

Principle of Mathematical Induction

Short Answer Type Questions

Q. 1 Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.

Sol. Let the statement $P(n): 3n < n!$

For $n = 1, 3 \times 1 < 1!$ [false]

For $n = 2, 3 \times 2 < 2! \Rightarrow 6 < 2$ [false]

For $n = 3, 3 \times 3 < 3! \Rightarrow 9 < 6$ [false]

For $n = 4, 3 \times 4 < 4! \Rightarrow 12 < 24$ [true]

For $n = 5, 3 \times 5 < 5! \Rightarrow 15 < 5 \times 4 \times 3 \times 2 \times 1 \Rightarrow 15 < 120$ [true]

Q. 2 Give an example of a statement $P(n)$ which is true for all n . Justify your answer.

Sol. Consider the statement

$$P(n): 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\text{For } n = 1, \quad 1 = \frac{1(1+1)(2 \times 1 + 1)}{6}$$

$$\Rightarrow \quad 1 = \frac{2(3)}{6}$$

$$\Rightarrow \quad 1 = 1$$

$$\text{For } n = 2, \quad 1 + 2^2 = \frac{2(2+1)(4+1)}{6}$$

$$\Rightarrow \quad 5 = \frac{30}{6} \Rightarrow 5 = 5$$

$$\text{For } n = 3, \quad 1 + 2^2 + 3^2 = \frac{3(3+1)(7)}{6}$$

$$\Rightarrow \quad 1 + 4 + 9 = \frac{3 \times 4 \times 7}{6}$$

$$\Rightarrow \quad 14 = 14$$

Hence, the given statement is true for all n .

Prove each of the statements in the following questions from by the Principle of Mathematical Induction.

Q. 3 $4^n - 1$ is divisible by 3, for each natural number n .

Thinking Process

In step I put $n=1$, the obtained result should be a divisible by 3. In step II put $n=k$ and take $P(k)$ equal to multiple of 3 with non-zero constant say q . In step III put $n=k+1$, in the statement and solve till it becomes a multiple of 3.

Sol. Let $P(n) : 4^n - 1$ is divisible by 3 for each natural number n .
Step I Now, we observe that $P(1)$ is true.

$$P(1) = 4^1 - 1 = 3$$

It is clear that 3 is divisible by 3.
Hence, $P(1)$ is true.

Step II Assume that, $P(n)$ is true for $n = k$
 $P(k) : 4^k - 1$ is divisible by 3

$$4^k - 1 = 3q$$

Step III Now, to prove that $P(k+1)$ is true.

$$\begin{aligned} P(k+1) : 4^{k+1} - 1 &= 4^k \cdot 4 - 1 \\ &= 4^k \cdot 3 + 4^k - 1 \\ &= 3 \cdot 4^k + 3q & [\because 4^k - 1 = 3q] \\ &= 3(4^k + q) \end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.
Hence, by the principle of mathematical induction $P(n)$ is true for all natural number n .

Q. 4 $2^{3n} - 1$ is divisible by 7, for all natural numbers n .

Sol. Let $P(n) : 2^{3n} - 1$ is divisible by 7

Step I We observe that $P(1)$ is true.

$$P(1) : 2^{3 \times 1} - 1 = 2^3 - 1 = 8 - 1 = 7$$

It is clear that $P(1)$ is true.

Step II Now, assume that $P(n)$ is true for $n = k$,
 $P(k) : 2^{3k} - 1$ is divisible by 7.

$$\Rightarrow 2^{3k} - 1 = 7q$$

Step III Now, to prove $P(k+1)$ is true.

$$\begin{aligned} P(k+1) : 2^{3(k+1)} - 1 &= 2^{3k} \cdot 2^3 - 1 \\ &= 2^{3k} \cdot 8 - 1 \\ &= 2^{3k} (7 + 1) - 1 \\ &= 7 \cdot 2^{3k} + 2^{3k} - 1 \\ &= 7 \cdot 2^{3k} + 7q & [\text{from step II}] \\ &= 7(2^{3k} + q) \end{aligned}$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.
So, by the principle of mathematical induction $P(n)$ is true for all natural number n .

Q. 5 $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n .

Sol. Let $P(n) : n^3 - 7n + 3$ is divisible by 3, for all natural number n .

Step I We observe that $P(1)$ is true.

$$\begin{aligned} P(1) &= (1)^3 - 7(1) + 3 \\ &= 1 - 7 + 3 \\ &= -3, \text{ which is divisible by 3.} \end{aligned}$$

Hence, $P(1)$ is true.

Step II Now, assume that $P(n)$ is true for $n = k$.

$$\therefore P(k) = k^3 - 7k + 3 = 3q$$

Step III To prove $P(k + 1)$ is true

$$\begin{aligned} P(k + 1) &: (k + 1)^3 - 7(k + 1) + 3 \\ &= k^3 + 1 + 3k(k + 1) - 7k - 7 + 3 \\ &= k^3 - 7k + 3 + 3k(k + 1) - 6 \\ &= 3q + 3[k(k + 1) - 2] \end{aligned}$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

[from step II]

So, by the principle of mathematical induction $P(n)$ is true for all natural number n .

Q. 6 $3^{2n} - 1$ is divisible by 8, for all natural numbers n .

Sol. Let $P(n) : 3^{2n} - 1$ is divisible by 8, for all natural numbers.

Step I We observe that $P(1)$ is true.

$$\begin{aligned} P(1) : 3^{2(1)} - 1 &= 3^2 - 1 \\ &= 9 - 1 = 8, \text{ which is divisible by 8.} \end{aligned}$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : 3^{2k} - 1 = 8q$$

Step III Now, to prove $P(k + 1)$ is true.

$$\begin{aligned} P(k + 1) &: 3^{2(k + 1)} - 1 \\ &= 3^{2k} \cdot 3^2 - 1 \\ &= 3^{2k} \cdot (8 + 1) - 1 \\ &= 8 \cdot 3^{2k} + 3^{2k} - 1 \\ &= 8 \cdot 3^{2k} + 8q \\ &= 8(3^{2k} + q) \end{aligned}$$

[from step II]

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for all natural numbers n .

Q. 7 For any natural numbers n , $7^n - 2^n$ is divisible by 5.

Sol. Consider the given statement is

$P(n) : 7^n - 2^n$ is divisible by 5, for any natural number n .

Step I We observe that $P(1)$ is true.

$$P(1) = 7^1 - 2^1 = 5, \text{ which is divisible by 5.}$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) = 7^k - 2^k = 5q$$

Step III Now, to prove $P(k + 1)$ is true,

$$\begin{aligned} P(k + 1) &: 7^{k + 1} - 2^{k + 1} \\ &= 7^k \cdot 7 - 2^k \cdot 2 \end{aligned}$$

$$\begin{aligned}
&= 7^k \cdot (5 + 2) - 2^k \cdot 2 \\
&= 7^k \cdot 5 + 2 \cdot 7^k - 2^k \cdot 2 \\
&= 5 \cdot 7^k + 2(7^k - 2^k) \\
&= 5 \cdot 7^k + 2(5q) \\
&= 5(7^k + 2q), \text{ which is divisible by 5.} \quad [\text{from step II}]
\end{aligned}$$

So, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for any natural number n .

Q. 8 For any natural numbers n , $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

Sol. Let $P(n) : x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

Step I We observe that $P(1)$ is true.

$$P(1) : x^1 - y^1 = x - y$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : x^k - y^k \text{ is divisible by } (x - y).$$

$\therefore$

$$x^k - y^k = q(x - y)$$

Step III Now, to prove $P(k + 1)$ is true.

$$\begin{aligned}
P(k + 1) : x^{k+1} - y^{k+1} \\
&= x^k \cdot x - y^k \cdot y \\
&= x^k \cdot x - x^k \cdot y + x^k \cdot y - y^k \cdot y \\
&= x^k(x - y) + y(x^k - y^k) \\
&= x^k(x - y) + yq(x - y) \\
&= (x - y)[x^k + yq], \text{ which is divisible by } (x - y). \quad [\text{from step II}]
\end{aligned}$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true. So, by the principle of mathematical induction $P(n)$ is true for any natural number n .

Q. 9 $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

Thinking Process

In step I put $n=2$, the obtained result should be divisible by 6. Then, follow the same process as in question no. 4.

Sol. Let $P(n) : n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

Step I We observe that $P(2)$ is true. $P(2) : (2)^3 - 2$

$$\Rightarrow 8 - 2 = 6, \text{ which is divisible by 6.}$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : k^3 - k \text{ is divisible by 6.}$$

$\therefore$

$$k^3 - k = 6q$$

Step III To prove $P(k + 1)$ is true

$$\begin{aligned}
P(k + 1) : (k + 1)^3 - (k + 1) \\
&= k^3 + 1 + 3k(k + 1) - (k + 1) \\
&= k^3 + 1 + 3k^2 + 3k - k - 1 \\
&= k^3 - k + 3k^2 + 3k \\
&= 6q + 3k(k + 1) \quad [\text{from step II}]
\end{aligned}$$

We know that, $3k(k + 1)$ is divisible by 6 for each natural number $n = k$.

So, $P(k + 1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true.

Q. 10 $n(n^2 + 5)$ is divisible by 6, for each natural number n .

Sol. Let $P(n) : n(n^2 + 5)$ is divisible by 6, for each natural number n .

Step I We observe that $P(1)$ is true.

$$P(1) : 1(1^2 + 5) = 6, \text{ which is divisible by 6.}$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : k(k^2 + 5) \text{ is divisible by 6.}$$

$$\therefore k(k^2 + 5) = 6q$$

Step III Now, to prove $P(k + 1)$ is true, we have

$$\begin{aligned} P(k + 1) &: (k + 1)[(k + 1)^2 + 5] \\ &= (k + 1)[k^2 + 2k + 1 + 5] \\ &= (k + 1)[k^2 + 2k + 6] \\ &= k^3 + 2k^2 + 6k + k^2 + 2k + 6 \\ &= k^3 + 3k^2 + 8k + 6 \\ &= k^3 + 5k + 3k^2 + 3k + 6 \\ &= k(k^2 + 5) + 3(k^2 + k + 2) \\ &= (6q) + 3(k^2 + k + 2) \end{aligned}$$

We know that, $k^2 + k + 2$ is divisible by 2, where, k is even or odd.

Since, $P(k + 1) : 6q + 3(k^2 + k + 2)$ is divisible by 6. So, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true.

Q. 11 $n^2 < 2^n$, for all natural numbers $n \geq 5$.

Sol. Consider the given statement

$$P(n) : n^2 < 2^n \text{ for all natural numbers } n \geq 5.$$

Step I We observe that $P(5)$ is true

$$\begin{aligned} P(5) &: 5^2 < 2^5 \\ &= 25 < 32 \end{aligned}$$

Hence, $P(5)$ is true.

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) = k^2 < 2^k \text{ is true.}$$

Step III Now, to prove $P(k + 1)$ is true, we have to show that

$$P(k + 1) : (k + 1)^2 < 2^{k+1}$$

Now,

$$\begin{aligned} k^2 < 2^k &= k^2 + 2k + 1 < 2^k + 2k + 1 \\ &= (k + 1)^2 < 2^k + 2k + 1 \end{aligned} \quad \dots (i)$$

Now, $(2k + 1) < 2^k$

$$\begin{aligned} &= 2^k + 2k + 1 < 2^k + 2^k \\ &= 2^k + 2k + 1 < 2 \cdot 2^k \\ &= 2^k + 2k + 1 < 2^{k+1} \end{aligned} \quad \dots (ii)$$

From Eqs. (i) and (ii), we get $(k + 1)^2 < 2^{k+1}$

So, $P(k + 1)$ is true, whenever $P(k)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all natural numbers $n \geq 5$.

Q. 12 $2n < (n + 2)!$ for all natural numbers n .

Sol. Consider the statement

$P(n) : 2n < (n + 2)!$ for all natural number n .

Step I We observe that, $P(1)$ is true. $P(1) : 2(1) < (1 + 2)!$

$$\Rightarrow 2 < 3! \Rightarrow 2 < 3 \times 2 \times 1 \Rightarrow 2 < 6$$

Hence, $P(1)$ is true.

Step II Now, assume that $P(n)$ is true for $n = k$,

$$P(k) : 2k < (k + 2)! \text{ is true.}$$

Step III To prove $P(k + 1)$ is true, we have to show that

$$P(k + 1) : 2(k + 1) < (k + 1 + 2)!$$

Now,

$$2k < (k + 2)!$$

$$2k + 2 < (k + 2)! + 2$$

$$2(k + 1) < (k + 2)! + 2$$

...(i)

Also,

$$(k + 2)! + 2 < (k + 3)!$$

...(ii)

From Eqs. (i) and (ii),

$$2(k + 1) < (k + 1 + 2)!$$

So, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by principle of mathematical induction $P(n)$ is true.

Q. 13 $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

Sol. Consider the statement

$P(n) : \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

Step I We observe that $P(2)$ is true.

$$P(2) : \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}, \text{ which is true.}$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : \sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} \text{ is true.}$$

Step III To prove $P(k + 1)$ is true, we have to show that

$$P(k + 1) : \sqrt{k + 1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k + 1}} \text{ is true.}$$

Given that,

$$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}}$$

$$\Rightarrow \sqrt{k} + \frac{1}{\sqrt{k + 1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k + 1}}$$

$$\Rightarrow \frac{(\sqrt{k})(\sqrt{k + 1}) + 1}{\sqrt{k + 1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k + 1}} \quad \dots(i)$$

$$\text{If } \sqrt{k + 1} < \frac{\sqrt{k}\sqrt{k + 1} + 1}{\sqrt{k + 1}}$$

$$\Rightarrow k + 1 < \sqrt{k}\sqrt{k + 1} + 1$$

$$\Rightarrow k < \sqrt{k(k + 1)} \Rightarrow \sqrt{k} < \sqrt{k + 1} \quad \dots(ii)$$

From Eqs. (i) and (ii),

$$\sqrt{k + 1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k + 1}}$$

So, $P(k + 1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 14 $2 + 4 + 6 + \dots + 2n = n^2 + n$, for all natural numbers n .

Sol. Let $P(n) : 2 + 4 + 6 + \dots + 2n = n^2 + n$

For all natural numbers n .

Step I We observe that $P(1)$ is true.

$$P(1) : 2 = 1^2 + 1$$

$$2 = 2 \text{ which is true.}$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$\therefore P(k) : 2 + 4 + 6 + \dots + 2k = k^2 + k$$

Step III To prove that $P(k + 1)$ is true.

$$\begin{aligned} P(k + 1) : 2 + 4 + 6 + 8 + \dots + 2k + 2(k + 1) \\ &= k^2 + k + 2(k + 1) \\ &= k^2 + k + 2k + 2 \\ &= k^2 + 2k + 1 + k + 1 \\ &= (k + 1)^2 + k + 1 \end{aligned}$$

So, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, $P(n)$ is true.

Q. 15 $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$ for all natural numbers n .

Sol. Consider the given statement

$$P(n) : 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1, \text{ for all natural numbers } n$$

Step I We observe that $P(0)$ is true.

$$P(1) : 1 = 2^{0+1} - 1$$

$$1 = 2^1 - 1$$

$$1 = 2 - 1$$

$$1 = 1, \text{ which is true.}$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$\text{So, } P(k) : 1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1 \text{ is true.}$$

Step III Now, to prove $P(k + 1)$ is true.

$$\begin{aligned} P(k + 1) : 1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} \\ &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

So, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, $P(n)$ is true.

Q. 16 $1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$, for all natural numbers n .

Sol. Let $P(n) : 1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$, for all natural numbers n .

Step I We observe that $P(1)$ is true.

$$P(1) : 1 = 1(2 \times 1 - 1), 1 = 2 - 1 \text{ and } 1 = 1, \text{ which is true.}$$

Step II Now, assume that $P(n)$ is true for $n = k$.

$$\text{So, } P(k) : 1 + 5 + 9 + \dots + (4k - 3) = k(2k - 1) \text{ is true.}$$

Step III Now, to prove $P(k + 1)$ is true.

$$\begin{aligned}
 P(k + 1) &: 1 + 5 + 9 + \dots + (4k - 3) + 4(k + 1) - 3 \\
 &= k(2k - 1) + 4(k + 1) - 3 \\
 &= 2k^2 - k + 4k + 4 - 3 \\
 &= 2k^2 + 3k + 1 \\
 &= 2k^2 + 2k + k + 1 \\
 &= 2k(k + 1) + 1(k + 1) \\
 &= (k + 1)(2k + 1) \\
 &= (k + 1)[2k + 1 + 1 - 1] \\
 &= (k + 1)[2(k + 1) - 1]
 \end{aligned}$$

So, $P(k + 1)$ is true, whenever $p(k)$ is true, hence $P(n)$ is true.

Long Answer Type Questions

Use the Principle of Mathematical Induction in the following questions.

Q. 17 A sequence $a_1, a_2, a_3, \dots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$, for all natural numbers $k \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

Sol. A sequence $a_1, a_2, a_3, \dots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$, for all natural numbers $k \geq 2$.

Let $P(n) : a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

Step I We observe $P(2)$ is true.

For $n = 2$, $a_2 = 3 \cdot 7^{2-1} = 3 \cdot 7^1 = 21$ is true.

As $a_1 = 3, a_k = 7a_{k-1}$

$\Rightarrow a_2 = 7 \cdot a_{2-1} = 7 \cdot a_1$

$\Rightarrow a_2 = 7 \times 3 = 21$ [$\because a_1 = 3$]

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : a_k = 3 \cdot 7^{k-1}$$

Step III Now, to prove $P(k + 1)$ is true, we have to show that

$$P(k + 1) : a_{k+1} = 3 \cdot 7^{k+1-1}$$

$$\begin{aligned}
 a_{k+1} &= 7 \cdot a_{k+1-1} = 7 \cdot a_k \\
 &= 7 \cdot 3 \cdot 7^{k-1} = 3 \cdot 7^{k-1+1}
 \end{aligned}$$

So, $P(k + 1)$ is true, whenever $p(k)$ is true. Hence, $P(n)$ is true.

Q. 18 A sequence $b_0, b_1, b_2, \dots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all natural numbers k . Show that $b_n = 5 + 4n$, for all natural number n using mathematical induction.

Sol. Consider the given statement,

$P(n) : b_n = 5 + 4n$, for all natural numbers given that $b_0 = 5$ and $b_k = 4 + b_{k-1}$

Step I $P(1)$ is true.

$$P(1) : b_1 = 5 + 4 \times 1 = 9$$

As $b_0 = 5, b_1 = 4 + b_0 = 4 + 5 = 9$

Hence, $P(1)$ is true.

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : b_k = 5 + 4k$$

Step III Now, to prove $P(k + 1)$ is true, we have to show that

$$\therefore P(k + 1) : b_{k+1} = 5 + 4(k + 1)$$

$$b_{k+1} = 4 + b_{k+1-1}$$

$$= 4 + b_k$$

$$= 4 + 5 + 4k = 5 + 4(k + 1)$$

So, by the mathematical induction $P(k + 1)$ is true whenever $P(k)$ is true, hence $P(n)$ is true.

Q. 19 A sequence $d_1, d_2, d_3, \dots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$, for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$, for all $n \in N$.

Sol. Let $P(n) : d_n = \frac{2}{n!}, \forall n \in N$, to prove $P(2)$ is true.

$$\text{Step I} \quad P(2) : d_2 = \frac{2}{2!} = \frac{2}{2 \times 1} = 1$$

As, given

$$d_1 = 2$$

$$\Rightarrow d_k = \frac{d_{k-1}}{k}$$

$$\Rightarrow d_2 = \frac{d_1}{2} = \frac{2}{2} = 1$$

Hence, $P(2)$ is true.

Step II Now, assume that $P(k)$ is true.

$$P(k) : d_k = \frac{2}{k!}$$

Step III Now, to prove that $P(k + 1)$ is true, we have to show that $P(k + 1) : d_{k+1} = \frac{2}{(k + 1)!}$

$$\begin{aligned} d_{k+1} &= \frac{d_{k+1-1}}{k} = \frac{d_k}{k} \\ &= \frac{2}{k!k} = \frac{2}{(k + 1)!} \end{aligned}$$

So, $P(k + 1)$ is true. Hence, $P(n)$ is true.

Q. 20 Prove that for all $n \in N$

$$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos[\alpha + (n - 1)\beta]$$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Thinking Process

To prove this, use the formula $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$ and

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)$$

Sol. Let $P(n) : \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos[\alpha + (n-1)\beta]$

$$= \frac{\cos \left[\alpha + \left(\frac{n-1}{2} \right) \beta \right] \sin \left(\frac{n\beta}{2} \right)}{\sin \frac{\beta}{2}}$$

Step I We observe that $P(1)$

$$P(1) : \cos \alpha = \frac{\cos \left[\alpha + \left(\frac{1-1}{2} \right) \beta \right] \sin \frac{\beta}{2}}{\sin \frac{\beta}{2}} = \frac{\cos(\alpha + 0) \sin \frac{\beta}{2}}{\sin \frac{\beta}{2}}$$

$$\cos \alpha = \cos \alpha$$

Hence, $P(1)$ is true.

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos[\alpha + (k-1)\beta]$$

$$= \frac{\cos \left[\alpha + \left(\frac{k-1}{2} \right) \beta \right] \sin \frac{k\beta}{2}}{\sin \frac{\beta}{2}}$$

Step III Now, to prove $P(k+1)$ is true, we have to show that

$$P(k+1) : \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos[\alpha + (k-1)\beta] + \cos(\alpha + k\beta) = \frac{\cos \left(\alpha + \frac{k\beta}{2} \right) \sin(k+1)\frac{\beta}{2}}{\sin \frac{\beta}{2}}$$

$$\text{LHS} = \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos[\alpha + (k-1)\beta] + \cos(\alpha + k\beta)$$

$$= \frac{\cos \left[\alpha + \left(\frac{k-1}{2} \right) \beta \right] \sin \frac{k\beta}{2}}{\sin \frac{\beta}{2}} + \cos(\alpha + k\beta)$$

$$= \frac{\cos \left[\alpha + \left(\frac{k-1}{2} \right) \beta \right] \sin \frac{k\beta}{2} + \cos(\alpha + k\beta) \sin \frac{\beta}{2}}{\sin \frac{\beta}{2}}$$

$$= \frac{\sin \left(\alpha + \frac{k\beta}{2} - \frac{\beta}{2} + \frac{k\beta}{2} \right) - \sin \left(\alpha + \frac{k\beta}{2} - \frac{\beta}{2} - \frac{k\beta}{2} \right) + \sin \left(\alpha + k\beta + \frac{\beta}{2} \right) - \sin \left(\alpha + k\beta - \frac{\beta}{2} \right)}{2 \sin \frac{\beta}{2}}$$

$$= \frac{\sin \left(\alpha + k\beta + \frac{\beta}{2} \right) - \sin \left(\alpha - \frac{\beta}{2} \right)}{2 \sin \frac{\beta}{2}}$$

$$= \frac{2 \cos \frac{1}{2} \left(\alpha + \frac{\beta}{2} + k\beta + \alpha - \frac{\beta}{2} \right) \sin \frac{1}{2} \left(\alpha + \frac{\beta}{2} + k\beta - \alpha + \frac{\beta}{2} \right)}{2 \sin \frac{\beta}{2}}$$

$$= \frac{\cos \left(\frac{2\alpha + k\beta}{2} \right) \sin \left(\frac{k\beta + \beta}{2} \right)}{\sin \frac{\beta}{2}} = \frac{\cos \left(\alpha + \frac{k\beta}{2} \right) \sin(k+1)\frac{\beta}{2}}{\sin \frac{\beta}{2}} = \text{RHS}$$

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Q. 21 Prove that $\cos \theta \cos 2\theta \cos 2^2\theta \dots \cos 2^{n-1}\theta = \frac{\sin 2^n \theta}{2^n \sin \theta}, \forall n \in N.$

Sol. Let $P(n) : \cos \theta \cos 2\theta \dots \cos 2^{n-1}\theta = \frac{\sin 2^n \theta}{2^n \sin \theta}$

$$\begin{aligned} \text{Step I For } n = 1, P(1) : \cos \theta &= \frac{\sin 2^1 \theta}{2^1 \sin \theta} \\ &= \frac{\sin 2\theta}{2 \sin \theta} = \frac{2 \sin \theta \cos \theta}{2 \sin \theta} = \cos \theta \end{aligned}$$

which is true.

Step II Assume that $P(n)$ is true, for $n = k$.

$$P(k) : \cos \theta \cdot \cos 2\theta \cdot \cos 2^2\theta \dots \cos 2^{k-1}\theta = \frac{\sin 2^k \theta}{2^k \sin \theta} \text{ is true.}$$

Step III To prove $P(k+1)$ is true.

$$\begin{aligned} P(k+1) : \cos \theta \cdot \cos 2\theta \cdot \cos 2^2\theta \dots \cos 2^{k-1}\theta \cdot \cos 2^k\theta \\ &= \frac{\sin 2^k \theta}{2^k \sin \theta} \cdot \cos 2^k\theta \\ &= \frac{2 \sin 2^k\theta \cdot \cos 2^k\theta}{2 \cdot 2^k \sin \theta} \\ &= \frac{\sin 2 \cdot 2^k\theta}{2^{k+1} \sin \theta} = \frac{\sin 2^{(k+1)}\theta}{2^{k+1} \sin \theta} \end{aligned}$$

which is true.

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Q. 22 Prove that, $\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\frac{\sin n\theta}{2} \sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}},$

for all $n \in N.$

Thinking Process

To use the formula of $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$ and

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \cdot \sin \frac{B-A}{2} \text{ also } \cos(-\theta) = \cos \theta.$$

Sol. Consider the given statement

$$\begin{aligned} P(n) : \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta \\ &= \frac{\frac{\sin n\theta}{2} \sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}}, \text{ for all } n \in N \end{aligned}$$

Step I We observe that $P(1)$ is

$$P(1) : \sin \theta = \frac{\frac{\sin \theta}{2} \cdot \sin \frac{(1+1)\theta}{2}}{\sin \frac{\theta}{2}} = \frac{\frac{\sin \theta}{2} \cdot \sin \theta}{\sin \frac{\theta}{2}}$$

$$\sin \theta = \sin \theta$$

Hence, $P(1)$ is true.

Step II Assume that $P(n)$ is true, for $n = k$.

$$P(k) : \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin k\theta \\ = \frac{\sin \frac{k\theta}{2} \sin \left(\frac{k+1}{2} \theta \right)}{\sin \frac{\theta}{2}} \text{ is true.}$$

Step III Now, to prove $P(k+1)$ is true.

$$P(k+1) : \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin k\theta + \sin (k+1)\theta \\ = \frac{\sin \frac{(k+1)\theta}{2} \sin \left(\frac{k+1+1}{2} \theta \right)}{\sin \frac{\theta}{2}}$$

$$\text{LHS} = \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin k\theta + \sin (k+1)\theta$$

$$= \frac{\sin \frac{k\theta}{2} \sin \left(\frac{k+1}{2} \theta \right)}{\sin \frac{\theta}{2}} + \sin (k+1)\theta = \frac{\sin \frac{k\theta}{2} \sin \left(\frac{k+1}{2} \theta \right) + \sin (k+1)\theta \cdot \sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} \\ = \frac{\cos \left[\frac{k\theta}{2} - \left(\frac{k+1}{2} \theta \right) \right] - \cos \left[\frac{k\theta}{2} + \left(\frac{k+1}{2} \theta \right) \right] + \cos \left[(k+1)\theta - \frac{\theta}{2} \right] - \cos \left[(k+1)\theta + \frac{\theta}{2} \right]}{2 \sin \frac{\theta}{2}} \\ = \frac{\cos \frac{\theta}{2} - \cos \left(k\theta + \frac{\theta}{2} \right) + \cos \left(k\theta + \frac{\theta}{2} \right) - \cos \left(k\theta + \frac{3\theta}{2} \right)}{2 \sin \frac{\theta}{2}} \\ = \frac{\cos \frac{\theta}{2} - \cos \left(k\theta + \frac{3\theta}{2} \right)}{2 \sin \frac{\theta}{2}} = \frac{2 \sin \frac{1}{2} \left(\frac{\theta}{2} + k\theta + \frac{3\theta}{2} \right) \cdot \sin \frac{1}{2} \left(k\theta + \frac{3\theta}{2} - \frac{\theta}{2} \right)}{2 \sin \frac{\theta}{2}} \\ = \frac{\sin \left(\frac{k\theta + 2\theta}{2} \right) \cdot \sin \left(\frac{k\theta + \theta}{2} \right)}{\sin \frac{\theta}{2}} = \frac{\sin (k+1)\theta \cdot \sin \left(k+1 \right) \frac{\theta}{2}}{\sin \frac{\theta}{2}}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 23 Show that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number, for all $n \in N$.

Thinking Process

Here, use the formula $(a+b)^5 = a^5 + 5ab^4 + 10a^2b^3 + 10a^3b^2 + 5a^4b + b^5$

and $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

Sol. Consider the given statement

$$P(n) : \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15} \text{ is a natural number, for all } n \in N.$$

Step I We observe that $P(1)$ is true.

$$P(1) : \frac{(1)^5}{5} + \frac{1^3}{3} + \frac{7(1)}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1, \text{ which is a natural number. Hence, } P(1) \text{ is true.}$$

Step II Assume that $P(n)$ is true, for $n = k$.

$$P(k) : \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} \text{ is natural number.}$$

Step III Now, to prove $P(k+1)$ is true.

$$\begin{aligned}
 & \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15} \\
 &= \frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{k^3 + 1 + 3k(k+1)}{3} + \frac{7k+7}{15} \\
 &= \frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{k^3 + 1 + 3k^2 + 3k}{3} + \frac{7k+7}{15} \\
 &= \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} + \frac{5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{3k^2 + 3k + 1}{3} + \frac{7k+7}{15} \\
 &= \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} + k^4 + 2k^3 + 2k^2 + k + k^2 + k + \frac{1}{5} + \frac{1}{3} + \frac{7}{15} \\
 &= \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} + k^4 + 2k^3 + 3k^2 + 2k + 1, \text{ which is a natural number}
 \end{aligned}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 24 Prove that $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24}$, for all natural numbers $n > 1$.

Sol. Consider the given statement

$$P(n): \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24}, \text{ for all natural numbers } n > 1.$$

Step I We observe that, $P(2)$ is true,

$$\begin{aligned}
 P(2): \frac{1}{2+1} + \frac{1}{2+2} &> \frac{13}{24} \\
 \frac{1}{3} + \frac{1}{4} &> \frac{13}{24} \\
 \frac{4+3}{12} &> \frac{13}{24} \\
 \frac{7}{12} &> \frac{13}{24}, \text{ which is true.}
 \end{aligned}$$

Hence, $P(2)$ is true.

Step II Now, we assume that $P(n)$ is true,

For $n = k$,

$$P(k): \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} > \frac{13}{24}.$$

Step III Now, to prove $P(k+1)$ is true, we have to show that

$$\begin{aligned}
 P(k+1): \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} + \frac{1}{2(k+1)} &> \frac{13}{24} \\
 \text{Given, } \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} &> \frac{13}{24} \\
 \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{2k} + \frac{1}{2(k+1)} &> \frac{13}{24} + \frac{1}{2(k+1)} \\
 \frac{13}{24} + \frac{1}{2(k+1)} &> \frac{13}{24} \\
 \therefore \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} + \frac{1}{2(k+1)} &> \frac{13}{24}
 \end{aligned}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 25 Prove that number of subsets of a set containing n distinct elements is 2^n , for all $n \in N$.

Sol. Let $P(n)$: Number of subset of a set containing n distinct elements is 2^n , for all $n \in N$.

Step I We observe that $P(1)$ is true, for $n = 1$.

Number of subsets of a set contain 1 element is $2^1 = 2$, which is true.

Step II Assume that $P(n)$ is true for $n = k$.

$P(k)$: Number of subsets of a set containing k distinct elements is 2^k , which is true.

Step III To prove $P(k + 1)$ is true, we have to show that

$P(k + 1)$: Number of subsets of a set containing $(k + 1)$ distinct elements is 2^{k+1} .

We know that, with the addition of one element in the set, the number of subsets become double.

$\therefore$ Number of subsets of a set containing $(k + 1)$ distinct elements $= 2 \times 2^k = 2^{k+1}$.

So, $P(k + 1)$ is true. Hence, $P(n)$ is true.

Objective Type Questions

Q. 26 If $10^n + 3 \cdot 4^{n+2} + k$ is divisible by 9, for all $n \in N$, then the least positive integral value of k is

- (a) 5 (b) 3 (c) 7 (d) 1

Sol. (a) Let $P(n)$: $10^n + 3 \cdot 4^{n+2} + k$ is divisible by 9, for all $n \in N$.

For $n = 1$, the given statement is also true $10^1 + 3 \cdot 4^{1+2} + k$ is divisible by 9.

$$\begin{aligned} \therefore &= 10 + 3 \cdot 64 + k = 10 + 192 + k \\ &= 202 + k \end{aligned}$$

If $(202 + k)$ is divisible by 9, then the least value of k must be 5.

$$\therefore 202 + 5 = 207 \text{ is divisible by 9}$$

$$\Rightarrow \frac{207}{9} = 23$$

Hence, the least value of k is 5.

Q. 27 For all $n \in N$, $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by

- (a) 19 (b) 17 (c) 23 (d) 25

Sol. (b, c)

Given that, $3 \cdot 5^{2n+1} + 2^{3n+1}$

For $n = 1$,

$$\begin{aligned} &3 \cdot 5^{2(1)+1} + 2^{3(1)+1} \\ &= 3 \cdot 5^3 + 2^4 \\ &= 3 \times 125 + 16 = 375 + 16 = 391 \end{aligned}$$

Now, $391 = 17 \times 23$

which is divisible by both 17 and 23.

- Q. 28** If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is
- (a) 1 (b) 2 (c) 3 (d) 4

Sol. Let $P(n) : x^n - 1$ is divisible by $(x - k)$.
 For $n = 1$, $x^1 - 1$ is divisible by $(x - k)$.
 Since, if $x - 1$ is divisible by $x - k$. Then, the least possible integral value of k is 1.

Fillers

- Q. 29** If $P(n) : 2n < n!$, $n \in N$, then $P(n)$ is true for all $n \geq \dots\dots\dots$.

Sol. Given that, $P(n) : 2n < n!$, $n \in N$

For $n = 1$,	$2 < 1!$	[false]
For $n = 2$,	$2 \times 2 < 2! \quad 4 < 2$	[false]
For $n = 3$,	$2 \times 3 < 3!$	
	$6 < 3!$	
	$6 < 3 \times 2 \times 1$	
	$(6 < 6)$	[false]
For $n = 4$,	$2 \times 4 < 4!$	
	$8 < 4 \times 3 \times 2 \times 1$	
	$(8 < 24)$	[true]
For $n = 5$,	$2 \times 5 < 5!$	
	$10 < 5 \times 4 \times 3 \times 2 \times 1$	
	$(10 < 120)$	[true]

Hence, $P(n)$ is for all $n \geq 4$.

True/False

- Q. 30** Let $P(n)$ be a statement and let $P(k) \Rightarrow P(k + 1)$, for some natural number k , then $P(n)$ is true for all $n \in N$.

Sol. False
 The given statement is false because $P(1)$ is true has not been proved.