ISC SEMESTER 2 EXAMINATION
SAMPLE PAPER - 4
MATHEMATICS

Maximum Marks: 40

Time allowed: One and a half hour

*Candidates are allowed an additional **10 minutes** for **only** reading the paper.*

*They must **Not** start writing during this time.*

The Question Paper consists of three sections A, B and C.

*Candidates are required to attempt all questions from **Section A** and all questions **EITHER** from **Section B** **OR** **Section C***

All working, including rough work, should be done on the same sheet as, and adjacent to the rest of the answer.

Mathematical tables and graph papers are provided.

Section-A

Question 1.

Choose the correct option for the following questions.

(i) If $I = \int_4^5 |x-5|\, dx$, then the value of I is :

(a) $\frac{5}{4}$ (b) 0 (c) $\frac{49}{2}$ (d) $\frac{1}{2}$

(ii) If $P(B) = \frac{2}{5}$, $P(A) = \frac{3}{10}$ and $P(A \cup B) = \frac{3}{5}$, then $P(A/B) + P(B/A)$ equals :

(a) $\frac{1}{4}$ (b) $\frac{7}{12}$ (c) $\frac{5}{12}$ (d) $\frac{1}{3}$

(iii) If $\int \sin^3 x \cos^4 x\, dx = \frac{\cos^a x}{a} - \frac{\cos^b x}{b} + c$, where, a and b are integers, then the values of a and b are :

(a) 1, 1 (b) 5, 7 (c) 7, 5 (d) None of these

(iv) Find the sum of the order and degree of the differential equation $y = x\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2}$.

(a) 0 (b) 1 (c) 2 (d) 3

(v) Three numbers are chosen from 1 to 20. Find the probability that they are consecutive.

(a) 1 (b) 0 (c) $\frac{3}{190}$ (d) $\frac{18}{90}$

(vi) If $\int x \sin x\, dx = -x\cos x - a$; then a is equal to :

(a) $\sin x + c$ (b) $\cos x + c$ (c) $-\sin x + c$ (d) None of these

Question 2.

Evaluate : $\int_0^1 \frac{dx}{e^x + e^{-x}}$.

OR

Evaluate : $\int \sin^3 x \cos^2 x\, dx$.

Question 3.

Solve : $y\,dx - (x + 2y^2)\,dy = 0$.

OR

Solve the differential equation $(1 + x^2)\dfrac{dy}{dx} = 4x^2 - 2xy$.

Question 4.

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx =$$

Question 5.

Three persons A, B and C shot to hit a target. If in trials, A hits the target 4 times in 5 shots, B hits 3 times in 4 shots and C hits 2 times in 3 shots. Find the probability that :

(i) Exactly 2 persons hit the target

(ii) At least 2 persons hit the target.

OR

Bag A contains three red and four white balls, bag B contains two red and three white balls. If one ball is drawn from bag A and two balls from bag B, find the probability that :

(i) One ball is red and two balls are white.

(ii) All three balls are of the same colour.

Question 6.

Evaluate $\int e^x \left[\dfrac{1+\sin x}{1+\cos x}\right] dx$.

Question 7.

The compressors used in refrigerators are manufactured by three different factories at Pune, Nasik and Nagpur. It is known that the Pune factory produces twice as many as compressors as the Nasik one, which produces the same number of compressors as the Nagpur one (during the same period).

Experience also shows that 0·2% of the compressors produced at Pune as well as Nasik are defective and so are 0·4% of these produced at Nagpur. A quality controller chooses a compressor and finds it is a defective one. What is the probability that it was produced at Nasik ?

Question 8.

Evaluate : $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\,dx$.

OR

Prove that : $\int_0^{\pi} \dfrac{x}{1 - \cos\alpha \sin x}\,dx = \dfrac{\pi(\pi - \alpha)}{\sin\alpha}$.

Section-B

Question 9.

Choose the correct option for the following questions.

(i) The equation of the plane parallel to the lines $x - 1 = 2y - 5 = 2z$ and $3x = 4y - 11 = 3z - 4$ and passing through the point (2, 3, 3) is :

(a) $x - 4y + 2z + 4 = 0$ (B) $x + 4y + 2z + 4 = 0$ (c) $x - 4y + 2z - 4 = 0$ (d) $2x + y + 3z - 6 = 0$

(ii) Write the equation of a plane which is at a distance of $5\sqrt{3}$ units from the origin and the normal to which is equally inclined to coordinate axes :

(a) $x + y + z = 1$ (b) $x + y + z = 15$ (c) $x + y + z = 5\sqrt{3}$ (d) None of these

Question 10.

Find the equation of the plane passing through the point (2, – 3, 1) and perpendicular to the line joining the points (4, 5, 0) and (1, – 2, 4).

Question 11.

Using integration, find the area bounded by the tangent to the curve $4y = x^2$ at the point (2, 1) and the lines whose equations are $x = 2y$ and $x = 3y - 3$.

Section-C

Question 12.

Choose the correct option for the following questions.

(i) The two lines $4x + 2y - 3 = 0$ and $3x + 6y + 5 = 0$. Then the correlation coefficient between x and y is :

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\pm\frac{1}{2}$ (d) None of these

(ii) For the given lines of regression, $3x - 2y = 5$ and $x - 4y = 7$, find regression coefficients b_{yx} and b_{xy}

(a) $\frac{1}{4}, \frac{2}{3}$ (b) $\frac{2}{3}, \frac{1}{4}$ (c) $\frac{1}{4}, \frac{3}{2}$ (d) None of these

Question 13.

If $b_{xy} = \frac{216}{97}$, $\bar{x} = 25.6$ and $\bar{y} = 17.2$, then find the value of x for $y = 19$.

Question 14.

A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5,760 to invest and has space for at most 20 items. A fan and sewing machine costs ₹ 360 and ₹ 240 respectively. He can sell a fan at a profit of ₹ 22 and a sewing machine at a profit of ₹ 18. He can sell whatever he buys, how should he invest his money in order to maximise his profit ?

Answers

Section-A

Answers 1.

(i) (d) $\frac{1}{2}$

Explanation :

$$\int_4^5 |x-5|\,dx = \int_4^5 -(x-5)dx$$

$$= \int_4^5 (5-x)dx$$

$$= \left[5x - \frac{x^2}{2}\right]_4^5$$

$$= \left[5\times5 - \frac{5\times5}{2}\right] - \left[5\times4 - \frac{4\times4}{2}\right]$$

$$= \left[25 - \frac{25}{2}\right] - \left[20 - \frac{16}{2}\right]$$

$$= \frac{50-25}{2} - 12 = \frac{25}{2} - 12$$

$$= \frac{25-24}{2} = \frac{1}{2}$$

(ii) (b) $\frac{7}{12}$

Explanation :

Given : $P(A) = \frac{2}{5}, P(B) = \frac{3}{10}, P(A \cup B) = \frac{3}{5}$

We know, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad \frac{3}{5} = \frac{2}{5} + \frac{3}{10} - P(A \cap B)$

$\Rightarrow \quad P(A \cap B) = \frac{7}{10} - \frac{3}{5} = \frac{1}{10}$

Now,
$$P(A/B) + P(B/A) = \frac{P(A \cap B)}{P(B)} + \frac{P(A \cap B)}{P(A)}$$
$$= P(A \cap B)\left(\frac{1}{P(B)} + \frac{1}{P(A)}\right)$$
$$= \frac{1}{10}\left(\frac{10}{3} + \frac{5}{2}\right)$$
$$= \frac{1}{10}\left(\frac{35}{6}\right) = \frac{7}{12}$$

(iii) (c) 7, 5

Explanation :

Let
$$I = \int \sin^3 x \cos^4 x \, dx$$
$$= \int \sin^2 x . \cos^4 x \, (\sin x \, dx)$$
$$= \int (1 - \cos^2 x) \cos^4 x \, (\sin x \, dx)$$

Let $\cos x = t$

$\Rightarrow \quad -\sin x \, dx = dt$

or $\quad \sin x \, dx = -dt$

$\therefore$
$$I = \int (1 - t^2)\, t^4 \, (-dt)$$
$$= -\int (t^4 - t^6)\, dt$$
$$= \int (t^6 - t^4)\, dt$$
$$= \frac{t^7}{7} - \frac{t^5}{5} + c$$
$$= \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + c$$

$$\therefore \quad \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + c = \frac{\cos^a x}{a} - \frac{\cos^b x}{b} + c$$

$\Rightarrow a = 7, b = 5$

(iv) (d) 3

Explanation :

Order of the given differential equation is 2.
Degree of the given differential equation is 1.
$\therefore$ Sum of order and degree of the given D.E. = 2 + 1 = 3.

(v) (c) $\frac{3}{190}$

Explanation :

Total number of ways of choosing three numbers from 1 to 20 = $^{20}C_3$

$$= \frac{20 \times 19 \times 18}{3 \times 2 \times 1}$$

$\therefore$ $n(S) = 1140$

Three consecutive numbers from 1 to 20 = {(1, 2, 3), (2, 3, 4), (3, 4, 5), …, (17, 18, 19), (18, 19, 20)}

$\therefore$ $n(E) = 18$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{18}{1140} = \frac{3}{190}$

(vi) (a) $\sin x + c$

Explanation :

$$\int x \sin x \, dx = 1$$

Using integration by parts

$$I = x\int \sin x \, dx - \int\left[\frac{d}{dx}(x) \times \int \sin x \, dx\right] dx$$

$$= -x \cos x - \int (-\cos x) \, dx$$

$$= -x \cos x + \sin x + c$$

$\therefore$ $a = \sin x + c$

Answer 2.

Let $I = \int_0^1 \frac{dx}{e^x + e^{-x}}$

$\therefore$ $I = \int_0^1 \frac{dx}{e^x + \frac{1}{e^x}} = \int_0^1 \frac{e^x}{e^{2x} + 1} dx$

Let $e^x = t$

$\therefore$ $e^x \, dx = dt$

When $x = 0, t = 1$

When $x = 1, t = e$

$\therefore$ $I = \int_1^e \frac{1}{t^2 + 1} dt = [\tan^{-1} t]_1^e$

$\Rightarrow$ $I = \tan^{-1} e - \tan^{-1} 1$

$\Rightarrow$ $I = \tan^{-1} e - \frac{\pi}{4}$ **Ans.**

OR

Let $I = \int \sin^3 x \cos^2 x \, dx$

$$I = \int \sin^2 x . \cos^2 x \sin dx$$

$$I = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

Let $\cos x = t$

$\Rightarrow -\sin x \, dx = dt$

$\therefore$ $I = -\int (1 - t^2) t^2 \, dt$

$$I = -\int t^2 \, dt + \int t^4 \, dt$$

$$I = \frac{-t^3}{3} + \frac{t^5}{5} + c = \frac{-\cos^3 x}{3} + \frac{\cos^5 x}{5} + c.$$ **Ans.**

Answer 3.

Given : $y\,dx - (x + 2y^2)\,dy = 0$

$\Rightarrow \quad y\,dx = (x + 2y^2)\,dy$

$\Rightarrow \quad \dfrac{dx}{dy} = \dfrac{x + 2y^2}{y} = \dfrac{x}{y} + \dfrac{2y^2}{y}$

$\Rightarrow \quad \dfrac{dx}{dy} - \dfrac{x}{y} = 2y$

Comparing with $\dfrac{dx}{dy} + \mathrm{P}x = \mathrm{Q}(y)$, we get

$$\mathrm{P} = -\frac{1}{y}, \mathrm{Q}(y) = 2y$$

$$\text{I.F.} = e^{\int \mathrm{P}\,dy} = e^{\int -\frac{1}{y}\,dy} = e^{-\log y} = e^{\log y^{-1}}$$

$$\text{I.F.} = y^{-1} = \frac{1}{y}$$

∴ Solution of differential equation is

$$x \times \text{I.F.} = \int \text{I.F.} \times \mathrm{Q}(y)\,dy$$

$\Rightarrow \quad \dfrac{x}{y} = \displaystyle\int \frac{1}{y}\cdot 2y\,dy = \int 2\,dy$

$\Rightarrow \quad \dfrac{x}{y} = 2y + c$

$\Rightarrow \quad x = 2y^2 + cy.$ **Ans.**

OR

Given differential equation is,

$$(1 + x^2)\frac{dy}{dx} = 4x^2 - 2xy$$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{4x^2}{1+x^2} - \dfrac{2xy}{1+x^2}$

or $\quad \dfrac{dy}{dx} + \dfrac{2xy}{1+x^2} = \dfrac{4x^2}{1+x^2}$

This is of the form $\dfrac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$

where, $\quad \mathrm{P} = \dfrac{2x}{1+x^2}$ and $\mathrm{Q} = \dfrac{4x^2}{1+x^2}$

Thus, the solution of given differential equation is,

$$y \times \text{I.F.} = \int (\mathrm{Q} \times \text{I.F.})\,dx + c$$

Where, $\quad \text{I.F.} = e^{\int \mathrm{P}dx}$

$$= e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\log(1+x^2)} \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log[f(x)] + c\right]$$

$$= 1 + x^2$$

$\therefore \quad y \times (1 + x^2) = \int\left[\frac{4x^2}{1+x^2}\times(1+x^2)\right]dx + c$

$\Rightarrow \quad (1 + x^2)y = \int 4x^2\, dx + c$

$\Rightarrow \quad (1 + x^2)y = \frac{4x^3}{3} + c$ **Ans.**

Answer 4.

Let $\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx = I$ ...(i)

Using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Here $a = 0$, $b = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\left[\sin\left(\frac{\pi}{2}-x\right)\right]^3}{\left[\sin\left(\frac{\pi}{2}-x\right)\right]^3 + \left[\cos\left(\frac{\pi}{2}-x\right)\right]^3}\, dx$$

$$= \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\, dx \quad ...(ii)$$

Adding equations (i) and (ii)

$$2I = \int_0^{\pi/2}\left(\frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x}\right)dx = \int_0^{\pi/2} dx$$

$\Rightarrow \quad 2I = [x]_0^{\pi/2} = \left(\frac{\pi}{2} - 0\right)$

$\therefore \quad I = \frac{\pi}{4}$ **Ans.**

Answer 5.

Given : $P(A) = \frac{4}{5}$, $P(B) = \frac{3}{4}$ and $P(C) = \frac{2}{3}$

$\therefore \quad P(\overline{A}) = \frac{1}{5}$, $P(\overline{B}) = \frac{1}{4}$ and $P(\overline{C}) = \frac{1}{3}$

(i) P(Exactly 2 persons hit the target) = $P(A).P(B).P(\overline{C}) + P(A).P(\overline{B}).P(C) + P(\overline{A}).P(B).P(C)$

$$= \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{5}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)$$

$$= \frac{1}{5} + \frac{2}{15} + \frac{1}{10}$$

$$= \frac{6+4+3}{30} = \frac{13}{30}$$

$\therefore$ Required probability = $\frac{13}{30}$ **Ans.**

(ii) Probability at least two persons hit the target

= Probability of 2 persons hit the target

+ Probability all persons hit the target

$$= \frac{13}{30} + P(A).P(B).P(C)$$

$$= \frac{13}{30} + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}$$

$$= \frac{13}{30} + \frac{2}{5}$$

$$= \frac{25}{30} = \frac{5}{6}$$

$\therefore$ Required probability $= \frac{5}{6}$ **Ans.**

OR

Bag A contains 3 red and 4 white balls

Bag B contains 2 red and 3 white balls

(i) Probability (one red ball and two white balls)

= P(one red ball from bag A and 2 white balls from bag B)

+ P(one white ball from bag A and one red and one white ball from bag B)

$$= \frac{3}{7} \times \frac{^3C_2}{^5C_2} + \frac{4}{7} \times \frac{^2C_1 \times {^3C_1}}{^5C_2}$$

$$= \frac{3}{7} \times \frac{3}{10} + \frac{4}{7} \times \frac{2 \times 3}{10}$$

$$= \frac{9}{70} + \frac{24}{70} = \frac{33}{70}$$ **Ans.**

(ii) Probability all 3 balls are of the same colour

= Probability (one red ball from A and 2 red balls from bag B)

+ Probability (one white ball from bag A and 2 white balls from bag B)

$$= \frac{3}{7} \times \frac{^2C_2}{^5C_2} + \frac{4}{7} \times \frac{^3C_2}{^5C_2}$$

$$= \frac{3}{7} \times \frac{1}{10} + \frac{4}{7} \times \frac{3}{10}$$

$$= \frac{3}{70} + \frac{12}{70} = \frac{15}{70}$$

$\therefore$ Required probability $= \frac{3}{14}$ **Ans.**

Answer 6.

Consider, $\int e^x \left[\frac{1+\sin x}{1+\cos x}\right] dx$

$$= \int e^x \frac{(1+\sin x)(1-\cos x)}{(1+\cos x)(1-\cos x)} dx$$

$$= \int \frac{e^x (1-\cos x+\sin x-\sin x\cos x)}{1-\cos^2 x} dx$$

$$= \int \frac{e^x}{\sin^2 x} [1 - \cos x + \sin x - \sin x \cos x]\, dx$$

$$= \int e^x.\text{cosec}^2\, x\, dx - \int e^x.\ \text{cosec}\ x.\ \cot x\ .\ dx + \int e^x.\text{cosec}\ x\ dx - \int e^x\ .\ \cot x.\ dx$$

On integrating by parts

$$= e^x\ .\int \text{cosec}^2\, x\ .\ dx - \int e^x\ .\ (-\cot x)\ .\ dx - e^x\ .\int -\text{cosec}\ x.\ \cot x.\ dx$$
$$- \int e^x.\ \text{cosec}\ x\ .\ dx + \int e^x.\ \text{cosec}\ x\ .\ dx - \int e^x \cot x.\ dx.$$

$$= e^x\ .\int \text{cosec}^2\, x.\ dx + \int e^x \cot x\ dx + e^x\ .\ \text{cosec}\ x - \int e^x\ \text{cosec}\ x\ .\ dx$$
$$+ \int e^x\ \text{cosec}\ x\ .\ dx - \int e^x \cot x\ .\ dx$$

$$= e^x\ .\ (-\cot x) + e^x\ \text{cosec}\ x + c$$

Where c is constant of integration.

$$= e^x [\text{cosec } x - \cot x] + c \qquad \textbf{Ans.}$$

Answer 7.

Let factory at Nasik manufactures x compressors.

$\because$ number of compressors manufactured at Pune factory = $2x$

and number of compressors manufactured at Nagpur factory = x

$$\text{Probability (compressor is manufactured in Nasik)} = \frac{x}{4x}$$

$$\therefore \quad P(E_1) = \frac{1}{4}$$

$$\text{Probability (compressor is manufactured in Pune)} = \frac{2x}{4x}$$

$$P(E_2) = \frac{1}{2}$$

$$\text{Probability (compressor is manufactured in Nagpur)} = \frac{x}{4x}$$

$$P(E_3) = \frac{1}{4}$$

Let A be the event that the compressor is defective

$$\therefore \quad P(A/E_1) = 0.2\% = \frac{0.2}{100}$$

$$P(A/E_2) = 0.2\% = \frac{0.2}{100}$$

$$P(A/E_3) = 0.4\% = \frac{0.4}{100}$$

Required probability,

$$\text{P(defective compressor is from Nasik)} = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2) + P(E_3).P(A/E_3)}$$

$$= \frac{\frac{1}{4} \times \frac{0.2}{100}}{\frac{1}{4} \times \frac{0.2}{100} + \frac{1}{2} \times \frac{0.2}{100} + \frac{1}{4} \times \frac{0.4}{100}}$$

$$= \frac{0.2}{0.2 + 0.4 + 0.4}$$

$$= \frac{0.2}{1} = \frac{2}{10} = \frac{1}{5}$$

$$\therefore \quad \text{Required probability} = \frac{1}{5} \text{ or } 20\%. \qquad \textbf{Ans.}$$

Answer 8.

Let
$$I = \int_{-\pi/4}^{\pi/4} \log (\sin x + \cos x)\, dx$$

$$\Rightarrow \quad I = \int_{-\pi/4}^{\pi/4} \log \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right) dx$$

$$\Rightarrow \quad I = \int_{-\pi/4}^{\pi/4} \log \sqrt{2} \left(\sin x \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos x \right) dx$$

$$\Rightarrow \quad I = \int_{-\pi/4}^{\pi/4} \log \sqrt{2} \sin \left(x + \frac{\pi}{4} \right) dx$$

$$\Rightarrow \quad I = \int_{-\pi/4}^{\pi/4} \log \sqrt{2}\, dx + \int_{-\pi/4}^{\pi/4} \log \sin \left(x + \frac{\pi}{4} \right) dx$$

$$\Rightarrow \quad I = \left[x \log \sqrt{2} \right]_{-\pi/4}^{\pi/4} + \int_{-\pi/4}^{\pi/4} \log \sin \left(x + \frac{\pi}{4} \right) dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \log \sqrt{2} + \int_{-\pi/4}^{\pi/4} \log \sin \left(x + \frac{\pi}{4} \right) dx$$

Let $x + \frac{\pi}{4} = t$

$dx = dt$

When $x = -\frac{\pi}{4}$ then $t = 0$

When $x = \frac{\pi}{4}$ then $t = \frac{\pi}{2}$

$\therefore \quad I = \frac{\pi}{4} \log 2 + \int_0^{\pi/4} \log \sin t \, dt$

$\Rightarrow \quad I = \frac{\pi}{4} \log 2 - \frac{\pi}{2} \log 2$

$\therefore \quad I = -\frac{\pi}{4} \log 2.$ **Ans.**

OR

Let $I = \int_0^{\pi} \frac{x}{1 - \cos \alpha \sin x} dx$

$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{1 - \cos \alpha \sin (\pi - x)} dx \quad \left[\because \int_0^a f(x)dx = \int f(a-x)dx\right]$

$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{1 - \cos \alpha \sin x} dx$

$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi}{1 - \cos \alpha \sin x} dx - \int_0^{\pi} \frac{x}{1 - \cos \alpha \sin x} dx$

$\Rightarrow \quad I = \pi \int_0^{\pi} \frac{dx}{1 - \cos \alpha \sin x} - I$

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{dx}{1 - \cos \alpha \sin x} = \pi \int_0^{\pi} \frac{dx}{1 - \cos \alpha \frac{2 \tan x/2}{1 + \tan^2 x/2}}$

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\left(1 + \tan^2 \frac{x}{2}\right)}{1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} \cos \alpha} dx$

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 2 \cos \alpha \tan \frac{x}{2}} dx$

Let $\tan \frac{x}{2} = t$

$\Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt$

When $x = 0$, $t = 0$

When $x = \pi$, $t = \infty$

$\therefore \quad 2I = \pi \int_0^{\infty} \frac{2}{1 + t^2 - 2t \cos \alpha} dt$

$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{dt}{t^2 - 2t \cos \alpha + \cos^2 \alpha + 1 - \cos^2 \alpha}$

$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{dt}{(t - \cos \alpha)^2 + (\sin \alpha)^2}$

$\Rightarrow \quad I = \pi \frac{1}{\sin \alpha} \left[\tan^{-1} \frac{t - \cos \alpha}{\sin \alpha} \right]_0^{\infty}$

$\Rightarrow \quad I = \frac{\pi}{\sin \alpha} \left[\tan^{-1} \infty - \tan^{-1} (-\cot \alpha) \right]$

$\Rightarrow \quad I = \frac{\pi}{\sin \alpha} \left(\frac{\pi}{2} + \tan^{-1} (\cot \alpha) \right)$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \tan^{-1}\tan\left(\frac{\pi}{2} - \alpha\right)\right]$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha}\left(\frac{\pi}{2} + \frac{\pi}{2} - \alpha\right)$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha}(\pi - \alpha).$$

Hence Proved.

Section-B

Answer 9.

(i) (a) $x - 4y + 2z + 4 = 0$

Explanation :

$x - 4y + 2z + 4 = 0$ satisfy the point (2, 3, 3).

(ii) (c) $x + y + z = 5\sqrt{3}$

Explanation :

Normal of the plane is equally inclined of coordinate axes *i.e.*, $\alpha = \beta = \gamma$.

$$\because \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$\therefore \quad \cos\alpha = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$$

or

$$a = b = c = \frac{1}{\sqrt{3}}$$

Equation of the plane passing through origin and perpendicular distance from origin is $5\sqrt{3}$ units.

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = d$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}}(x - 0) + \frac{1}{\sqrt{3}}(y - 0) + \frac{1}{\sqrt{3}}(z - 0) = 5\sqrt{3}$$

$$\Rightarrow \quad \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 5\sqrt{3}$$

$$\therefore \quad x + y + z = 15$$

Hence equation of the plane is $x + y + z = 15$.

Answer 10.

The general equation of a plane is,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

So, the equation of the plane passing through (2, – 3, 1) is,

$$a(x - 2) + b(y + 3) + c(z - 1) = 0 \qquad \text{...(i)}$$

The direction ratios of the line joining the points (4, 5, 0) and (1, – 2, 4) are (1 – 4, – 2 – 5, 4 – 0) *i.e.*, (– 3, – 7, 4).

Since, the plane is perpendicular to the line whose direction rations are – 3, – 7 and 4.

So, direction ratios of the normal to the plane are – 3, – 7 and 4.

Putting these values in equation (i), we get

$$-3(x - 2) + (-7)(y + 3) + 4(z - 1) = 0$$

$$\Rightarrow \quad -3x + 6 - 7y - 21 + 4z - 4 = 0$$

$$\Rightarrow \quad -3x - 7y + 4z - 19 = 0$$

or $3x + 7y - 4z + 19 = 0$

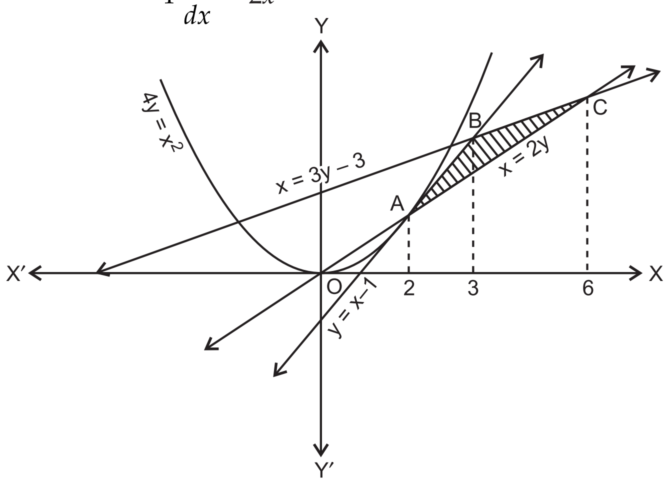
which is the required equation of plane. **Ans.**

Answer 11.

Given curve is, $4y = x^2$

Differentiate w.r.t. x, we get

$$4\frac{dy}{dx} = 2x$$



$\therefore$ At (2, 1), $\frac{dy}{dx} = \frac{2}{x} = \frac{2}{2} = 1$

Equation of the tangent at (2, 1),

$$y - 1 = 1(x - 2)$$

$\Rightarrow$ $y = x - 1$

Intersection point of lines $y = x - 1$ and $x = 2y$ is A(2, 1).

Intersection point of lines $y = x - 1$ and $x = 3y - 3$ is B(3, 2).

Intersection point of lines $x = 2y$ and $x = 3y - 3$ is C(6, 3).

$$\therefore \quad \text{ar}(\Delta ABC) = \int_2^3 (x-1)\,dx + \int_3^6 \left(\frac{x+3}{3}\right) dx - \int_2^6 \frac{x}{2}\,dx$$

$$= \left[\frac{x^2}{2} - x\right]_2^3 + \frac{1}{3}\left[\frac{x^2}{2} + 3x\right]_3^6 - \frac{1}{2}\left[\frac{x^2}{2}\right]_2^6$$

$$= \left[\frac{9}{2} - 3 - 2 + 2\right] + \frac{1}{3}\left[18 + 18 - \frac{9}{2} - 9\right] - \frac{1}{4}[36 - 4]$$

$$= \frac{3}{2} + \frac{15}{2} - \frac{32}{4}$$

$$= 9 - 8 = 1$$

$\therefore$ Required area = 1 unit^2. **Ans.**

Section-C

Answer 12.

(i) (b) $-\frac{1}{2}$

Explanation :

Given : $4x + 2y - 3 = 0$

$\Rightarrow$ $4x = -2y + 3$

$\Rightarrow \quad x = -\frac{1}{2}y + \frac{3}{4}$

$\therefore \quad b_{xy} = -\frac{1}{2}$

Also, $\quad 3x + 6y + 5 = 0$

$6y = -3x - 5$

$y = \frac{-1}{2}x - \frac{5}{6}$

$\therefore \quad b_{yx} = -\frac{1}{2}$

$\because \quad r = \pm\sqrt{b_{xy} \times b_{yx}} = \pm\sqrt{\frac{-1}{2} \times \frac{-1}{2}} = \pm\frac{1}{2}$

$\because \quad b_{xy} < 0, b_{yx} < 0$

$\therefore \quad r < 0$

$\therefore \quad r = -\frac{1}{2}$

(ii) (a) $\frac{1}{4}, \frac{2}{3}$

Explanation :

Given : $\quad x - 4y = 7$

$\Rightarrow \quad 4y = x - 7$

$\Rightarrow \quad y = \frac{1}{4}x - \frac{7}{4}$

$\therefore \quad b_{yx} = \frac{1}{4}$

Also, $\quad 3x - 2y = 5$

$\Rightarrow \quad 3x = 2y + 5$

$\Rightarrow \quad x = \frac{2}{3}y + \frac{5}{3}$

$\therefore \quad b_{xy} = \frac{2}{3}$

$\therefore \quad r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{\frac{2}{3} \times \frac{1}{4}} = \sqrt{\frac{1}{6}}$

$\because \quad r < 1$

Hence, $\quad b_{yx} = \frac{1}{4}$

and $\quad b_{xy} = \frac{2}{3}$

Answer 13.

Regression equation of x on y

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

$$\Rightarrow \quad x - 25.6 = \frac{216}{97}(y - 17.2)$$

$\Rightarrow \quad x = \frac{216}{97} y - \frac{17.2 \times 216}{97} + 25.6$

$\Rightarrow \quad x = 2.227y - 38.30 + 25.6$

$\Rightarrow \quad x = 2.227y - 12.70$

When $y = 19$

$\Rightarrow \quad x = 2.227 \times 19 - 12.70$

$\Rightarrow \quad x = 42.31 - 12.70$

$\Rightarrow \quad x = 29.61$ **Ans.**

Answer 14.

Let he purchase x units of fan and y units of sewing machines.

Object : To maximise profit

$$Z = 22x + 18y$$

Constraints : $360x + 240y \leq 5760$

or $3x + 2y \leq 48$

$x + y \leq 20$

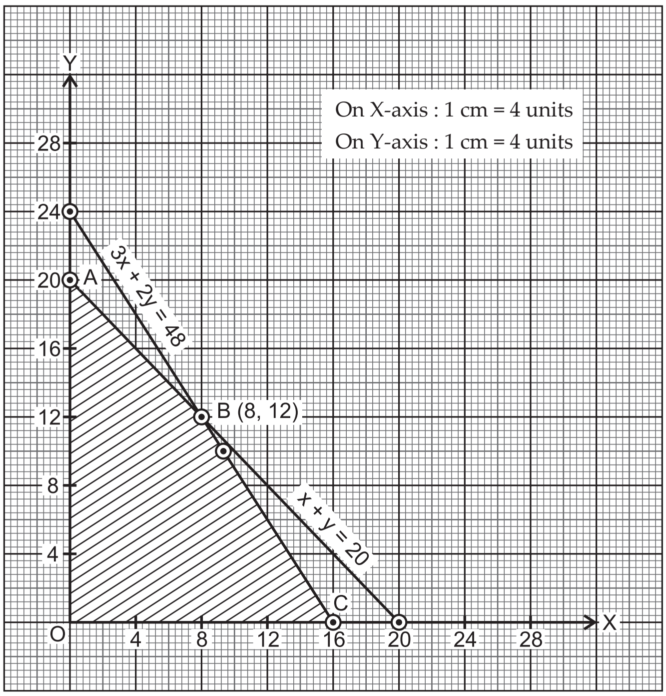
$x \geq 0, y \geq 0$

$3x + 2y = 48$

x	0	16	8
y	24	0	12

$x + y = 20$

x	20	0	10
y	0	20	10



The shaded region is the required feasible region.

$$Z = 22x + 18y$$

At A(0, 20) $\quad Z = 22 \times 0 + 18 \times 20 = ₹\ 360$

At B(8, 12) $\quad Z = 22 \times 8 + 18 \times 12 = ₹\ 392$

At C(16, 0) $\quad Z = 22 \times 16 + 18 \times 0 = ₹\ 352$

Hence, profit is maximum when he purchased 8 units of fan and 12 units of sewing machine. **Ans.**

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