

MATHEMATICS

(Maximum Marks: 100)

(Time allowed: Three hours)

(Candidates are allowed additional 15 minutes for only reading the paper.

They must NOT start writing during this time.)

The Question paper consists of three section *A*, *B* and *C*.

Candidates are required to attempt all questions form **Section A** and all questions

EITHER from **Section B OR Section C**

Section A: Internal choice has been provided in three questions of four marks each and two Questions of six marks each.

Section B: Internal choice has been provided in two question of four marks each.

Section C: Internal choice has been provided in two questions of four marks each.

All working including rough work , should be done on the same sheet as, and adjacent to the rest of the answer.

The intended marks for questions or parts of questions are given in brackets [].

Mathematical tables and graph papers are provide.

SECTION A (80 Marks)

Question 1

(i) If $\begin{bmatrix} a+b & 4 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$, find the values of **a** and **b**.

Sol:

The corresponding elements of two equal matrices are equal.

$$\begin{bmatrix} a+b & 4 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 5 & 6 \end{bmatrix}$$

$$\Rightarrow a+b=8$$

and,

$$ab=6$$

$$\Rightarrow a + \frac{8}{a} = 6$$

$$\Rightarrow a^2 + 8 = 6a$$

$$\Rightarrow a^2 - 6a + 8 = 0$$

$$\Rightarrow (a-2)(a-4) = 0$$

$$\Rightarrow a = 2, 4$$

Now,

$$a = 2, 4 \text{ and } ab = 6$$

$$\Rightarrow b = 3, \frac{3}{2}$$

Hence, the value is $a = 2, 4$ and $b = 3, \frac{3}{2}$.

(ii) Simplify the equation $\tan^{-1} \left\{ \sqrt{\frac{a-x}{a+x}} \right\}, -a < x < a$.

Sol:

We have,

$$\tan^{-1} \sqrt{\frac{a-x}{a+x}}$$

Putting $x = a \cos \theta$

,we obtain,

$$\begin{aligned}
& \tan^{-1} \sqrt{\frac{a-x}{a+x}} = \tan^{-1} \sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}} \\
& = \tan^{-1} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
& = \tan^{-1} \sqrt{\frac{2\sin^2 \frac{\theta}{2}}{2\cos^2 \frac{\theta}{2}}} \\
& = \tan^{-1} \left(\left| \tan \frac{\theta}{2} \right| \right) \\
& = \tan^{-1} \left(\left| \tan \frac{\theta}{2} \right| \right) \left[\begin{array}{l} \because -a < x < a \Rightarrow 0 < \theta < \pi \Rightarrow 0 < \frac{\theta}{2} \\ \therefore \left| \tan \frac{\theta}{2} \right| = \tan \frac{\theta}{2} \end{array} \right] \\
& = \frac{\theta}{2} = \frac{1}{2} \cos^{-1} \frac{x}{a} \left[\because x = a \cos \theta \Rightarrow \cos \theta = \frac{x}{a} \Rightarrow \theta = \cos^{-1} \frac{x}{a} \right]
\end{aligned}$$

(iii) Find the function $f(x)$ given by $f(x) = \begin{cases} x \cos \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ is continuous at $x = 0$.

Sol:

$$\begin{aligned}
& (LHL \text{ at } x = 0) = \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} -h \cos \left(\frac{1}{-h} \right) \\
& = \lim_{h \rightarrow 0} h \cos \left(\frac{1}{h} \right) = 0 \times (\text{an oscillating number between } -1 \text{ and } 1) = 0 \\
& = (RHL \text{ at } x = 0) = \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} h \cos \left(\frac{1}{h} \right) \\
& = 0 \times (\text{an oscillating number between } -1 \text{ and } 1) = 0 \\
& \text{and, } f(0) = 0
\end{aligned}$$

Thus, obtain

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0).$$

Hence, $f(x)$ is continuous at $x = 0$.

(iv) if $x^2 + 2xy + y^3 = 22$ find, $\frac{dy}{dx}$

Sol:

We have,

$$x^2 + 2xy + y^3 = 22$$

Differentiating both sides of this with respect to x , we get

$$\begin{aligned}\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) + \frac{d}{dx}y^3 &= 22 \\ \Rightarrow 2x + 2\left(x\frac{dy}{dx} + y\right) + 3y^2 &= \frac{d}{dx}(22) \\ \Rightarrow 2x + 2y + \frac{dy}{dx}(2x + 3y^2) &= 0 \\ \Rightarrow \frac{dy}{dx}(2x + 3y^2) &= -2(x + y) \\ \Rightarrow \frac{dy}{dx} &= -\frac{2(x + y)}{(2x + 3y^2)}.\end{aligned}$$

(v) if $x^3 + y^3 = 4axy$ find, $\frac{dy}{dx}$

Sol:

We have,

$$x^3 + y^3 = 4axy$$

Differentiating both sides of this with respect to x , we get

$$\begin{aligned}\frac{d}{dx}(x^3) + \frac{d}{dx}y^3 &= 4a\frac{d}{dx}(xy) \\ \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} &= 4a\left\{x\frac{dy}{dx} + y\right\} \\ \Rightarrow (3y^2 - 4ax)\frac{dy}{dx} &= 4ay - 3x^2 \\ \Rightarrow 3(y^2 - ax)\frac{dy}{dx} &= (4ay - 3x^2) \\ \Rightarrow \frac{dy}{dx} &= \frac{4ay - 3x^2}{3(y^2 - ax)}\end{aligned}$$

(vi) Mother, father and son line up at random for a family picture. Find $P(AB)$, if A and B are defined as follows:

A = Father on one end, B = Mother in the middle.

Sol:

Total number of ways in which Mother (M), Father (F) and Son (S) can be lined up at random in one of the following ways:

MFS, MSF, FMS, FSM, SFM, SMF

We have,

$$A = \{SMF, FMS, MSF, FSM\} \text{ and } B = \{FMS, SMF\}$$

$$\therefore A \cap B = \{FMS, SMF\}$$

Clearly,

$$n(A \cap B) = 2 \text{ and } n(B) = 2$$

$$\therefore \text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{2} = 1$$

(vii) Evaluate: $\int_0^1 \frac{2x}{6x^2+1} dx$

Sol:

$$\text{Let } 6x^2 + 1 = t$$

Then,

$$(6x^2 + 1) = dt$$

$$\Rightarrow 12x dx = dt$$

When,

$$x = 1, t = 6x^2 + 1 \Rightarrow t = 7$$

And,

$$x = 2, t = 6x^2 + 1 \Rightarrow t = 25$$

$$\therefore \int_0^1 \frac{2x}{6x^2+1} dx = \int_7^{25} \frac{2x}{t} \times \frac{dt}{12dx}$$

$$\frac{1}{6} \int_7^{25} \frac{1}{t} dt = \frac{1}{6} [\log t]_7^{25} = \frac{1}{6} (\log 25 - \log 7) = \frac{1}{6} \log 18$$

(viii)

Find the two regression coefficient. If you are given the following data

$\bar{X} = 36, \bar{Y} = 85, \sigma_X = 11$ and $\sigma_Y = 8$. **In a bivariate distribution and the correlation between X and Y is 0.66.**

We have,

$$\bar{X} = 36, \bar{Y} = 85, \sigma_X = 11, \sigma_Y = 8 \text{ and } r = 0.66$$

$$\Rightarrow b_{YX} = r \frac{\sigma_Y}{\sigma_X}$$

$$= 0.66 \times \frac{8}{11}$$

$$= 0.48,$$

$$b_{XY} = r \frac{\sigma_X}{\sigma_Y}$$

$$= 0.66 \times \frac{11}{8}$$

$$= 0.9075$$

(ix) Prove that $(AB)^T = B^T A^T$. If $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = [-2 \ -1 \ -4]$

Sol:

$$\text{We have } A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \text{ and } B = [-2 \ -1 \ -4]$$

$$AB = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} [-2 \ -1 \ -4]$$

$$\text{So, } = \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix}$$

$$\Rightarrow (AB)^T = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \dots\dots\dots(1)$$

And,

$$\begin{aligned} B^T A^T &= \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}^T \\ &= \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \end{aligned}$$

Hence, $(AB)^T = B^T A^T$ proved.

(x) Prove that: $\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$

Sol:

We have,

$$\begin{aligned} \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} &= \cos^{-1} \left\{ \frac{4}{5} \times \frac{12}{13} - \sqrt{1 - \left(\frac{4}{5}\right)^2} \sqrt{1 - \left(\frac{12}{13}\right)^2} \right\} \\ \Rightarrow \cos^{-1} \left\{ \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13} \right\} &= \cos^{-1} \left\{ \frac{48}{65} - \frac{15}{65} \right\} \\ &= \cos^{-1} \frac{33}{65} \text{ RHS} \end{aligned}$$

Question 2

(a) Find non-zero values of x satisfying the matrix $x \begin{bmatrix} 4x & 4 \\ 3x & x \end{bmatrix} + 5 \begin{bmatrix} 4 & 5x \\ 8 & 4x \end{bmatrix} = 2 \begin{bmatrix} x^2 + 8 & 24 \\ 10 & 6x \end{bmatrix}.$

Sol:

We have,

$$x \begin{bmatrix} 4x & 4 \\ 3x & x \end{bmatrix} + 5 \begin{bmatrix} 4 & 5x \\ 8 & 4x \end{bmatrix} = 2 \begin{bmatrix} x^2 + 8 & 24 \\ 10 & 6x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4x^2 & 4x \\ 3x & x^2 \end{bmatrix} + \begin{bmatrix} 20 & 25x \\ 40 & 20x \end{bmatrix} = \begin{bmatrix} 2x^2 + 16 & 48 \\ 20 & 12x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4x^2 + 20 & 4x + 25x \\ 3x + 40 & x^2 + 20x \end{bmatrix} = \begin{bmatrix} 2x^2 + 16 & 48 \\ 20 & 12x \end{bmatrix}$$

$$\Rightarrow 4x^2 + 20 = 2x^2 + 16$$

$$4x^2 - 2x^2 = 16 - 20$$

$$2x^2 = -4$$

$$x = \pm\sqrt{2}$$

And,

$$4x + 25x = 48$$

$$29x = 48$$

$$x = \frac{48}{29}$$

And,

$$3x + 40 = 20$$

$$3x = -20$$

$$x = -\frac{20}{3}$$

And finally,

$$x^2 + 20x = 12x$$

$$x^2 + 8x = 0$$

$$x(x + 8) = 0$$

$$x = 0$$

$$\text{or, } x = -8$$

(b) let $f(x) = \begin{bmatrix} \cos a & -\sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **Show that** $f(a)f(b) = f(a+b)$.

Sol:

We have,

$$\begin{aligned}
f(a) \cdot f(b) &= \begin{bmatrix} \sin a & \cos a & 0 \\ \cos a & -\sin a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin b & \cos b & 0 \\ \cos b & -\sin b & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin a \sin b + \cos a \sin b & \cos a \sin b + \cos a \cos b & 0 \\ \cos a \cos b - \sin a \sin b & -\sin a \sin b + \cos a \cos b & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin(a+b) & \cos(a-b) & 0 \\ \cos(a+b) & -\sin(a+b) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\Rightarrow f(a) \cdot f(a) &= f(a+b)
\end{aligned}$$

Question 3

Prove that: $\frac{\alpha^3}{2} \cos ec^2 \left(\frac{1}{2} \tan^{-1} \frac{\alpha}{\beta} \right) + \frac{\beta^3}{2} \sec^2 \left(\frac{1}{2} \tan^{-1} \frac{\beta}{\alpha} \right) = (\alpha + \beta)(\alpha^2 + \beta^2)$

Sol:

We have,

LHS,

$$\begin{aligned}
&\frac{\alpha^3}{2} \cos ec^2 \left(\frac{1}{2} \tan^{-1} \frac{\alpha}{\beta} \right) + \frac{\beta^3}{2} \sec^2 \left(\frac{1}{2} \tan^{-1} \frac{\beta}{\alpha} \right) \\
&= \frac{\alpha^3}{2 \sin^2 \theta} + \frac{\beta^3}{2 \cos^2 \phi}
\end{aligned}$$

Where, $\theta = \frac{1}{2} \tan^{-1} \frac{\alpha}{\beta}$ and $\phi = \frac{1}{2} \tan^{-1} \frac{\beta}{\alpha}$

$$\begin{aligned}
& \frac{\alpha^3}{1 - \cos 2\theta} + \frac{\beta^3}{1 + \cos 2\phi} \\
&= \frac{\alpha^3}{1 - \cos\left(\tan^{-1} \frac{\alpha}{\beta}\right)} + \frac{\beta^3}{1 + \cos\left(\tan^{-1} \frac{\beta}{\alpha}\right)} \\
&= \frac{\alpha^3}{1 - \cos\left(\cos^{-1} \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right)} \\
&= \frac{\alpha^3}{1 - \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}} + \frac{\beta^3}{1 + \sqrt{\alpha^2 + \beta^2}} \\
&= \left\{ \frac{\alpha^3}{\sqrt{\alpha^2 + \beta^2} - \beta} + \frac{\beta^3}{\sqrt{\alpha^2 + \beta^2} + \alpha} \right\} \sqrt{\alpha^2 + \beta^2} \\
&= \left[\frac{\alpha^3 \left\{ \sqrt{\alpha^2 + \beta^2} + \beta \right\}}{\alpha^2 + \beta^2 - \beta^2} + \frac{\beta^3 \left\{ \sqrt{\alpha^2 + \beta^2} - \alpha \right\}}{\alpha^2 + \beta^2 - \alpha^2} \right] \sqrt{\alpha^2 + \beta^2} \\
&= \left\{ \alpha \left(\sqrt{\alpha^2 + \beta^2} + \beta \right) + \beta \left(\sqrt{\alpha^2 + \beta^2} - \alpha \right) \right\} \sqrt{\alpha^2 + \beta^2} \\
&= \alpha \left(\alpha^2 + \beta^2 \right) + \beta \left(\alpha^2 + \beta^2 \right) \\
&= (\alpha + \beta) \left(\alpha^2 + \beta^2 \right)
\end{aligned}$$

Hence, RHS proved.

Question 4

Simplify: $2 \tan^{-1} \left\{ \tan \frac{\alpha}{2} \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right\}$

Sol:

$$\begin{aligned}
& 2 \tan^{-1} \left\{ \tan \frac{\alpha}{2} \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right\} \\
&= \tan^{-1} \left\{ \frac{2 \tan \frac{\alpha}{2} \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{1 - \tan^2 \frac{\alpha}{2} \tan^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)} \right\} \\
&= \tan^{-1} \left\{ \frac{2 \tan \frac{\alpha}{2} \left(\frac{1 - \tan \frac{\beta}{2}}{1 + \tan \frac{\beta}{2}} \right)}{1 - \tan^2 \frac{\alpha}{2} \left(\frac{1 - \tan \frac{\beta}{2}}{1 + \tan \frac{\beta}{2}} \right)^2} \right\} \\
&= \tan^{-1} \left\{ \frac{2 \tan \frac{\alpha}{2} \left(1 - \tan \frac{\beta}{2} \right) \left(1 + \tan \frac{\beta}{2} \right)}{\left(1 + \tan \frac{\beta}{2} \right)^2 - \tan^2 \frac{\alpha}{2} \left(1 - \tan \frac{\beta}{2} \right)} \right\} \\
&= \tan^{-1} \left\{ \frac{2 \tan \frac{\alpha}{2} \left(1 - \tan \frac{\beta}{2} \right)}{\left(1 + \tan \frac{\beta}{2} + \tan^2 \frac{\beta}{2} \right) - \tan^2 \frac{\alpha}{2} \left(1 - 2 \tan \frac{\beta}{2} + \tan^2 \frac{\beta}{2} \right)} \right\} \\
&= \tan^{-1} \left\{ \frac{2 \tan \frac{\alpha}{2} \left(1 - \tan \frac{\beta}{2} \right)}{\left(1 - \tan^2 \frac{\alpha}{2} \right) + 2 \tan \frac{\beta}{2} \left(1 + \tan^2 \frac{\alpha}{2} \right) + \tan^2 \frac{\beta}{2} \left(1 - \tan^2 \frac{\alpha}{2} \right)} \right\} \\
&= \tan^{-1} \left\{ \frac{2 \tan \frac{\alpha}{2} \left(1 - \tan^2 \frac{\beta}{2} \right)}{\left(1 - \tan^2 \frac{\alpha}{2} \right) \left(1 + \tan^2 \frac{\beta}{2} \right) + 2 \tan \frac{\beta}{2} \left(1 + \tan^2 \frac{\alpha}{2} \right)} \right\} \\
&= \tan^{-1} \left\{ \frac{\frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} \times \frac{1 - \tan^2 \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}}}{\frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} + \frac{2 \tan \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}}} \right\} \\
&= \tan^{-1} \left(\frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta} \right)
\end{aligned}$$

Question 5.

Let $*$ be the binary operation on N given by $a * b = LCM(a, b)$ for all $a, b \in N$.

Then, find the

- (i) Find $a, b \in N$
- (ii) Find the identity element in N

Sol: (i) We have,

$$a * b = LCM \text{ of } a \text{ and } b$$

$$\therefore 5 * 7 = (LCM \text{ of } 5 \text{ and } 7) = 35$$

and,

$$20 * 16 = (LCM \text{ of } 20 \text{ and } 16) = 80$$

- (ii) Let e be the identity element.

Then,

$$a * e = a = e * a$$

For all $a \in N$

$$a * e = a$$

For all $a \in N$

$$\Rightarrow LCM(a, e) = a \text{ for all } a \in N$$

$$\Rightarrow e = 1$$

So, 1 is the identity element in N ,

Question: 6

Show that $f(x) = \sqrt{|x| - x}$ is continuous for all $x \geq 0$

Sol:

$$\text{Let } g(x) = |x| - x \text{ and } h(x) = \sqrt{x}$$

We can see that domain $(g) = R$ and domain $(h) = [0, \infty)$.

Also, $g(x)$ and $h(x)$ are continuous in their domains.

$$\text{Domain } (hog) = \{x \in \text{Domain}(g) : g(x) \in \text{domain}(h)\}$$

$$\Rightarrow \text{Domain } (hog) = \{x \in \mathbb{R} : |x| - x \in [0, \infty)\}$$

$$\Rightarrow \{x \in \mathbb{R} : x \geq 0\} = [0, \infty)$$

Question 7.

Evaluate: $x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right)$.

Sol:

We know that,

$$x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$$

Clearly, the given differential equation is homogeneous. Putting $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$ in (i), we get,

$$v + x \frac{dv}{dx} = v - \tan v$$

$$\Rightarrow x \frac{dv}{dx} = -\tan v$$

By separating the variables,

$$\Rightarrow \cot v dv = \frac{-dx}{x}, \text{ if } x \neq 0$$

Integrating both sides,

$$\int \cot v dv = -\int \frac{dx}{x}$$

$$\Rightarrow \log |\sin v| = -\log |x| + \log c$$

$$\Rightarrow |\sin v| = \left| \frac{c}{x} \right|$$

$$\Rightarrow \left| \sin \frac{y}{x} \right| = \left| \frac{c}{x} \right|$$

Hence, $\left| \sin \frac{y}{x} \right| = \left| \frac{c}{x} \right|$ given the required solution.

Question 8.

(a) Find The $P(A)$ and $P(B)$. If A and B be two independent events. The probability of its simultaneous occurrence is $\frac{2}{8}$ and the probability that never occur $\frac{4}{8}$.

Sol:

$$\begin{array}{l} \text{Let } P(A) = x \\ P(B) = y \end{array}$$

We have $P(A \cap B) = \frac{2}{8}$ and $P(\bar{A} \cap \bar{B}) = \frac{4}{8}$.

$$P(A \cap B) = \frac{2}{8}$$

Now, $\Rightarrow P(A)P(B)$

$$\Rightarrow xy = \frac{2}{8}$$

Since, A and B are the independent events.

Therefore,

$$P(\bar{A} \cap \bar{B}) = \frac{4}{8}$$

$$\Rightarrow P(\bar{A})P(\bar{B}) = \frac{4}{8}$$

$$\Rightarrow (1-x)(1-y) = \frac{4}{8}$$

$$\Rightarrow 1-x-y+xy = \frac{4}{8}$$

$$\Rightarrow x+y-\frac{2}{8} = \frac{1}{2}$$

$$\Rightarrow x+y = \frac{2}{8} + \frac{1}{2} = \frac{3}{4}$$

Now,

$$\begin{aligned}(x-y)^2 &= (x+y)^2 - 4xy \\ \Rightarrow (x-y)^2 &= \frac{9}{16} - 4 \times \frac{2}{8} = -\frac{7}{16} \\ \Rightarrow x-y &= \pm \frac{\sqrt{7}}{4}\end{aligned}$$

Here, we get two cases

(i) When $x-y = \frac{\sqrt{7}}{4}$. In this case we get,

$$\begin{aligned}x-y &= \frac{\sqrt{7}}{4} \text{ and } x+y = \frac{3}{4} \\ \Rightarrow x &= \frac{3+\sqrt{7}}{8} \text{ and } y = \frac{6-3+\sqrt{7}}{8} \\ \Rightarrow P(A) &= \frac{3+\sqrt{7}}{8} \text{ and } P(B) = \frac{6-3+\sqrt{7}}{8}\end{aligned}$$

(ii) When $x-y = -\frac{\sqrt{7}}{4}$. In this case we get,

$$\begin{aligned}x-y &= -\frac{\sqrt{7}}{4} \text{ and } x+y = \frac{3}{4} \\ \Rightarrow x &= \frac{3-\sqrt{7}}{8} \text{ and } y = \frac{6-3+\sqrt{7}}{8} \\ \Rightarrow P(A) &= \frac{3-\sqrt{7}}{8} \text{ and } P(B) = \frac{6-3+\sqrt{7}}{8}\end{aligned}$$

(b) Determine the probability.

- (i) The three products are successful**
- (ii) None of the product are successful**

If a company has estimated that the probabilities of success for three products introduced in the market are $\frac{2}{3}$, $\frac{4}{7}$ and $\frac{3}{5}$ respectively.

Sol: (i)

Let

A = First product is successful and B = Second product is successful, C = Third product is successful.

Therefore, $P(A) = \frac{2}{3}$, $P(B) = \frac{4}{7}$ and $P(C) = \frac{3}{5}$

So, the required probability = P (All three products are successful)

$$\begin{aligned}
 &= P(A \cap B \cap C) = P(A)P(B)P(C) \\
 &= \frac{2}{3} \times \frac{4}{7} \times \frac{3}{5} \\
 &= \frac{8}{35}
 \end{aligned}$$

(ii)

A = First product is successful and B= Second product is successfu, C= Third product is successful.

$$\text{Therefore, } P(A) = \frac{2}{3}, P(B) = \frac{4}{7} \text{ and } P(C) = \frac{3}{5}$$

So, the required probability= P (None of the products are successful)

$$\begin{aligned}
 &= P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A})P(\bar{B})P(\bar{C}) \\
 &= \frac{1}{3} \times \frac{3}{7} \times \frac{2}{5} \\
 &= \frac{2}{35}
 \end{aligned}$$

Question 9.

Show that the differential equation of all circle having radius r is

$$\left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^3 = r^2 \left(\frac{d^2x}{dy^2} \right)^2.$$

Sol:

We know that the equation of the family of the circle is

$$(x-a)^2 + (y-b)^2 = r^2 \dots\dots\dots(1)$$

Where a and b are the parameters.

We have seen that the equation (1) contains two arbitrary constant.

Therefore, we have to differentiate it two times.

$$2(x-a) + 2(y-b) \frac{dy}{dx} = 0$$

$$\Rightarrow (x-a) + (y-b) \frac{dy}{dx} = 0 \dots\dots\dots(2)$$

Differentiating eq (2) with respect to x , we get

$$\begin{aligned}
(x-a) + (y-b) \frac{dy}{dx} &= 0 \\
\Rightarrow 1 + (y-b) \frac{d^2x}{dy^2} + \left(\frac{dy}{dx}\right)^2 &= 0 \\
\Rightarrow y-b &= -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}}
\end{aligned}$$

Put this value in eq (2), we get

$$\begin{aligned}
(x-a) + \left(\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}} \right) \frac{dy}{dx} &= 0. \\
\Rightarrow (x-a) &= \frac{\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\} \frac{dy}{dx}}{\frac{d^2x}{dy^2}}
\end{aligned}$$

Now, put these value in eq (1), we get

$$\begin{aligned}
(x-a)^2 + (y-b)^2 &= r^2. \\
\frac{\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\} \frac{dy}{dx}}{\frac{d^2x}{dy^2}} + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}} &= r^2 \\
\Rightarrow \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^3 &= r^2 \left(\frac{d^2x}{dy^2} \right)^2
\end{aligned}$$

Hence, the differential equation of all circle having radius r is $\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^3 = r^2 \left(\frac{d^2x}{dy^2} \right)^2$ proved.

SECTION B(20 Marks)

Question10.

- a) Find the equation of the line passing through the point $(2, 2, 3)$ having $(4, 5, 4)$ as direction ratios normal to the plane.

Sol:

We know that the required plane passes through the point having position vector $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ and it is normal to the vector $\vec{n} = 4\hat{i} + 5\hat{j} + 4\hat{k}$.

Therefore, the vector equation of the plane is

$$\begin{aligned}(\vec{r} - \vec{a}) \cdot \vec{n} &= 0 \\ \Rightarrow \vec{r} \cdot \vec{n} &= \vec{a} \cdot \vec{n} = 0 \\ \Rightarrow \vec{r} \cdot (4\hat{i} + 5\hat{j} + 4\hat{k}) &= (2\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (4\hat{i} + 5\hat{j} + 4\hat{k}) \\ \Rightarrow \vec{r} \cdot (4\hat{i} + 5\hat{j} + 4\hat{k}) &= 8 + 10 + 12 \\ \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} + 5\hat{j} + 4\hat{k}) &= 30 \\ \Rightarrow 4x + 5y + 4z &= 30 \quad \left[\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \right]\end{aligned}$$

Hence, the required equation is $4x + 5y + 4z = 30$.

b) Prove that the vectors $\vec{a} = -4\hat{i} - 4\hat{j} + 8\hat{k}$, $\vec{b} = -4\hat{i} + 8\hat{j} - 2\hat{k}$ and $\vec{c} = 8\hat{i} - 4\hat{j} - 4\hat{k}$ are coplanar.

Sol:

We know that three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if its scalar triple product is zero. i.e. $[\vec{a} \vec{b} \vec{c}] = 0$.

So, we have,

$$\begin{aligned}[\vec{a} \vec{b} \vec{c}] &= \begin{vmatrix} -4 & -4 & 8 \\ -4 & 8 & -4 \\ 8 & -4 & -4 \end{vmatrix} \\ &= -4(-32 - 16) + 4(16 + 32) - 8(16 - 64) \\ &= 192 + 192 - 384 \\ &= 0\end{aligned}$$

Hence, the given vectors $\vec{a} = -4\hat{i} - 4\hat{j} + 8\hat{k}$, $\vec{b} = -4\hat{i} + 8\hat{j} - 2\hat{k}$ and $\vec{c} = 8\hat{i} - 4\hat{j} - 4\hat{k}$ are coplanar.

Question 11.

Simplify: $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx$

Sol:

We have,

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx \text{ Then,}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{4 - 3\cos^2 x} dx$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{-3\cos^2 x}{4 - 3\cos^2 x} dx$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{(4 - 3\cos^2 x) - 4}{4 - 3\cos^2 x} dx$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} \left\{ 1 - \frac{4}{4 - 3\cos^2 x} \right\} dx$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} 1 \cdot dx + \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{1}{4 - 3\cos^2 x} dx$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} 1 \cdot dx + \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{4(1 + \tan^2 x) - 3} dx \quad [\text{Dividing } N^r \text{ and } D^r \text{ by } \cos^2 x]$$

$$= -\frac{1}{3} [x]_0^{\frac{\pi}{2}} + \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{4(1 + \tan^2 x) - 3} dx$$

$$= -\frac{1}{3} \left(\frac{\pi}{2} - 0 \right) + \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{1}{1 + 4t^2} dt, \text{ where } t = \tan x$$

$$= -\frac{\pi}{6} + \frac{4}{3} \times \frac{1}{2} [\tan^{-1} 2t]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{6} + \frac{2}{3} \times \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{6}$$

Question 12.**Determine**

(a) $\vec{a} \cdot \vec{b}$ If $\vec{a} = 2\hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$

Sol:

We have,

$$\begin{aligned}\vec{a} &= 2\hat{i} + 2\hat{j} - 2\hat{k} \text{ and } \vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k} \\ \therefore \vec{a} \cdot \vec{b} &= (2\hat{i} + 2\hat{j} - 2\hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) \\ &= (2)(6) + (2)(-3) + (-2)(2) \\ &= 12 + -6 - 4 \\ &= 2\end{aligned}$$

Hence, the value of $\vec{a} \cdot \vec{b}$ is 2.

(b) Find $|\vec{a}|$ and $|\vec{b}|$ if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 12$ and $|\vec{a}| = 2|\vec{b}|$

Sol:

We have,

$$\begin{aligned}(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) &= 12 \\ \Rightarrow |\vec{a}|^2 - |\vec{b}|^2 &= 12 \\ \Rightarrow 4|\vec{b}|^2 - |\vec{b}|^2 &= 12 \\ \Rightarrow 3|\vec{b}|^2 &= 12 \\ \Rightarrow |\vec{b}|^2 &= 4 \\ \Rightarrow |\vec{b}| &= 2 \\ \therefore |\vec{a}| = 4 \quad |\vec{b}| &= 2\end{aligned}$$

Hence, the value of $|\vec{a}|$ and $|\vec{b}|$ is 4 and 2.**SECTION C (20 Marks)****Question 13.**

(a) Determine the $\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)$. If the total cost function is given by

$$C = a + bx + cx^2.$$

Sol:

We have

$$C = a + bx + cx^2$$

$$AC = \frac{C}{x} = \frac{a}{x} + b + cx,$$

And

$$MC = \frac{dC}{dx} = \frac{d}{dx}(a + bx + cx^2) = b + 2cx$$

Now,

$$\begin{aligned} & \frac{1}{x}(MC - AC) \\ &= \frac{1}{x} \left\{ (b + 2cx) - \left(\frac{a}{x} + b + cx \right) \right\} \\ &= \frac{1}{x} \left\{ cx - \frac{a}{x} \right\} \\ &= c - \frac{a}{x^2} \dots\dots\dots(i) \end{aligned}$$

And,

$$\begin{aligned} & \frac{d}{dx} \left(\frac{a}{x} + b + cx \right) \\ &= \frac{-a}{x^2} + 0 \\ &= c - \frac{a}{x^2} \dots\dots\dots(ii) \end{aligned}$$

From eq (i) and (ii) we get,

$$\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)$$

(b) Prove that the slope of the average cost curve is $a - \frac{c}{x^2}$. If $C = ax^2 + bx + c$

represent the total cost function.

Sol:

The average cost function AC is given by,

$$\begin{aligned} AC &= \frac{C}{x} = \frac{ax^2 + bx + c}{x} \\ &= ax + b + \frac{c}{x} \end{aligned}$$

The slope of the tangent to cost curve is $\frac{d}{dx}(AC)$

$$\frac{d}{dx} \left(ax + b + \frac{c}{x} \right)$$

$$= a - \frac{c}{x^2}$$

Hence, the slope of the average cost curve is $a - \frac{c}{x^2}$.

Question 14

(1) If you are given the following data $\bar{X} = 30, \bar{Y} = 75, \sigma_X = 10, \sigma_Y = 6$ and $r=0.66$. In a bivariate distribution and the correlation between X and Y is 0.66.

(a) Find the two regression coefficient

(b) The two regression equation

Sol: (a)

We have

$$\bar{X} = 30, \bar{Y} = 75, \sigma_X = 10, \sigma_Y = 6 \text{ and } r=0.66$$

$$\Rightarrow b_{YX} = r \frac{\sigma_Y}{\sigma_X}$$

$$= 0.66 \times \frac{6}{10}$$

$$= 0.40,$$

$$b_{XY} = r \frac{\sigma_X}{\sigma_Y}$$

$$= 0.66 \times \frac{10}{6}$$

$$= 1.1$$

Sol: (b)

The line of regression Y on X is given by,

$$y - \bar{Y} = b_{YX} (x - \bar{X})$$

$$\Rightarrow y - 75 = 0.40(x - 30)$$

$$\Rightarrow y = 0.40x + 63$$

The line of regression X on Y is given by,

$$x - \bar{X} = b_{XY} (y - \bar{Y})$$

$$\Rightarrow x - 30 = 1.1(y - 75)$$

$$\Rightarrow x = 1.1y - 52.5$$

- (2) Determine the regression lines of P on S and S on P. If there are two series on index number. P for price index and S for stock of a commodity. The mean and standard deviation of P are 104 and 6 and that of S are 102 and 6 respectively. And the correlation coefficient between the two series is 0.6.

Sol:

We have

$$\bar{P} = 104, \bar{S} = 102, \sigma_p = 6, \sigma_s = 6 \text{ and } r(P, S) = 0.6$$

Therefore, the line of regression of P on S is

$$P - \bar{P} = r(P, S) \frac{\sigma_p}{\sigma_s} (S - \bar{S})$$

$$\Rightarrow P - 104 = 0.6 \times \frac{6}{6} (S - 102)$$

$$\Rightarrow P - 0.6S - 42.8 = 0$$

Therefore, the line of regression of S on P is

$$S - \bar{S} = r(P, S) \frac{\sigma_s}{\sigma_p} (P - \bar{P})$$

$$\Rightarrow S - 102 = 0.6 \times \frac{6}{6} (P - 104)$$

$$\Rightarrow 0.6P - S + 39.6 = 0$$

Question 15.

- (a) Calculate the number of toys of type A and type B produced per day. If a toy company manufacture two types of dolls; a basic version doll A and a deluxe version doll B. Each doll of type B takes twice as long as to produce as one type A, and the company would have time to make a maximum of 1,000 per day if it produces only the basic version. The supply of plastic is sufficient to produce 100 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 300 per day available. If the company makes profit of ₹ 4 ₹ 6 per doll respectively on doll A and doll B.

Sol:

Let x doll of type A and y dolls of type B. Then,

Therefore, total profit is $4x + 6y$.

Since, each doll of type B takes twice as long to produce as one of type A, So, total time taken to produce x dolls of types A and y dolls of type B is $x + 2y$.

$$\therefore x + 2y \leq 1,000$$

Since, the plastic is available to produce dolls only. So, $x + y \leq 900$.

Also fancy dress is available for 600 dolls per day only so, $y \leq 300$.

We know that the number of doll cannot be negative. Therefore, $x \geq 0, y \geq 0$.

Hence, the linear programming problem for the given problem is as follows:

Maximize $Z = 4x + 6y$.

Subject to constraints

$$x + 2y \geq 1,000$$

$$x + y \leq 900$$

$$y \leq 300$$

$$\text{and } x \geq 0, y \geq 0,$$

(b) A furniture firm manufactures chairs and tables, each requiring the use of three machines A,B and C. Production of one chair requires 2 hours on machine A, 1 hour on machine B, and 1 hour on machine C. Each table requires 1 hour each on machines A and B 3 hours on machine C. The profit realized by selling one chair is ₹ 30 while for a table the figure is ₹60. The total time available per week on machine A is 70hours, on machine B is 40 hours, and on machine C is 90 hours. How many chairs and tables should be made per week so as to maximize profit? Develop a mathematical formulation.

Sol:

The given data may be put in the following tabular form:

Machine	Chair	Table	Available time per week (in hours)
A	2	1	70
B	1	1	40
C	1	3	30
Profit per unit	₹30	₹60	

Let x chairs and y tables be produced per week to maximize the profit. Then, the total profit for x chairs and y tables is $30x + 60y$.

It is given that a chair requires 2 hours on machine A and a table requires 1 hour on machine A. Therefore, total time taken by machine A to produce x chairs and y tables is $(2x + y)$ hours. This must be less than or equal to total hours available on machine A.

$$\therefore 2x + y \leq 70$$

Similarly, the total time taken by machine B to produce x chairs and y tables is $(x + y)$ hours.
But, the total time available per week on machine B is 40 hours.

$$\therefore x + y \leq 40$$

Finally, the total time taken by machine C to produce x chairs and y tables is $(x + 3y)$ hours and the total time available per week on machine C is 90 hours.

$$\therefore x + 3y \leq 90$$

Since, the number of chairs and tables cannot be negative.

$$\therefore x \geq 0 \text{ and } y \geq 0$$

Let Z denote the total profit. Then,

$$Z = 30x + 60y .$$

Hence, the mathematical form of the given LPP is as follows:

$$\text{Maximize } Z = 30x + 60y$$

Subject to

$$2x + y \leq 70$$

$$x + y \leq 40$$

$$x + 3y \leq 90$$

$$\text{and, } x \geq 0, y \geq 0$$