_{x=4} = -2\pi (4 + 4) = -16\pi < 0$$

Thus, V is maximum when $x = 4$

Putting $x = 4$ in (1), we obtain

$$\therefore \text{Height of the cone} = x + R = 4 + 12 = 16 \text{ cm}$$



Question 13

(a) Evaluate : $\int \frac{x-1}{\sqrt{x^2-x}} dx$

Ans. Let $I = \int \frac{x-1}{\sqrt{x^2-x}} dx$

$$\therefore x-1 = A \frac{d}{dx} (x^2-x) + B$$

$$x-1 = A(2x-1) + B$$

$$1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$$

$$-1 = -A + B \Rightarrow -1 = \frac{-1}{2} + B \Rightarrow B = \frac{-1}{2}$$

$$\begin{aligned}
 I &= \int \frac{\frac{1}{2}(2x-1)dx}{\sqrt{x^2-x}} - \int \frac{1}{2} \frac{dx}{\sqrt{x^2-x}} \\
 &= \int \frac{\frac{1}{2}(2x-1)dx}{\sqrt{x^2-x}} - \frac{1}{2} \int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \\
 &= \frac{1}{2} \times 2\sqrt{x^2-x} - \frac{1}{2} \times \log \left| \left(x-\frac{1}{2}\right) + \sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + C \\
 &= \sqrt{x^2-x} - \frac{1}{2} \log \left| x - \frac{1}{2} + \sqrt{x^2-x} \right| + C
 \end{aligned}$$

OR

(b) Evaluate : $\int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin x \cos x} dx$

Ans. $I = \int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin x \cos x} \cdot dx \quad \dots(1)$

Using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\begin{aligned}
 I &= \int_0^{\pi/2} \frac{\cos^2 \left(\frac{\pi}{2} - x \right)}{1 + \sin \left(\frac{\pi}{2} - x \right) \cos \left(\frac{\pi}{2} - x \right)} dx \\
 &= \int_0^{\pi/2} \frac{\sin^2 x}{1 + \cos x \cdot \sin x} dx \quad \dots(2)
 \end{aligned}$$

Adding eq. (1) & (2)

$$\begin{aligned}
 2I &= \int_0^{\pi/2} \frac{\cos^2 x + \sin^2 x}{1 + \sin x \cos x} dx \\
 &= \int_0^{\pi/2} \frac{1}{1 + \sin x \cos x} dx \\
 &= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + \tan x} dx
 \end{aligned}$$

$$2I = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{1 + \tan^2 x + \tan x}$$

Put $\tan x = t$, $\sec^2 x \, dx = dt$

when $x = 0$, $t = 0$

when $x = \frac{\pi}{2}$, $t = \infty$

$$2I = \int_0^{\infty} \frac{dt}{t^2 + 2t \cdot \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + 1}$$

$$= \int_0^{\infty} \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \left[\tan^{-1} \left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \right]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left[\frac{2t+1}{\sqrt{3}} \right]_0^{\infty}$$

$$2I = \frac{2}{\sqrt{3}} \left[\frac{\pi}{2} - \frac{\pi}{6} \right]$$

$$I = \frac{1}{\sqrt{3}} \left[\frac{3\pi - \pi}{6} \right]$$

$$= \frac{1}{\sqrt{3}} \left[\frac{2\pi}{6} \right] = \frac{\pi}{3\sqrt{3}}$$

Question 14

From a lot of 6 items containing 2 defective items, a sample of 4 items are drawn at random. Let the random variable X denote the number of defective items in the sample.

If the sample is drawn without replacement, find :

- (a) The probability distribution of X
- (b) Mean of X
- (c) Variance of X

Ans. In 6 items 2 defective and 4 non-defective
Let P is the probability of defective items

Let x = number of defective items

$\therefore x = 0, 1, 2$

$$\therefore P(x=0) = \frac{{}^4C_4}{{}^6C_4} = \frac{1}{15}$$

$$\therefore P(x=1) = \frac{{}^2C_1 \times {}^4C_3}{{}^6C_4} = \frac{8}{15}$$

$$\therefore P(x=2) = \frac{{}^2C_2 \times {}^4C_2}{{}^6C_4} = \frac{6}{15}$$

<u>X</u>	<u>P(x)</u>	<u>xP(x)</u>	<u>x²P(x)</u>
<u>0</u>	<u>$\frac{1}{15}$</u>	<u>0</u>	<u>0</u>
<u>1</u>	<u>$\frac{8}{15}$</u>	<u>$\frac{8}{15}$</u>	<u>$\frac{8}{15}$</u>
<u>2</u>	<u>$\frac{6}{15}$</u>	<u>$\frac{12}{15}$</u>	<u>$\frac{24}{15}$</u>

$$\begin{aligned} \text{(b) Mean}(\bar{X}) &= \sum P_i X_i \\ &= \frac{20}{15} = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} \text{(c) Variance}(\sigma^2) &= \sum P_i X_i^2 - \left(\sum P_i X_i\right)^2 \\ &= \frac{32}{15} - \left(\frac{4}{3}\right)^2 = \frac{16}{45} = 0.35 \end{aligned}$$

SECTION - B (20 Marks)

Question 15

(a) Find λ if the scalar projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.

Ans. Projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 4$

$$= \frac{(\lambda\hat{i} + \hat{j} + 4\hat{k}) \cdot (2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{2^2 + 6^2 + 3^2}} = 4$$

$$\therefore \frac{2\lambda + 6(1) + 4(3)}{\sqrt{49}} = 4$$

$$\therefore \frac{2\lambda + 18}{7} = 4$$

$$\therefore 2\lambda = 28 - 18$$

$$\therefore 2\lambda = 10$$

$$\lambda = 5$$

- (b) The Cartesian equation of line is : $2x - 3 = 3y + 1 = 5 - 6z$. Find the vector equation of a line passing through $(7, -5, 0)$ and parallel to the given line.

Ans. Cartesian equation of a line is

$$2x - 3 = 3y + 1 = 5 - 6z$$

$$\text{i.e., } 2\left(x - \frac{3}{2}\right) = 3\left(y + \frac{1}{3}\right) = -6\left(z - \frac{5}{6}\right)$$

Dividing by -6 throughout

$$\text{i.e., } \frac{x - \frac{3}{2}}{-3} = \frac{y + \frac{1}{3}}{-2} = \frac{z - \frac{5}{6}}{1}$$

$\therefore$ D.r.s of the above line is $-3, -2, 1$

Now, equation of a line passing through point $(7, -5, 0)$ and parallel to the above line whose d.r.s is $-3, -2, 1$ is

$$\vec{r} = (7\hat{i} - 5\hat{j}) + \lambda(-3\hat{i} - 2\hat{j} + \hat{k})$$

$$\therefore \vec{r} = (7\hat{i} - 5\hat{j}) + \lambda(3\hat{i} + 2\hat{j} - \hat{k})$$

- (c) Find the equation of the plane through the intersection of the planes

$$\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 9 \text{ and } \vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3 \text{ and passing through the origin.}$$

Ans. Equation of I plane is $\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 9$

$$\text{i.e., } (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 9$$

$$x + 3y - z = 9$$

$$x + 3y - z - 9 = 0 \quad \dots(\text{i})$$

Equation of II plane is $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$

$$\text{i.e., } (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$$

$$\text{i.e., } 2x - y + z = 3$$

$$\text{i.e., } 2x - y + z - 3 = 0 \quad \dots(\text{ii})$$

Now, equation of a plane passing through intersection of given planes is

$$(x + 3y - z - 9) + \lambda(2x - y + z - 3) = 0$$

$$(1 + 2\lambda)x + (3 - \lambda)y + (-1 + \lambda)z - 9 - 3\lambda = 0$$

Since plane is passing through the origin (0, 0, 0)

$$-9 - 3\lambda = 0$$

$$-3\lambda = 9$$

$$\lambda = -3$$

Question 16

- (a) If A, B, C are three non-collinear points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, respectively, then show that

$$\text{the length of the perpendicular from C on AB is } \frac{\left| (\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) \right|}{\left| \vec{b} - \vec{a} \right|}$$

Ans. Let ABC be a triangle and let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of its vertices A, B, C respectively. Let CM be the perpendicular from C on AB. Then,

$$\text{Area of } \triangle ABC = \frac{1}{2} (\overline{AB}) \cdot CM = \frac{1}{2} |\overline{AB}| (CM)$$

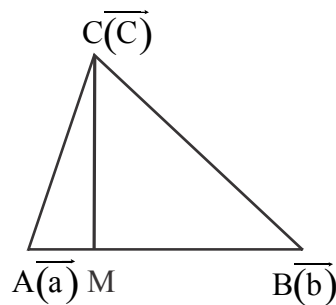
$$\text{Also, Area of } \triangle ABC = \frac{1}{2} |\overline{AB} \times \overline{AC}|$$

$$\text{Area of } \triangle ABC = \frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

$$\therefore \frac{1}{2} |\overline{AB}| (CM) = \frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

$$\Rightarrow CM = \frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{|\overline{AB}|}$$

$$\Rightarrow CM = \frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{|\vec{b} - \vec{a}|}$$



OR

- (b) Show that the four points A, B, C and D with position vectors

$$4\hat{i} + 5\hat{j} + \hat{k}, \quad -\hat{j} - \hat{k}, \quad 3\hat{i} + 9\hat{j} + 4\hat{k} \quad \text{and} \quad 4(-\hat{i} + \hat{j} + \hat{k})$$
 respectively, are coplanar.

Ans. $\vec{a} = 4\hat{i} + 5\hat{j} + \hat{k}$

$$\vec{b} = -\hat{i} - \hat{j}$$

$$\vec{c} = 3\hat{i} + 9\hat{j} + 4\hat{k}$$

$$\vec{d} = -4\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\overline{AB} = \vec{b} - \vec{a} = -4\hat{i} - 6\hat{j} - 2\hat{k}$$

$$\overline{AC} = \overline{c} - \overline{a} = -\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\overline{AD} = \overline{d} - \overline{a} = -8\hat{i} - \hat{j} + 3\hat{k}$$

$$\overline{AB}, \overline{AC} \text{ \& } \overline{AD} \text{ are coplanar if } \begin{vmatrix} \overline{AB} & \overline{AC} & \overline{AD} \end{vmatrix} = 0 \text{ i.e., } \overline{AB} \cdot (\overline{AC} \times \overline{AD}) = 0$$

$$= \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= -4(12+3) + 6(-3+24) - 2(1+32)$$

$$= -4(15) + 6(21) - 2(66)$$

$$= -60 + 126 - 66$$

$$= 0$$

$$\therefore \overline{AB}, \overline{AC} \text{ \& } \overline{AD} \text{ are coplanar}$$

$$\therefore \text{Points A, B, C and D are coplanar}$$

Question 17

- (a) Draw a rough sketch of the curve and find the area of the region bounded by curve $y^2 = 8x$ and the line $x = 2$.

Ans.

$$\text{Given equation is } y^2 = 8x$$

$$\text{Comparing with } y^2 = 4ax$$

$$\text{We get } 4a = 8$$

$$\text{i.e. } a = 2$$

$$\text{Given } y^2 = 4(2)x$$

$$y^2 = 8x$$

$$\therefore y = \sqrt{8x}$$

$$\text{Also, } x = 2 \text{ meets } y^2 = 8x$$

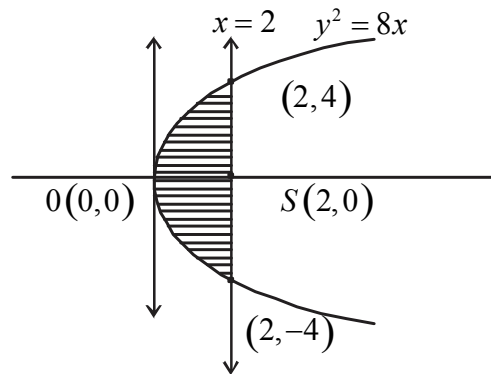
$$\therefore y^2 = 16$$

$$\therefore y = \pm 4$$

$$\therefore (2, 4) \text{ and } (2, -4) \text{ are their point of intersection.}$$

$$\therefore \text{Required are } A = 2 \int_0^2 \sqrt{8x} \, dx$$

$$= 2\sqrt{8} \int_0^2 x^{1/2} \, dx$$



$$\begin{aligned}
 &= 4\sqrt{2} = \left[\frac{x^{3/2}}{3/2} \right]_0^2 \\
 &= \frac{8\sqrt{2}}{3} [2^{3/2} - 0] \\
 &= \frac{8\sqrt{2}}{3} \times \sqrt{8} \\
 &= \frac{8\sqrt{2} \times 2\sqrt{2}}{3} = \frac{32}{3} \text{ sq. units}
 \end{aligned}$$

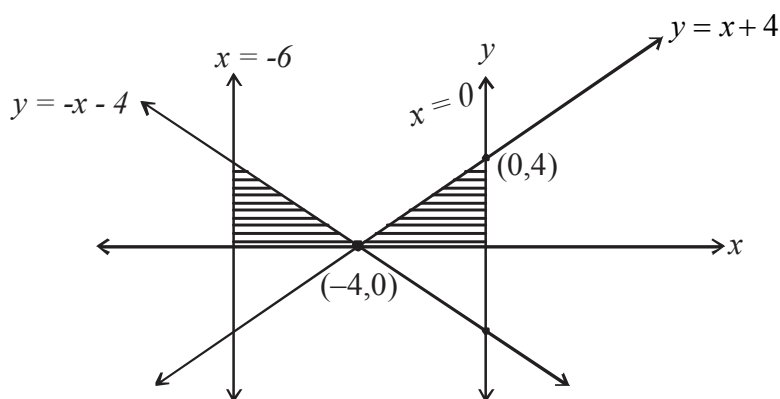
OR

- (b) Sketch the graph of $y = |x + 4|$. Using integration, find the area of the region bounded by the curve $y = |x + 4|$ and $x = -6$ and $x = 0$.

Ans.

$$y = x + 4, \text{ if } x > -4$$

$$\& y = -(x + 4), \text{ if } x < -4$$



For $y = x + 4$

when $x = 0, y = 4$

& when $y = 0, x = -4$

Points are

$\therefore (0, 4) \& (-4, 0)$

$\therefore$ Required area

$$= \int_{-6}^{-4} -(x + 4) dx + \int_{-4}^0 (x + 4) dx$$

$$= -\left[\frac{x^2}{2} + 4x \right]_{-6}^{-4} + \left[\frac{x^2}{2} + 4x \right]_{-4}^0$$

$$= -\left[\frac{(-4)^2}{2} + 4(-4) - \left(\frac{(-6)^2}{2} + 4(-6) \right) \right] + \left[0 + 0 - \left(\frac{(-4)^2}{2} + 4(-4) \right) \right]$$

For $y = -x - 4$,

when $x = 0, y = -4$

when $y = 0, x = -4$

$\therefore$ Point are

$(0, -4) \& (-4, 0)$

$$\begin{aligned}
&= -\left[\frac{16}{2} - 16 - \left[\frac{36}{2} - 24\right]\right] + \left[-\left(\frac{16}{2} - 16\right)\right] \\
&= -[-8 + 6] + [8] \\
&2 + 8 = 10 \text{ sq. unit}
\end{aligned}$$

Question 18

Find the image of a point having position vector : $3\hat{i} - 2\hat{j} + \hat{k}$ in the plane $\vec{r} \cdot (3\hat{i} - \hat{j} + 4\hat{k}) = 2$.

Ans. Let B be root of pont $A(3\hat{i} - 2\hat{j} + \hat{k})$ in the plane $\vec{r} \cdot (3\hat{i} - \hat{j} + 4\hat{k}) = 2$ can of AB : is

$$\vec{r}(3\hat{i} - 2\hat{j} + \hat{k}) + \lambda(3\hat{i} - \hat{j} + 4\hat{k})$$

$$\therefore \frac{x-3}{3} = \frac{y+2}{-1} = \frac{z-1}{4} = \lambda$$

$$\therefore x = 3\lambda + 3, y = -\lambda - 2, z = 4\lambda + 1$$

subtitute x, y & z in plane $3x - y + 4z = 2$

$$\therefore 3(3\lambda + 3) - (-\lambda - 2) + 4(4\lambda + 1) = 2$$

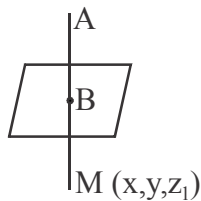
$$9\lambda + 9 + \lambda + 2 + 16\lambda + 4 = 2$$

$$26\lambda + 13 = 0 \Rightarrow \lambda = -\frac{1}{2}$$

$$\therefore x = -\frac{3}{2} + 3, y = \frac{1}{2} - 2, z = -2 + 1$$

$$x = \frac{3}{2}, y = -\frac{3}{2}, z = -1$$

$\therefore$ by midpt formula,



$$\frac{3}{2} = \frac{3+x_1}{2}, \quad \frac{-3}{2} = \frac{-2+y_1}{2}, \quad -1 = \frac{1-z_1}{2}$$

$$\boxed{x_1 = 0}, \quad \boxed{y_1 = -1}, \quad -z = 1 - z_1 \quad \boxed{z_1 = 1 + 2 = 3}$$