21. Some Special Series

Exercise 21.1

1. Question

Find the sum of the following series to n terms:

$1^3 + 3^3 + 5^3 + 7^3 + \ldots\ldots\ldots$

Answer

nth term would be = 2n - 1

We know, $1^3 + 2^3 + 3^3 + 4^3 + 5^3 \ldots\ldots\ldots\ldots\ldots n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Therefore,

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 \ldots\ldots\ldots\ldots\ldots(2n)^3 = \left[\frac{2n(2n+1)}{2}\right]^2$............equation 1

$$(1^3 + 3^3 + 5^3 \ldots\ldots\ldots(2n-1)^3) + (2^3 + 4^3 + 6^3 \ldots\ldots\ldots(2n)^3) = \left[\frac{2n(2n+1)}{2}\right]^2$$

$(1^3 + 3^3 + 5^3 \ldots\ldots(2n-1)^3) + 2^3(1^3 + 2^3 + 3^3 \ldots\ldots(2n)^3) = \left[\frac{2n(2n+1)}{2}\right]^2$...............equation 2

From equation 1

$2^3(1^3 + 2^3 + 3^3 \ldots\ldots\ldots n^3) = 2^3\left[\frac{n(n+1)}{2}\right]^2$

[replace 2n by n]

$2^3(1^3 + 2^3 + 3^3 \ldots\ldots\ldots n^3) = 2^3\left[\frac{n(n+1)}{2}\right]^2$

Substituting in equation 2

$$(1^3 + 3^3 + 5^3 \ldots\ldots\ldots(2n-1)^3) + 2^3\left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{2n(2n+1)}{2}\right]^2$$

$$(1^3 + 3^3 + 5^3 \ldots\ldots\ldots(2n-1)^3) = \left[\frac{n(2n+1)}{1}\right]^2 - 2^3\left[\frac{n(n+1)}{2}\right]^2$$

$$(1^3 + 3^3 + 5^3 \ldots\ldots\ldots(2n-1)^3) = \frac{(n)^2(2n+1)^2}{1} - 2^3\left[\frac{n(n+1)}{2}\right]^2$$

$$(1^3 + 3^3 + 5^3 \ldots\ldots\ldots(2n-1)^3) = \frac{(n)^2}{1}\left[(2n+1)^2 - \frac{2(n+1)^2}{1}\right]$$

$$(1^3 + 3^3 + 5^3 \ldots\ldots\ldots(2n-1)^3) = \frac{(n)^2}{1}[4n^2 + 1 + 4n - 2n^2 - 2 - 4n]$$

$$(1^3 + 3^3 + 5^3 \ldots\ldots\ldots(2n-1)^3) = n^2[2n^2 - 1]$$

2. Question

Find the sum of the following series to n terms:

$2^3 + 4^3 + 6^3 + 8^3 + \ldots\ldots\ldots$

Answer

nth term would be 2n

We know $1^3 + 2^3 + 3^3 + 4^3 + 5^3 \ldots\ldots\ldots\ldots\ldots n^3 = \left[\frac{n(n+1)}{2}\right]^2$......(1)

Therefore,

$(2^3 + 4^3 + 6^3 \ldots\ldots\ldots\ldots (2n)^3) = 2^3(1^3 + 2^3 + 3^3 \ldots\ldots\ldots\ldots n^3)$

Substituting the value from 1

$$(2^3 + 4^3 + 6^3 \ldots\ldots\ldots\ldots (2n)^3) = 2^3\left[\frac{n(n+1)}{2}\right]^2$$

3. Question

Find the sum of the following series to n terms:

1.2.5 + 2.3.6 + 3.4.7 + ...

Answer

The nth term be n(n + 1)(n + 4)

Thus we can write 1.2.5 + 2.3.6 + 3.4.7 + ...

The general term would be r(r + 1)(r + 4)

$$\sum_{r=1}^{n} r(r+1)(r+4)$$

$$\sum_{r=1}^{n} r^3 + 5r^2 + 4r$$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}\ldots\ldots d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}\ldots\ldots\ldots + d_0\sum 1$

Thus,

$\sum_{r=1}^{n} r^3 + 5r^2 + 4r = \sum_{n=1}^{n} r^3 + 5\sum_{n=1}^{n} r^2 + 4\sum_{n=1}^{n} r$......(1)

We know

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \ldots\ldots\ldots + n^3 = [\frac{n(n+1)}{2}]^2$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

Substituting in (1)

$$\sum_{r=1}^{n} n^3 + 5\sum_{r=1}^{n} n^2 + 4\sum_{r=1}^{n} n$$

$$= [\frac{n(n+1)}{2}]^2 + 5[\frac{n(n+1)(2n+1)}{6}] + 4[\frac{n(n+1)}{2}]$$

$= [\frac{n^2(n+1)^2}{2^2}] + 5[\frac{n(n+1)(2n+1)}{6}] + 4[\frac{n(n+1)}{2}]$

$= \frac{3n^2(n+1)^2 + 10n(n+1)(2n+1) + 24n(n+1)}{12}$

$= \frac{3n^4 + 26n^3 + 57n^2 + 34n}{12}$

4. Question

Find the sum of the following series to n terms:

1.2.4 + 2.3.7 + 3.4.10 + ...

Answer

The nth term be n(n + 1)(3n + 1)

1.2.4 + 2.3.7 + 3.4.10 + ...$=\sum_{r=1}^{n} r(r+1)(3r+1)$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

$$\sum_{r=1}^{n} r(r+1)(3r+1) = \sum_{r=1}^{n} 3r^3 + 4r^2 + r$$

$\sum_{r=1}^{n} 3r^3 + 4r^2 + r = 3\sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$ .....(1)

We know

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \ldots\ldots\ldots + n^3 = [\frac{n(n+1)}{2}]^2$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

Thus from (1)

$= 3[\frac{n(n+1)}{2}]^2 + 4\left[\frac{n(n+1)(2n+1)}{6}\right] + [\frac{n(n+1)}{2}]$

$= \frac{9n^2(n+1)^2 + 8n(n+1)(2n+1) + 6n(n+1)}{12}$

$= \frac{9n^4 + 34n^3 + 39n^2 + 14}{12}$

5. Question

Find the sum of the following series to n terms:

1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + ...

Answer

The nth term be $\sum_{k=0}^{n} n - k$

Where $\sum_{k=0}^{n} n - k$ = (n - 0) + (n - 1) + (n - 2) + ... + (n - n)

$$\sum_{k=0}^{n} n - k = n\sum_{k=0}^{n} 1 - \sum_{k=0}^{n} k$$

$$n\sum_{k=0}^{n} 1 = 1(k = 0th) + 1(k = 1th) + \ldots\ldots\ldots\ldots 1(k = nth) = n(n + 1)$$

Since,

$$\sum_{k=0}^{n} k = 1 + 2 + 3 + 4 \ldots n = \frac{n(n + 1)}{2}$$

$\sum_{k=0}^{n} n - k = n(n + 1) - \frac{n(n+1)}{2}$ ....(1)

1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + ...$= \sum_{n=1}^{n}\sum_{k=0}^{n} n - k$

From (1)

1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + ...$= \sum_{r=1}^{n} r(r + 1) - \frac{r(r+1)}{2}$

Thus, solving $\sum_{r=1}^{n} r(r + 1) - \frac{r(r+1)}{2}$

$$\sum_{r=1}^{n} r(r + 1) - \frac{r(r + 1)}{2} = \sum_{n=1}^{n} \frac{r(r + 1)}{2}$$

$$\sum_{r=1}^{n} r(r + 1) - \frac{r(r + 1)}{2} = \sum_{r=1}^{n} \frac{(r^2 + r)}{2}$$

Solving $\sum_{r=1}^{n} \frac{(r^2+r)}{2}$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}$.......$d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}$........ $+ d_0\sum 1$

Thus,

$$\sum_{r=1}^{n} \frac{(r^2 + r)}{2} = \frac{1}{2}(\sum_{r=1}^{n} r^2 + \sum_{n=1}^{n} r)$$

We know,

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n + 1)(2n + 1)}{6}]$$

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n + 1)}{2}]$$

Substituting

$$\sum_{r=1}^{n} \frac{(r^2 + r)}{2} = \frac{1}{2}([\frac{n(n + 1)(2n + 1)}{6}] + [\frac{n(n + 1)}{2}])$$

$$= \frac{1}{2}[\frac{n(n + 1)(2n + 1)}{6}] + [\frac{n(n + 1)}{2}]$$

$= \frac{1n(n + 1)}{2(2)}[\frac{(2n + 1)}{3} + 1]$

$= \frac{1n(n + 1)}{2(2)}[\frac{(2n + 4)}{3}]$

$= \frac{n(n + 1)(n + 2)}{6}$

Thus the answer is $\frac{n(n+1)(n+2)}{6}$

6. Question

Find the sum of the following series to n terms:

1 × 2 + 2 × 3 + 3 × 4 + 4 × 5 + ...

Answer

The last term be n(n + 1)

The generalized equation be

$\sum_{r=1}^{n} r(r + 1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$..................(1)

Since We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2} ....d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2} .... + d_0\sum 1$

We know

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n + 1)(2n + 1)}{6}]$$

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n + 1)}{2}]$$

Thus substituting in (1)

$= [\frac{n(n + 1)(2n + 1)}{6}] + [\frac{n(n + 1)}{2}]$

$= \frac{n(n + 1)}{2}[\frac{(2n + 1)}{3} + 1]$

$= \frac{n(n + 1)}{2}[\frac{(2n + 4)}{3}]$

$= \frac{n(n + 1)}{1}[\frac{(n + 2)}{3}]$

7. Question

Find the sum of the following series to n terms:

$3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 +$ ....

Answer

The nth term will be $n^2 \times (2n + 1)$

The generalized equation be

$$\sum_{r=1}^{n} r^2 \times (2r + 1) = \sum_{n=1}^{n} 2r^3 + r^2$$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

Thus

$\sum_{r=1}^{n} 2r^3 + r^2 = 2\sum_{n=1}^{n} r^3 + \sum_{n=1}^{n} r^2$.........(1)

We know

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \ldots\ldots\ldots + n^3 = [\frac{n(n+1)}{2}]^2$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

Substituting the values in (1)

$\sum_{r=1}^{n} 2r^3 + r^2 = 2\,[\frac{n(n+1)}{2}]^2 + [\frac{n(n+1)(2n+1)}{6}]$

$\sum_{r=1}^{n} 2r^3 + r^2 = \frac{n(n+1)}{2}[2\,[\frac{n(n+1)}{2}] + \frac{[2n+1]}{3}]$

$\sum_{r=1}^{n} 2r^3 + r = \frac{n(n+1)}{2}[\frac{3\,n(n+1) + 2n + 1}{3}]$

$\sum_{r=1}^{n} 2r^3 + r^2 = \frac{n(n+1)}{2}[\frac{3\,n^2 + 5n + 1}{3}]$

8 A. Question

Find the sum of the series whose nth term is :

$2n^3 + 3n^2 - 1$

Answer

1^{st} term = $2(1)^3 + 3(1)^2 - 1$

2^{nd} term = $2(2)^3 + 3(2)^2 - 1$

And so on

Nth term = $2n^3 + 3n^2 - 1$

General term be = $2r^3 + 3r^2 - 1$

Summation = 1^{st} term + 2^{nd} term + ... + nth term

= $2(1)^3 + 3(1)^2 - 1 + 2(2)^3 + 3(2)^2 - 1 + ...2n^3 + 3n^2 - 1$ ...(1)

We know,

$$\sum_{x=1}^{n} f(x) = f(1) + f(2) + \ldots\ldots\ldots\ldots\ldots\ldots f(n)$$

Thus

From (1) we have

Summation = $\sum_{r=1}^{n} 2r^3 + 3r^2 - 1$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

Thus

$\sum_{r=1}^{n} 2r^3 + 3r^2 - 1 = 2\sum_{x=1}^{n} r^3 + 3\sum_{x=1}^{n} r^2 - \sum_{x=1}^{n} 1$ ......(2)

We know

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \ldots\ldots\ldots + n^3 = [\frac{n(n+1)}{2}]^2$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + \ldots\ldots..n \text{ times} = n$$

Thus substituting the above values in (2)

Summation = $\sum_{r=1}^{n} 2n^3 + 3n^2 - 1 = 2[\frac{n(n+1)}{2}]^2 + 3\left[\frac{n(n+1)(2n+1)}{6}\right] - n$

$$= 2[\frac{n(n+1)}{2}]^2 + 3\left[\frac{n(n+1)(2n+1)}{6}\right] - n$$

$$= \frac{n(n+1)}{2}[2[\frac{n(n+1)}{2}] + \frac{3[2n+1]}{3}] - n$$

$$= \frac{n(n+1)}{2}[\frac{n(n+1) + 2n + 1}{1}] - n$$

$$= \frac{n(n+1)}{2}[\frac{n^2 + 3n + 1}{1}] - n$$

8 B. Question

Find the sum of the series whose nth term is :

$n^3 - 3^n$

Answer

Generalized term be $n^3 - 3^n$

1^{st} term = $(1)^3 - 3^{(1)}$

2^{nd} term = $(2)^3 - 3^{(2)}$

And so on

nth term= $n^3 - 3^n$

general term= $r^3 - 3^r$

Summation=1^{st} term + 2^{nd} term + ... + nth term

$=(1)^3 - 3^{(1)} + (2)^3 - 3^{(2)} + ... + n^3 - 3^n$.........(1)

We know

$$\sum_{x=1}^{n} f(x) = f(1) + f(2) + \dots\dots\dots\dots f(n)$$

Thus

From (1) we have

Summation = $\sum_{r=1}^{n} r^3 - 3^r$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}\dots\dots d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}\dots\dots + d_0\sum 1$

Thus

$\sum_{r=1}^{n} r^3 - 3^r = \sum_{r=1}^{n} r^3 - \sum_{x=1}^{n} 3^r$ ....(2)

We know,

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \dots\dots\dots + n^3 = [\frac{n(n+1)}{2}]^2$$

$$\sum_{r=1}^{n} 3^r = 3^1 + 3^2 + 3^3 \dots\dots\dots\dots\dots + 3^n = \frac{3(3^n - 1)}{3 - 1}$$

Since, $\frac{a_1(r^n - 1)}{r - 1} = a_1 + a_2 + a_3 \dots\dots\dots\dots a_n$ where $r = \frac{a_2}{a_1}$ if $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots \dots = \frac{a_n}{a_{n-1}}$

Thus substituting the above values in (2)

$$\sum_{r=1}^{n} r^3 - 3^r = [\frac{n(n+1)}{2}]^2 - \frac{3(3^n - 1)}{3 - 1}$$

Summation= $[\frac{n(n+1)}{2}]^2 - \frac{3(3^n - 1)}{3 - 1}$

$$= [\frac{n(n+1)}{2}]^2 - \frac{3(3^n - 1)}{2}$$

8 C. Question

Find the sum of the series whose nth term is :

n(n + 1) (n + 4)

Answer

Generalized term be r(r + 1) (r + 4)

1^{st} term = (1)((1) + 1) ((1) + 4)

2^{nd} term =(2)((2) + 1) ((2) + 4)

And so on

nth term = n(n + 1) (n + 4) = $n^3 + 5n^2 + 4n$

Summation = 1^{st} term + 2^{nd} term + ... + nth term

=(1)((1) + 1) ((1) + 4) + (2)((2) + 1) ((2) + 4)........ + $n^3 + 5n^2 + 4n$ ......(1)

We know,

$$\sum_{x=1}^{n} f(x) = f(1) + f(2) + \dots\dots\dots\dots f(n)$$

Thus

From (1) we have

Summation = $\sum_{r=1}^{n} r^3 + 5r^2 + 4r$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

Thus

$\sum_{r=1}^{n} r^3 + 5r^2 + 4r = \sum_{x=1}^{n} r^3 + 5\sum_{x=1}^{n} r^2 + 4\sum_{x=1}^{n} r$ (2)

We know,

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \ldots\ldots\ldots + n^3 = [\frac{n(n+1)}{2}]^2$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

Thus substituting the above values in (2)

$$\sum_{r=1}^{n} r^3 + 5r^2 + 4r = [\frac{n(n+1)}{2}]^2 + 5\left[\frac{n(n+1)(2n+1)}{6}\right] + 4[\frac{n(n+1)}{2}]$$

Summation = $[\frac{n(n+1)}{2}]^2 + 5\left[\frac{n(n+1)(2n+1)}{6}\right] + 4[\frac{n(n+1)}{2}]$

$$= [\frac{n(n+1)}{2}]^2 + 5\left[\frac{n(n+1)(2n+1)}{6}\right] + 4[\frac{n(n+1)}{2}]$$

$$= \frac{n(n+1)}{2}\{[\frac{n(n+1)}{2}] + 5\left[\frac{(2n+1)}{3}\right] + 4[1]\}$$

$$= \frac{n(n+1)}{2}\{\frac{3n(n+1) + 10(2n+1) + 24}{6}\}$$

$$= \frac{n(n+1)}{2}\{\frac{3n^2 + 23n + 34}{6}\}$$

8 D. Question

Find the sum of the series whose nth term is :

$(2n - 1)^2$

Answer

Generalized term be $(2r - 1)^2 = 4r^2 + 1 - 4r$

1^{st} term = $4(1)^2 + 1 - 4(1)$

2^{nd} term $= 4(2)^2 + 1 - 4(2)$

And so on

nth term= $4n^2 + 1 - 4n$

Summation=1^{st} term + 2^{nd} term + …….. + nth term

$= 4(1)^2 + 1 - 4(1) + 4(2)^2 + 1 - 4(2)$ ....$4n^2 + 1 - 4n$ ......(1)

We know ,

$$\sum_{x=1}^{n} f(x) = f(1) + f(2) + \ldots\ldots\ldots\ldots\ldots..f(n)$$

Thus

From (1) we have

Summation = $\sum_{r=1}^{n} 4r^2 + 1 - 4r$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

Thus

$\sum_{r=1}^{n} 1 + 4r^2 - 4r = \sum_{r=1}^{n} 1 + 4\sum_{r=1}^{n} r^2 - 4\sum_{x=1}^{n} r$ (2)

We know

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \ldots\ldots\ldots + n^3 = [\frac{n(n+1)}{2}]^2$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

$$\sum_{r=1}^{n} 1 = n$$

Thus substituting above values in (2)

$$\sum_{r=1}^{n} 1 + 4r^2 - 4r = n + 4\left[\frac{n(n+1)(2n+1)}{6}\right] - 4\left[\frac{n(n+1)}{2}\right]$$

$$= n + 4\left[\frac{n(n+1)(2n+1)}{6}\right] - 4\left[\frac{n(n+1)}{2}\right]$$

$$= n + \frac{4n(n+1)}{2}\{\left[\frac{(2n+1)}{3}\right] - [1]\}$$

$$= n + \frac{4n(n+1)}{2}\left[\frac{2n-2}{6}\right]$$

$$= n + \frac{2n(n+1)}{1}\left[\frac{n-1}{3}\right]$$

9. Question

Find the 20^{th} term and the sum of 20 terms of the series :

2 × 4 + 4 × 6 + 6 × 8 + ...

Answer

Given: 2 × 4 + 4 × 6 + 6 × 8 + ...

The nth term would be from given series 2n × (2n + 2)

The general term would be from given series 2r × (2r + 2)

Thus 20th term be 2(20) {2(20) + 2} = 40 × 42 = 1680

Summation=1st term + 2nd term + ... + 20th term

= 2 × 4 + 4 × 6 + 6 × 8 + ... 40 × 42 (1)

We know

$$\sum_{x=1}^{n} f(x) = f(1) + f(2) + \ldots\ldots\ldots\ldots\ldots..f(n)$$

Thus

From (1) we have

Summation = $\sum_{r=1}^{20} 2r \times (2r + 2)$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

$$\sum_{r=1}^{20} 2r \times (2r + 2) = \sum_{r=1}^{20} (4r^2 + 4r)$$

Thus,

$\sum_{r=1}^{20} (4r^2 + 4r) = 4\sum_{r=1}^{20} r^2 + 4\sum_{r=1}^{20} r$ (2)

We know

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

Thus

$$\sum_{r=1}^{20} r^2 = \left[\frac{20(20+1)(2(20)+1)}{6}\right] = \left[\frac{20(21)(41)}{6}\right]$$

$$\sum_{r=1}^{20} r = \left[\frac{20(20+1)}{2}\right] = \left[\frac{20(21)}{2}\right]$$

Thus substituting in above equation in (2)

$$\sum_{r=1}^{20} (4r^2 + 4r) = 4\left[\frac{20(21)(41)}{6}\right] + 4\left[\frac{20(21)}{2}\right]$$

=4(10)(7)(41) + 4(10)(21)

=12320

1. Question

If the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ subtends an angle α at the origin O, prove that : OP. OQ cos α = $x_1 x_2 + y_1 y_2$.

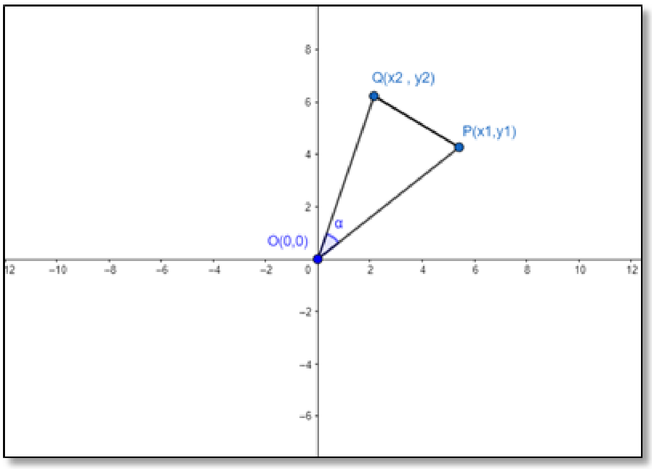
Answer

Key points to solve the problem:

• **The idea of distance formula-** Distance between two points P(x_1,y_1) and Q(x_2,y_2) is given by- **PQ =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Given,

Two points P and Q subtends an angle α at the origin as shown in figure:



From the figure we can see that points O,P and Q forms a triangle.

Clearly in ΔOPQ we have:

$\cos\alpha = \frac{OP^2+OQ^2-PQ^2}{2OP.OQ}$ {from cosine formula in a triangle}

⇒ $2\,OP.OQ\cos\alpha = OP^2+OQ^2-PQ^2$ .....**equation 1**

From distance formula we have-

OP = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

As, coordinates of O are (0, 0) ⇒ x_2 = 0 and y_2 = 0

Coordinates of P are (x_1, y_1) ⇒ x_1 = x_1 and y_1 = y_1

$= \sqrt{(x_1-0)^2+(y_1-0)^2}$

$= \sqrt{x_1^2+y_1^2}$

Similarly, OQ $= \sqrt{(x_2-0)^2+(y_2-0)^2}$

$= \sqrt{x_2^2+y_2^2}$

And, $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

∴ OP^2 + OQ^2 - PQ^2 = $\left(\sqrt{x_1^2+y_1^2}\right)^2+\left(\sqrt{x_2^2+y_2^2}\right)^2-\left(\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right)^2$

⇒ OP^2 + OQ^2 - PQ^2 = $x_1^2+y_1^2+\ x_2^2+y_2^2-\{(x_2-x_1)^2+(y_2-y_1)^2\}$

Using $(a-b)^2 = a^2 + b^2 - 2ab$

∴ OP^2 + OQ^2 - PQ^2 = $2x_1x_2 + 2y_1y_2$ ....**equation 2**

From equation 1 and 2 we have:

$2OP.OQ \cos\alpha = 2x_1x_2 + 2y_1y_2$

⇒ OP.OQ cos α = x_1 x_2 + y_1 y_2 …Proved.

2. Question

The vertices of a triangle ABC are A(0, 0), B (2, -1) and C (9, 0). Find cos B.

Answer

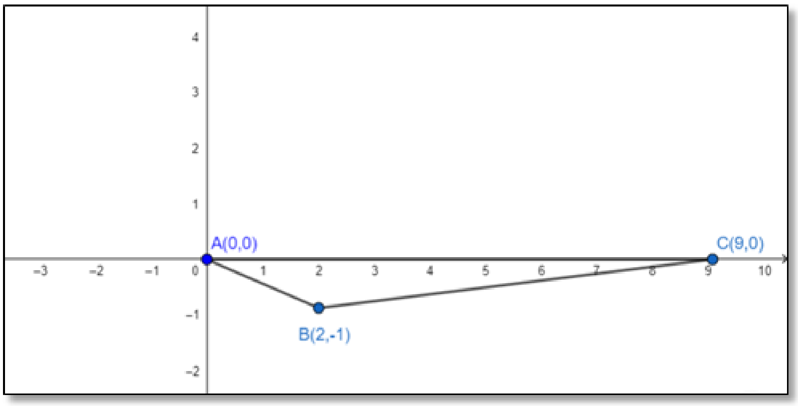
Key points to solve the problem:

• **The idea of distance formula-** Distance between two points P(x_1,y_1) and Q(x_2,y_2) is given by- **PQ =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Given,

Coordinates of the triangle and we need to find cos B which can be easily found using cosine formula.

See the figure:



From cosine formula in ΔABC , We have:

cos B = $\frac{AB^2+BC^2-AC^2}{2AB.BC}$

using distance formula we have:

AB = $\sqrt{(2-0)^2+(-1-0)^2} = \sqrt{5}$

BC = $\sqrt{(9-2)^2+(0-(-1))^2} = \sqrt{7^2+1^2} = \sqrt{50}$

And, AC = $\sqrt{(9-0)^2+(0-0)^2} = 9$

∴ cos B = $\frac{(\sqrt{5})^2+(\sqrt{50})^2-9^2}{2\sqrt{5}\sqrt{50}} = \frac{55-81}{2\sqrt{5}\sqrt{2\times25}} = \frac{-26}{10\sqrt{10}} = \frac{-13}{5\sqrt{10}}$ **…ans**

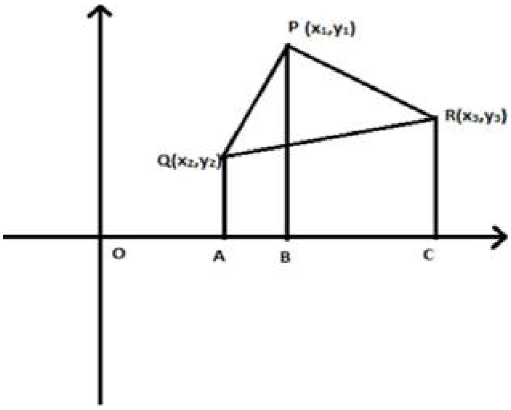
3. Question

Four points A (6, 3), B(-3, 5), C(4, -2) and D(x, 3x) are given in such a way that $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$, find x.

Answer

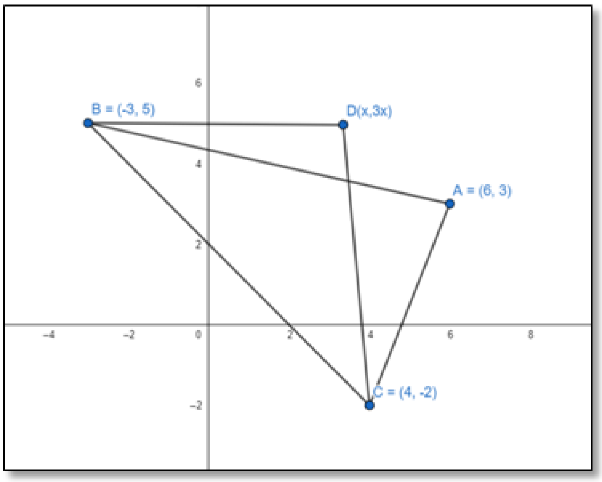
Key points to solve the problem:

• **The idea of distance formula-** Distance between two points P(x_1,y_1) and Q(x_2,y_2) is given by- **PQ =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



•**Area of a ΔPQR –** Let P(x_1,y_1) , Q(x_2,y_2) and R(x_3,y_3) be the 3 vertices of ΔPQR.

Ar(ΔPQR) = $\frac{1}{2}[x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)]$



Given, coordinates of the triangle as shown in the figure.

Also, $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$

ar(ΔDBC) = $\frac{1}{2}[x(5 - (-2)) + (-3)(-2 - 3x) + 4(3x - 5)]$

$= \frac{1}{2}[7x + 6 + 9x + 12x - 20] = 14x - 7$

Similarly, ar(ΔABC) = $\frac{1}{2}[6(5 - (-2)) + (-3)(-2 - 3) + 4(3 - 5)]$

$= \frac{1}{2}[42 + 15 - 8] = \frac{49}{2} = 24.5$

$\therefore \frac{\Delta DBC}{\Delta ABC} = \frac{1}{2} = \frac{14x-7}{24.5}$

⇒ 24.5 = 28x – 14

⇒ 28x = 38.5

⇒ x = 38.5/28 = 1.375 **...ans**

4. Question

The points A (2, 0), B(9, 1), C (11, 6) and D (4, 4) are the vertices of a quadrilateral ABCD. Determine whether ABCD is a rhombus or not.

Answer

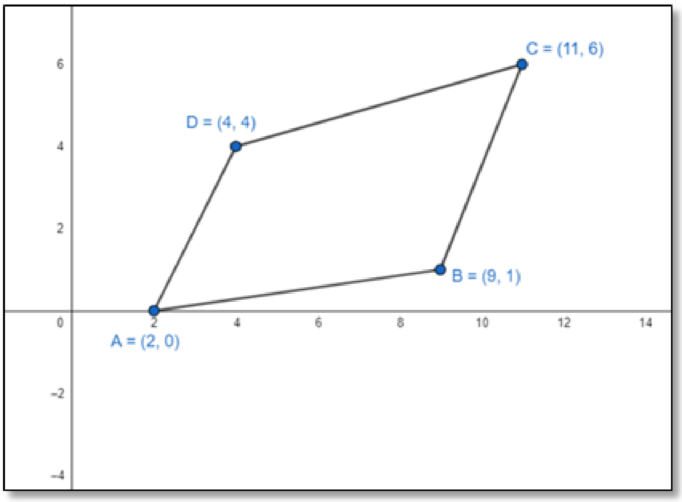
Key points to solve the problem:

• **The idea of distance formula-** Distance between two points P(x_1,y_1) and Q(x_2,y_2) is given by- **PQ =**

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

• **The idea of Rhombus –** It is a quadrilateral with all four sides equal.

Given, coordinates of 4 points that form a quadrilateral as shown in fig:



Using distance formula, we have:

AB = $\sqrt{(9-2)^2 + (1-0)^2} = \sqrt{7^2 + 1} = \sqrt{50}$

BC = $\sqrt{(11-9)^2 + (6-1)^2} = \sqrt{2^2 + 5^2} = \sqrt{29}$

Clearly, AB ≠ BC ⇒ quad ABCD does not have all 4 sides equal.

∴ ABCD is not a Rhombus ...**ans**

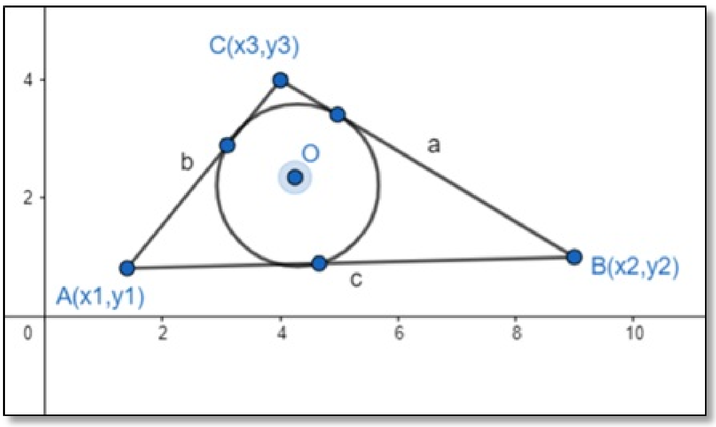
5. Question

Find the coordinates of the centre of the circle inscribed in a triangle whose vertices are (-36, 7), (20, 7) and (0, -8).

Answer

Key points to solve the problem:

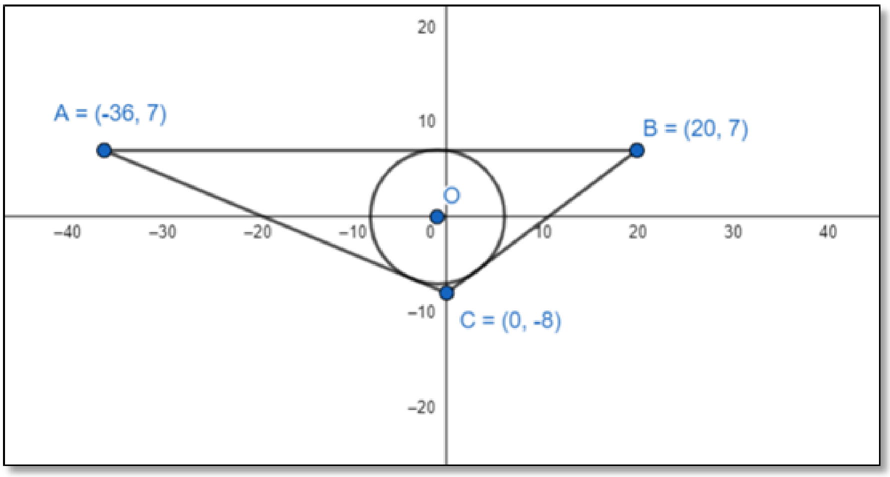
• **The idea of distance formula-** Distance between two points P(x_1,y_1) and Q(x_2,y_2) is given by- **PQ =** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

• 

Incentre of a triangle - Let A(x_1,y_1) , B(x_2,y_2) and C(x_3,y_3) be the 3 vertices of ΔABC and O be the centre of the circle inscribed in ΔABC

O = $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$ where a, b and c are length of sides opposite to ∠ A , ∠ B and ∠ C respectively.

Given, coordinates of vertices of the triangle as shown in figure:



We need to find the coordinates of O:

Before that, we have to find a ,b and c. We will use the distance formula to find the same.

As, a = BC = $\sqrt{(20-0)^2+(7-(-8))^2} = \sqrt{20^2+15^2} = 25$

b = AC = $\sqrt{(-36-0)^2+(7-(-8))^2} = \sqrt{36^2+15^2} = \sqrt{1521} = 39$

and c = AB = $\sqrt{(-36-20)^2+(7-7)^2} = 56$

∴ coordinates of O = $\left(\frac{25(-36)+39(20)+56(0)}{25+39+56}, \frac{25(7)+39(7)+56(-8)}{25+39+56}\right)$

$= \left(\frac{-1}{120}, \frac{0}{120}\right) = (-1\,,0)$ **...ans**

6. Question

The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle.

Answer

Key points to solve the problem:

• **The idea of distance formula-** Distance between two points P(x_1,y_1) and Q(x_2,y_2) is given by- **PQ =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

• **Equilateral triangle-** triangle with all 3 sides equal.

• **Coordinates of the midpoint of a line segment –** Let P(x_1,y_1 �) and Q(x_2,y_2) be the end points of line segment PQ. Then coordinated of the midpoint of PQ is given by – $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Given, an equilateral triangle with base along y axis and midpoint at (0,0)

∴ coordinates of triangle will be A(0,y_1) B(0,y_2) and C(x,0)

As midpoint is at origin ⇒ y_1+y_2 = 0 ⇒ y_1 = -y_2 .....**eqn 1**

Also length of each side = 2a (given)

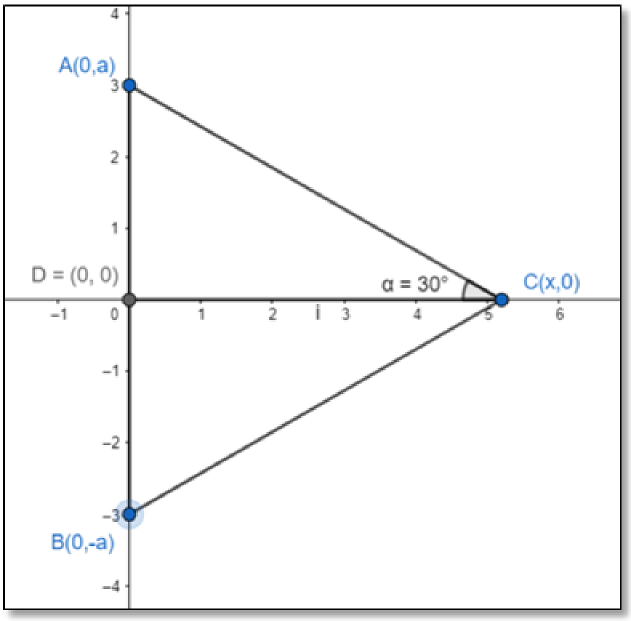
∴ AB = $\sqrt{(0-0)^2+(y_2-y_1)^2} = y_2 - y_1 = 2a$ **....eqn 2**

∴ from eqn 1 and 2:

y_1 = a and y_2 = -a

∴ 2 coordinates are – A(0,a) and B(0,-a)

See the figure:



Clearly from figure:

DC = x

Also in ΔADC: cos 30° $= \frac{DC}{AC} = \frac{x}{\sqrt{(0-x)^2+(a-0)^2}}$

$\therefore \frac{\sqrt{3}}{2} = \frac{x}{\sqrt{x^2+a^2}}$

Squaring both sides:

$3(x^2 + a^2) = 4x^2 \Rightarrow x^2 = 3a^2$

$\therefore x = \pm\sqrt{3}a$

∴ Coordinates of C are (√3a,0) or (-√3a,0) **....ans**

7. Question

Find the distance between P(x_1, y_1) and Q(x_2, y_2) when (i) PQ is parallel to the y-axis (ii) PQ is parallel to the x-axis.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Given, P(x_1, y_1) and Q(x_2, y_2) are two points.

i) When PQ is parallel to the y-axis

This implies that x – coordinate is constant ⇒ $x_2 = x_1$

∴ from distance formula:

PQ = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{0+(y_2-y_1)^2}$ **=** $|y_2 - y_1|$**...ans**

ii) When PQ is parallel to the x-axis

This implies that y – coordinate is constant ⇒ $y_2 = y_1$

∴ from distance formula:

PQ = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{0+(x_2-x_1)^2}$ **=** $|x_2 - x_1|$**...ans**

Note: we take modulus because square root gives both positive and negative values, but distance is always positive so we make it positive using modulus function.

8. Question

Find a point on the x-axis, which is equidistant from the point (7, 6) and (3, 4).

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

As the point is on the x-axis so y-coordinate is 0.

Let the coordinate be (x,0)

Given distance of (x,0) from (7,6) and (3,4) is same.

∴ using distance formula we have:

$$\sqrt{(x-7)^2 + (0-6)^2} = \sqrt{(x-3)^2 + (0-4)^2}$$

$\textit{squaring both sides, we have}$:

$(x - 7)^2 + (0 - 6)^2 = (x - 3)^2 + (0 - 4)^2$

$\Rightarrow x^2 + 49 - 14x + 36 = x^2 + 9 - 6x + 16$

$$\Rightarrow 8x = 60 \ \Rightarrow x = \frac{60}{8} = \frac{15}{2} = 7.5$$

∴ point on x-axis is (7.5,0) …ans

Exercise 21.2

1. Question

Sum the following series to n terms :

3 + 5 + 9 + 15 + 23 + …………

Answer

Let s=3 + 5 + 9 + 15 + 23 + ………… + n

By shifting each term by one

S = 3 + 5 + 9 + 15 + 23 + ………… + nth ………(1)

S = 3 + 5 + 9 + 15 + ………… + (n - 1)th + nth ……..(2)

by (1) - (2) we get

0 = 3 + 2 + 4 + 6 + 8 + …….nth - (n - 1)th - n

Nth = 3 + 2 + 4 + 6 + 8 + …….2(n - 1)th

Nth = 3 + 2(1 + 2 + 3 + 4 + …….(n - 1)th) ……….(3)

we know

$$\sum_{r=1}^{n-1} r = 1 + 2 + 3 \ldots\ldots\ldots + n - 1 = [\frac{n(n-1)}{2}]$$

Substituting the above-given value in (3)

$nth = 3 + 2[\frac{n(n-1)}{2}]$

nth = 3 + n^2 - n

general term = 3 + r^2 - r

thus

S = 3 + 5 + 9 + 15 + 23 + ... + nth $= \sum_{r=1}^{n} 3 + r^2 - r$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

$S = \sum_{r=1}^{n} 3 + r^2 - r = 3\sum_{n=1}^{n} 1 + \sum_{n=1}^{n} r^2 - \sum_{n=1}^{n} r$ (4)

We know

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \dots\dots\dots + n = [\frac{n(n+1)}{2}]$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \dots\dots\dots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + \dots\dots..n \text{ times} = n$$

Thus substituting the above values in(4)

$$S = 3n + [\frac{n(n+1)(2n+1)}{6}] - [\frac{n(n+1)}{2}]$$

$$S = 3n + \frac{n(n+1)}{2}\{[\frac{(2n+1)}{3}] - [1]\}$$

$$S = 3n + \frac{n(n+1)}{2}\{\frac{(2n-2)}{3}\}$$

$$S = 3n + \frac{n(n+1)}{1}\{\frac{(n-1)}{3}\}$$

2. Question

Sum the following series to n terms :

2 + 5 + 10 + 17 + 26 + ...

Answer

Let S = 2 + 5 + 10 + 17 + 26 + ... + n

By shifting each term by one

S = 2 + 5 + 10 + 17 + 26 + ... + nth ....(1)

S = 2 + 5 + 10 + 17 + ... + (n - 1)th + nth ....(2)

by (1) - (2) we get

0 = 2 + 3 + 5 + 7 + 9 + ...nth - (n - 1)th - nth

Nth = 2 + (3 + 5 + 7 + 9 + ...2r + 1) ....(3)

Nth = 2 + (summation of first (n - 1)th term) we know,

$$\sum_{r=1}^{n-1} 2n + 1 = 3 + 5 + 7 \ldots\ldots\ldots + 2n + 1 = [n^2 - 1]$$

Substituting the above given value in (3)

nth=n^2 - 1 + 2

general term=r^2 - 1 + 2

thus

S = 2 + 5 + 10 + 17 + 26 + ... + nth =$\sum_{r=1}^{n} 2 + r^2 - 1$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}$.......$d_0$=$a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}$........ + $d_0\sum 1$

$S = \sum_{r=1}^{n} 1 + n^2 = 1\sum_{r=1}^{n} 1 + \sum_{r=1}^{n} r^2$ (4)

We know

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n + 1)(2n + 1)}{6}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + \ldots\ldots..n \text{ times} = n$$

Thus substituting the above values in(4)

$$S = n + [\frac{n(n + 1)(2n + 1)}{6}]$$

$$S = \frac{6n + n(n + 1)(2n + 1)}{6}$$

3. Question

Sum the following series to n terms :

1 + 3 + 7 + 13 + 21 + ...

Answer

Let s=1 + 3 + 7 + 13 + 21 + ... + n

By shifting each term by one

s=1 + 3 + 7 + 13 + 21 + ... + nth (1)

s= 1 + 3 + 7 + 13 + ... + (n - 1)th + nth (2)

by (1) - (2) we get

0=1 + 2 + 4 + 6 + 8 + ...nth - (n - 1)th - nth

nth=1 + (2 + 4 + 6 + 8 + ...2r) (3)

nth=1 + (summation of first (n - 1)th term)

we know

$$\sum_{r=1}^{n-1} 2r = 2 + 4 + 6 + 8 \ldots\ldots\ldots + 2n - 2 = [n(n - 1)]$$

Substituting the above given value in (3)

nth=1 + n^2 - n

general term =1 + r^2 - r

thus

s=1 + 3 + 7 + 13 + 21 + ... + nth =$\sum_{r=1}^{n} 1 + r^2 - r$

We know by property that$\sum ax^n + bx^{n-1} + cx^{n-2} .... d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2} .... + d_0\sum 1$

$s = \sum_{r=1}^{n} 1 + r^2 - r = \sum_{r=1}^{n} 1 + \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$ (4)

We know

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \dots \dots \dots + n = [\frac{n(n + 1)}{2}]$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \dots \dots \dots + n^2 = [\frac{n(n + 1)(2n + 1)}{6}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + \dots \dots ..n \text{ times} = n$$

Thus substituting the above values in(4)

$$s = n + [\frac{n(n + 1)(2n + 1)}{6}] - [\frac{n(n + 1)}{2}]$$

$$s = n + \frac{n(n + 1)}{2}\{[\frac{(2n + 1)}{3}] - [1]\}$$

$$s = n + \frac{n(n + 1)}{2}\{\frac{(2n - 2)}{3}\}$$

$$s = n + \frac{n(n + 1)}{1}\{\frac{(n - 1)}{3}\}$$

4. Question

Sum the following series to n terms :

3 + 7 + 14 + 24 + 37 + ...

Answer

Let S = 3 + 7 + 14 + 24 + 37 +..........

By Shifting each term by one, we get,

S = 3 + 7 + 14 + 24 + 37 + ...+ nth term..........(1)

S = 3 + 7 + 14 + 24 +........ + (n – 1)th term + nth term ...(2)

Substracting equation 2 from equation 1 we get,

0 = 3 + 4 + 7 + 10 + 13 +......+ (nth term – (n – 1)th term) - nth term

Nth term = 3 + 4 + 7 + 10+...nth term – (n – 1)th term

We can see that 3, 4, 7,...is an A.P with first term = 3 and common difference = 3

Sum of this A.P $= \frac{n}{2}[2 \times 3 + (n - 1)3] = \frac{n}{2}(3n + 3)$

Therefore,

S $= \frac{n}{2}(3n+3) - (n-1)\text{th term}$

(n – 1)th term = a + (n – 2)d

(n – 1)th term = 3 + (n – 2)3

(n – 1)th term = 3n – 3

Therefore,

S $= \frac{n}{2}(3n+3) - (3n-3)$

S $= \frac{3n^2-3n+6}{2}$

5. Question

Sum the following series to n terms :

1 + 3 + 6 + 10 + 15 + ...

Answer

Let s = 1 + 3 + 6 + 10 + 15 + ... + n

By shifting each term by one

S = 1 + 3 + 6 + 10 + 15 + ... + nth .... (1)

S = 1 + 3 + 6 + 10 + ... + (n - 1)th + nth ....(2)

by (1) - (2) we get

0 = 1 + (2 + 3 + 4 + 5 + ...nth - (n - 1)th - n)

Nth = 1 + (2 + 3 + 4 + 5 + ...nth - (n - 1)th - nth)

Nth = 1 + (2 + 3 + 4 + ...r + 1) ....(3)

Nth = 1 + (summation upto (n - 1)th term)

we know

$$\sum_{r=1}^{n-1} r + 1 = 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n-1)}{2} + n - 1]$$

Substituting the above-given value in (3)

$$nth = 1 + [\frac{n(n-1)}{2} + n - 1]$$

$$nth = [\frac{n(n-1)}{2} + n]$$

$$nth = [\frac{n(n+1)}{2}]$$

thus

S =3 + 5 + 9 + 15 + 23 + ... + nth $= \sum_{r=1}^{n} [\frac{r(r+1)}{2}]$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2} \ldots\ldots d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2} \ldots\ldots + d_0\sum 1$

$s = \frac{1}{2}\sum_{r=1}^{n} r^2 + r = \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{=1}^{n} r$ ...... (4)

We know

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \dots\dots\dots + n = [\frac{n(n + 1)}{2}]$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \dots\dots\dots + n^2 = [\frac{n(n + 1)(2n + 1)}{6}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + \dots\dots..n \text{ times} = n$$

Thus substituting the above values in(4)

$$s = \frac{1}{2}[\frac{n(n + 1)(2n + 1)}{6}] + \frac{1}{2}[\frac{n(n + 1)}{2}]$$

$$s = \frac{n(n + 1)}{4}\{[\frac{(2n + 1)}{3}] + [1]\}$$

$$s = \frac{n(n + 1)}{4}\{\frac{(2n + 4)}{3}\}$$

$$s = \frac{n(n + 1)}{2}\{\frac{(n + 2)}{3}\}$$

6. Question

Sum the following series to n terms :

1 + 4 + 13 + 40 + 121 + ...

Answer

Let s=1 + 4 + 13 + 40 + 121 + ... + n

By shifting each term by one

S = 1 + 4 + 13 + 40 + 121 + ... + nth ....(1)

S = 1 + 4 + 13 + 40 + ... + (n - 1)th + nth ...(2)

by (1) - (2) we get

0 = 1 + (3 + 9 + 27 + 81 + ...nth - (n - 1)th - n)

Nth = 1 + (3 + 3^2 + 3^3 + 3^4 + ...nth - (n - 1)th - nth)

Nth = 1 + (3 + 3^2 + 3^3 + ...3^{n-1}) ....(3)

we know

$$\sum_{r=0}^{n-1} 3^r = 1 + 3 + 3^2 \dots\dots\dots + 3^{n-1} = [\frac{1(3^n - 1)}{3 - 1}]$$

Substituting the above-given value in (3)

$$nth = [\frac{1(3^n - 1)}{3 - 1}]$$

$$nth = [\frac{(3^n - 1)}{2}]$$

thus

s = 1 + 4 + 13 + 40 + 121 + ... +$[\frac{(3^n-1)}{2}]$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

$s = \frac{1}{2}\sum_{r=1}^{n} 3^r - 1 = \frac{1}{2}\sum_{r=1}^{n} 3^r - \frac{1}{2}\sum_{r=1}^{n} 1$ .... (4)

We know

$$\sum_{r=1}^{n} 3^r = 3 + 3^2 \ldots..3^n = (3^n - 1)\frac{3}{3-1}$$

$$\sum_{r=1}^{n} 3^r = 3 + 3^2 \ldots..3^n = (3^n - 1)\frac{3}{2}$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + \ldots\ldots..\text{n times} = n$$

Thus substituting the above values in(4)

$$s = \frac{1}{2}[(3^n - 1)\frac{3}{3-1}] - \frac{1}{2}[n]$$

$$s = \frac{1}{2}\{[(3^n - 1)\frac{3}{2}] - [n]\}$$

7. Question

Sum the following series to n terms :

4 + 6 + 9 + 13 + 18 + ...

Answer

Let s=4 + 6 + 9 + 13 + 18 + ... + n

shifting each term by one,

s=4 + 6 + 9 + 13 + 18 + ... + nth .....(1)

s= 4 + 6 + 9 + 13 + 18 + ... + (n - 1)th + nth ...(2)

by (1) - (2) we get

0 = 4 + (2 + 3 + 4 + 5 + ...nth - (n - 1)th - nth)

Nth = 4 + (2 + 3 + 4 + 5 + ...nth - (n - 1)th)

Nth = 4 + (2 + 3 + 4 + ...r + 1) ....(3)

Nth = 4 + (summation upto (n - 1)th term)

we know

$$\sum_{r=1}^{n-1} r + 1 = 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n-1)}{2} + n - 1]$$

Substituting the above-given value in (3)

$$nth = 4 + [\frac{n(n-1)}{2} + n - 1]$$

$$nth = [\frac{(n+2)(n-1)}{2} + 4]$$

thus

$s = 4 + 6 + 9 + 13 + 18 + \ldots\ldots\ldots\ldots + nth = \sum_{r=1}^{n}[\frac{(r+2)(r-1)}{2} + 4]$

$s = 4 + 6 + 9 + 13 + 18 + \ldots\ldots\ldots\ldots + nth = \sum_{r=1}^{n}[\frac{r^2+r}{2} + 3]$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}\ldots\ldots. d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}\ldots\ldots.. + d_0\sum 1$

$s = \sum_{r=1}^{n}\frac{r^2+r}{2} + 3 = \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{=1}^{n} r + 3\sum_{=1}^{n} 1$ (4)

We know

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + \ldots\ldots..n\ \text{times} = n$$

Thus substituting the above values in(4)

$$s = \frac{1}{2}[\frac{n(n+1)(2n+1)}{6}] + \frac{1}{2}\left[\frac{n(n+1)}{2}\right] + 3n$$

$$s = \frac{n(n+1)}{4}\{[\frac{(2n+1)+3}{3}]\} + 3n$$

$$s = \frac{n(n+1)}{2}\{\frac{(n+2)}{3}\} + 3n$$

$$s = \frac{n(n+1)}{2}\{\frac{(n+2)}{3}\} + 3n$$

8. Question

Sum the following series to n terms :

2 + 4 + 7 + 11 + 16 + …………

Answer

Let S = 2 + 4 + 7 + 11 + 16 + ………… + n

By shifting each term by one

S = 2 + 4 + 7 + 11 + 16 + ………………. + nth ….(1)

S = 2 + 4 + 7 + 11 + 16 + ………… + (n - 1)th + nth …(2)

by (1) - (2) we get

0 = 2 + (2 + 3 + 4 + 5 + …….nth - (n - 1)th - nth)

Nth = 2 + (2 + 3 + 4 + 5 + …….nth - (n - 1)th)

nth = 2 + (2 + 3 + 4 + …….r + 1) ………..(3)

nth = 2 + (summation upto (n - 1)th term)

we know

$$\sum_{r=1}^{n-1} r + 1 = 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n-1)}{2} + n - 1]$$

Substituting the above-given value in (3)

$$nth = 2 + [\frac{n(n-1)}{2} + n - 1]$$

$$nth = [\frac{(n+2)(n-1)}{2} + 2]$$

thus

s=2 + 4 + 7 + 11 + 16 + ... + nth $=\sum_{r=1}^{n}[\frac{(r+2)(r-1)}{2} + 2]$

s=2 + 4 + 7 + 11 + ... + nth $=\sum_{r=1}^{n}[\frac{r^2+r}{2} + 1]$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2} \ldots\ldots d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2} \ldots\ldots + d_0\sum 1$

$s = \sum_{r=1}^{n}\frac{r^2+r}{2} + 1 = \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{=1}^{n} r + \sum_{=1}^{n} 1$ ....(4)

We know

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + \ldots\ldots..n \text{ times} = n$$

Thus substituting the above values in(4)

$$s = \frac{1}{2}[\frac{n(n+1)(2n+1)}{6}] + \frac{1}{2}\left[\frac{n(n+1)}{2}\right] + n$$

$$s = \frac{n(n+1)}{4}\{[\frac{(2n+1)+3}{3}]\} + n$$

$$s = \frac{n(n+1)}{2}\{\frac{(n+2)}{3}\} + n$$

$$s = \frac{n(n+1)}{2}\{\frac{(n+2)}{3}\} + n$$

9. Question

Sum the following series to n terms :

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots\ldots\ldots..$$

Answer

The general term would be $\frac{1}{(3r-2).(3r+1)}$

The nth term would be $\frac{1}{(3n-2).(3n+1)}$

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}\dots\frac{1}{(3n-2)(3n+1)} = \left[\frac{4-1}{3.1.4}+\frac{7-4}{3.4.7}+\frac{10-7}{3.7.10}\dots\frac{(3n+1)-(3n-2)}{3.(3n-2)(3n+1)}\right]$$

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}\dots\frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\left[\frac{4-1}{1.4}+\frac{7-4}{4.7}+\frac{10-7}{7.10}\dots\frac{(3n+1)-(3n-2)}{(3n-2)(3n+1)}\right]$$

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}\dots\frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\left[1+\frac{-1}{4}+\frac{1}{4}+\frac{-1}{7}+\frac{-1}{7}+\frac{1}{10}\dots\frac{1}{3n-2}-\frac{1}{(3n+1)}\right]$$

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}\dots\frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\left[1-\frac{1}{(3n+1)}\right] = \frac{n}{3n+1}$$

10. Question

Sum the following series to n terms :

$$\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}+\frac{1}{16.21}+\dots\dots+\frac{1}{(5n-4)(5n+1)}$$

Answer

The general term would be $\frac{1}{(5r-4).(5r+1)}$

The nth term would be $\frac{1}{(5n-4).(5n+1)}$

$$\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}\dots\frac{1}{(5n-4)(5n+1)} = \left[\frac{6-1}{5.1.6}+\frac{11-6}{5.6.11}+\frac{16-11}{5.11.16}\dots\frac{(5n+1)-(5n-4)}{5.(5n-4)(5n+1)}\right]$$

$$\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}\dots\frac{1}{(5n-4)(5n+1)} = \frac{1}{5}\left[\frac{6-1}{1.6}+\frac{11-6}{6.11}+\frac{16-11}{11.16}\dots\frac{(5n+1)-(5n-2)}{(5n-4)(5n+1)}\right]$$

$$\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}\dots\frac{1}{(5n-4)(5n+1)} = \frac{1}{5}\left[1+\frac{-1}{6}+\frac{1}{6}+\frac{-1}{11}+\frac{-1}{16}+\frac{1}{11}\dots\frac{1}{5n-4}-\frac{1}{(5n+1)}\right]$$

$$\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}\dots\frac{1}{(5n-4)(5n+1)} = \frac{1}{5}\left[1-\frac{1}{(5n+1)}\right] = \frac{n}{5n+1}$$

1. Question

Find the locus of a point equidistant from the point (2, 4) and the y-axis.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- **AB =**

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k)

As we need to maintain the same distance of (h,k) from (2,4) and y-axis.

So we select a point (0,k) on the y-axis.

From distance formula:

Distance of (h,k) from (2,4) = $\sqrt{(h-2)^2 + (k-4)^2}$

Distance of (h,k) from (0,k) = $\sqrt{(h-0)^2 + (k-k)^2}$

According to question both distances are same.

$\therefore \sqrt{(h-2)^2 + (k-4)^2} = \sqrt{(h-0)^2 + (k-k)^2}$

Squaring both sides:

$(h-2)^2 + (k-4)^2 = (h-0)^2 + (k-k)^2$

$\Rightarrow h^2 + 4 - 4h + k^2 - 8k + 16 = h^2 + 0$

$\Rightarrow k^2 - 4h - 8k + 20 = 0$

Replace (h,k) with (x,y)

Thus, the locus of point equidistant from (2,4) and the y-axis is-

y^2 - 4x - 8y + 20=0 ….ans

2. Question

Find the equation of the locus of a point which moves such that the ratio of its distance from (2, 0) and (1, 3) is 5 : 4.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the point whose locus is to be determined to be (h,k)

Distance of (h,k) from (2,0) = $\sqrt{(h-2)^2 + (k-0)^2}$

Distance of (h,k) from (1,3) = $\sqrt{(h-1)^2 + (k-3)^2}$

According to the question:

$$\frac{\sqrt{(h-2)^2 + (k-0)^2}}{\sqrt{(h-1)^2 + (k-3)^2}} = \frac{5}{4}$$

Squaring both sides:

$16\{(h - 2)^2 + k^2\} = 25\{(h - 1)^2 + (k - 3)^2\}$

⇒ **16{h^2 + 4 - 4h + k^2 } = 25{h^2 - 2h + 1 + k^2 - 6k + 9}**

⇒ **9h^2 + 9k^2 + 14h - 150k + 186 = 0**

Replace (h,k) with (x,y)

Thus, the locus of a point which moves such that the ratio of its distance from (2, 0) and (1, 3) is 5 : 4 is –

9x^2 + 9y^2 + 14x - 150y + 186 = 0 ….ans

3. Question

A point moves as so that the difference of its distances from (ae, 0) and (-ae, 0) is 2a, prove that the equation to its locus is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where b^2 = a^2(e^2 – 1).

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the point whose locus is to be determined be (h,k)

Distance of (h,k) from (ae,0) = $\sqrt{(h-ae)^2+(k-0)^2}$

Distance of (h,k) from (-ae,0) = $\sqrt{(h-(-ae))^2+(k-0)^2}$

According to question:

$$\sqrt{(h-ae)^2+(k-0)^2}-\sqrt{(h-(-ae))^2+(k-0)^2}=2a$$

⇒ $\sqrt{(h-ae)^2+(k-0)^2}=2a+\sqrt{(h+ae)^2+(k-0)^2}$

Squaring both sides:

$$(h-ae)^2+(k-0)^2=\left\{2a+\sqrt{(h+ae)^2+(k-0)^2}\right\}^2$$

$$\Rightarrow h^2+a^2e^2-2aeh+k^2 = 4a^2+\{(h+ae)^2+k^2\}+4a\sqrt{(h+ae)^2+(k-0)^2}$$

$$\Rightarrow h^2+a^2e^2-2aeh+k^2 = 4a^2+h^2+2aeh+a^2e^2+k^2+4a\sqrt{(h+ae)^2+(k-0)^2}$$

⇒ $-4aeh-4a^2=4a\sqrt{(h+ae)^2+(k-0)^2}$

⇒ $-4a(eh+a)=4a\sqrt{(h+ae)^2+(k-0)^2}$

Again squaring both sides:

$$(eh+a)^2=(h+ae)^2+(k-0)^2$$

⇒ $e^2h^2+a^2+2aeh=h^2+a^2e^2+2aeh+k^2$

⇒ $h^2(e^2-1)-k^2=a^2(e^2-1)$

$$\therefore \frac{h^2}{a^2}-\frac{k^2}{a^2(e^2-1)}=1$$

$\Rightarrow \frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$

Replace (h,k) with (x,y)

Thus, the locus of a point such that the difference of its distances from (ae, 0) and (-ae, 0) is 2a:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$ **....proved**

4. Question

Find the locus of a point such that the sum of its distances from (0, 2) and (0, -2) is 6.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- **AB =** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the point whose locus is to be determined to be (h,k)

Distance of (h,k) from (0,2) = $\sqrt{(h-0)^2 + (k-2)^2}$

Distance of (h,k) from (0,-2) = $\sqrt{(h-0)^2 + (k-(-2))^2}$

According to the question:

$\sqrt{(h)^2 + (k-2)^2} + \sqrt{(h)^2 + (k+2)^2} = 6$

$\Rightarrow \sqrt{(h)^2 + (k-2)^2} = 6 - \sqrt{(h)^2 + (k+2)^2}$

Squaring both sides:

$h^2 + (k-2)^2 = \left\{6 - \sqrt{h^2 + (k+2)^2}\right\}^2$

$\Rightarrow h^2 + 4 - 4k + k^2 = 36 + \{h^2 + k^2 + 4k + 4\} - 12\sqrt{h^2 + (k+2)^2}$

$\Rightarrow -8k - 36 = -12\sqrt{h^2 + (k+2)^2}$

$\Rightarrow -4(2k+9) = -12\sqrt{h^2 + (k+2)^2}$

Again squaring both sides:

$(2k+9)^2 = \left\{3\sqrt{h^2 + (k+2)^2}\right\}^2$

$\Rightarrow 4k^2 + 81 + 36k = 9(h^2 + k^2 + 4k + 4)$

$\Rightarrow 9h^2 + 5k^2 = 45$

Replace (h,k) with (x,y)

Thus, the locus of a point such that sum of its distances from (0,2) and (0,-2) is 6:

$9x^2 + 5y^2 = 45$ **....proved**

5. Question

Find the locus of a point which is equidistant from (1, 3) and x-axis.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k)

As we need to maintain the same distance of (h,k) from (2,4) and x-axis.

So we select a point (h,0) on the x-axis.

From distance formula:

Distance of (h,k) from (1,3) = $\sqrt{(h-1)^2+(k-3)^2}$

Distance of (h,k) from (h,0) = $\sqrt{(h-h)^2+(k-0)^2}$

According to question both distances are same.

$\therefore \sqrt{(h-1)^2+(k-3)^2} = \sqrt{(h-h)^2+(k-0)^2}$

Squaring both sides:

$(h-1)^2+(k-3)^2 = (h-h)^2+(k-0)^2$

$\Rightarrow h^2+1-2h+k^2-6k+9 = k^2+0$

$\Rightarrow h^2-2h-6k+10 = 0$

Replace (h,k) with (x,y)

Thus, the locus of a point equidistant from (1,3) and x-axis is-

x^2 - 2x - 6y + 10 = 0 ....ans

6. Question

Find the locus of a point which moves such that its distance from the origin is three times is the distance from the x-axis.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k)

As we need to maintain a distance of (h,k) from origin such that it is 3 times the distance from the x-axis.

So we select a point (h,0) on the x-axis.

From distance formula:

Distance of (h,k) from (0,0) = $\sqrt{(h-0)^2+(k-0)^2}$

Distance of (h,k) from (h,0) = $\sqrt{(h-h)^2+(k-0)^2}$

According to question both distances are same.

$\therefore \sqrt{(h-0)^2+(k-0)^2} = 3\sqrt{(h-h)^2+(k-0)^2}$

Squaring both sides:

$\mathbf{h^2 + k^2 = 9k^2}$

$\Rightarrow \mathbf{h^2 = 8k^2}$

Replace (h,k) with (x,y)

Thus, the locus of a point is $\mathbf{x^2 = 8y^2}$ **…….ans**

7. Question

A(5, 3), B(3, -2) are two fixed points, find the equation to the locus of a point P which moves so that the area of the triangle PAB is 9 units.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- **AB =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

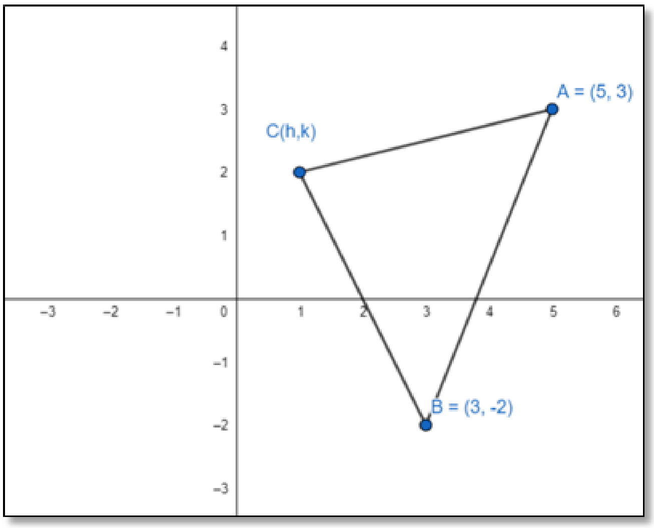
Area of a ΔPQR – Let $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(x_3,y_3)$ be the 3 vertices of ΔPQR.

Ar(ΔPQR) = $\frac{1}{2}|x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)|$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k). Name the moving point to be C

Given the area of ΔABC = 9



According to question:

$\mathbf{9} = \frac{1}{2}|5(-2-k)+3(k-3)+h\left((3-(-2)\right)|$

$\Rightarrow$ **18=|-10-5k+3k-9+3h+2h|**

$\Rightarrow$ **|5h-2k-19|=18**

$\therefore$ **5h-2k-19=18 or 5h-2k-19= -18**

$\Rightarrow$ **5h-2k-37=0 or 5h-2k-1=0**

Replace (h,k) with (x,y)

Thus, locus of point is **5x-2y-37=0 or 5x-2y-1=0 …….ans**

8. Question

Find the locus of a point such that the line segments having end points (2, 0) and (-2, 0) subtend a right angle at that point.

Answer

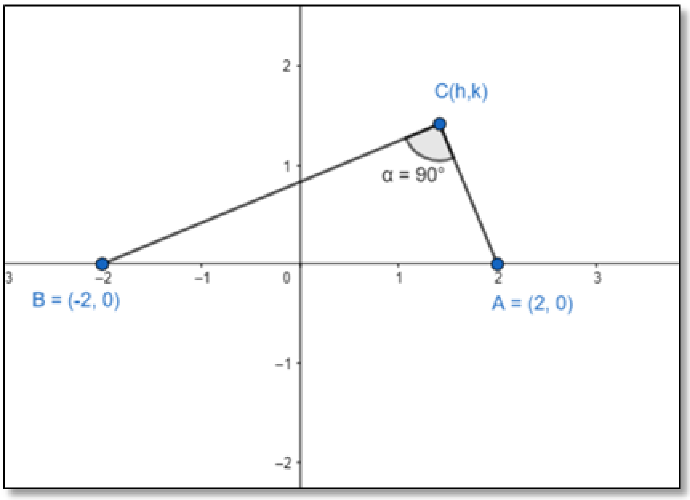
Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- **AB =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

• **Pythagoras theorem:** In right triangle ΔABC : the sum of the square of two sides is equal to the square of its hypotenuse.

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k) and name the moving point to be C.



According to a question on drawing the figure, we get a right triangle Δ ABC.

From Pythagoras theorem we have:

$BC^2 + AC^2 = AB^2$

From distance formula:

BC = $\sqrt{(h-(-2))^2+(k-0)^2}$

AC = $\sqrt{(h-2)^2+(k-0)^2}$

And AB = 4

$\therefore \left\{\sqrt{(h-(-2))^2+(k-0)^2}\right\}^2+\left\{\sqrt{(h-2)^2+(k-0)^2}\right\}^2=16$

$\Rightarrow (h+2)^2+k^2+(h-2)^2+k^2=16$

$\Rightarrow h^2+4+4h+k^2+h^2-4h+4+k^2=16$

$\Rightarrow 2h^2 + 2k^2 - 8 = 0$

$\Rightarrow h^2 + k^2 = 4$

Replace (h,k) with (x,y)

Thus, the locus of a point is **$x^2 + y^2 = 4$ ....ans**

9. Question

If A (-1, 1) and B (2, 3) are two fixed points, find the locus of a point P so that the area d ΔPAB = 8 sq. units.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

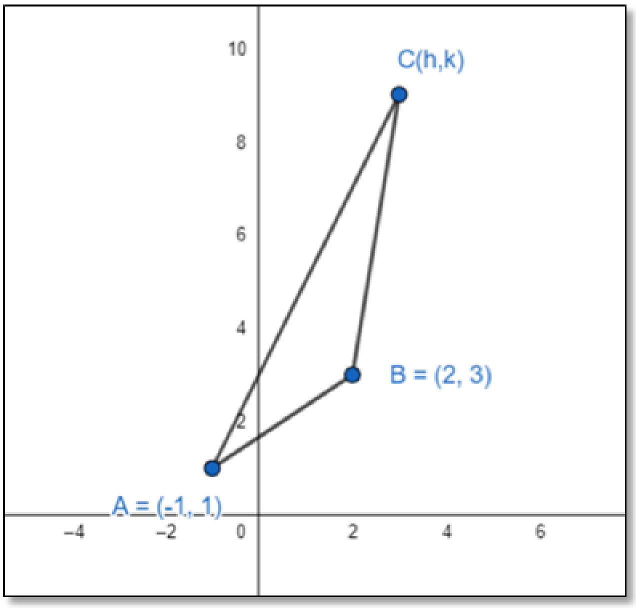
Area of a ΔPQR – Let P(x_1,y_1) , Q(x_2,y_2) and R(x_3,y_3) be the 3 vertices of ΔPQR.

Ar(ΔPQR) = $\frac{1}{2}|x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)|$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k). Name the moving point to be C

Given the area of ΔABC = 8



According to question:

$$8 = \frac{1}{2}|-1(3-k) + 2(k-1) + h((1-3)|$$

⇒ **16=|-3+k+2k-2+h-3h|**

⇒ **|3k-2h-5|=16**

∴ **3k-2h-5=16 or 3k-2h-5= -16**

⇒ **3k-2h-21=0 or 3k-2h+11=0**

Replace (h,k) with (x,y)

Thus, locus of point is **3y-2x-21=0 or 3y-2x+11=0 ....ans**

10. Question

A rod of length l slides between the two perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1 : 2.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

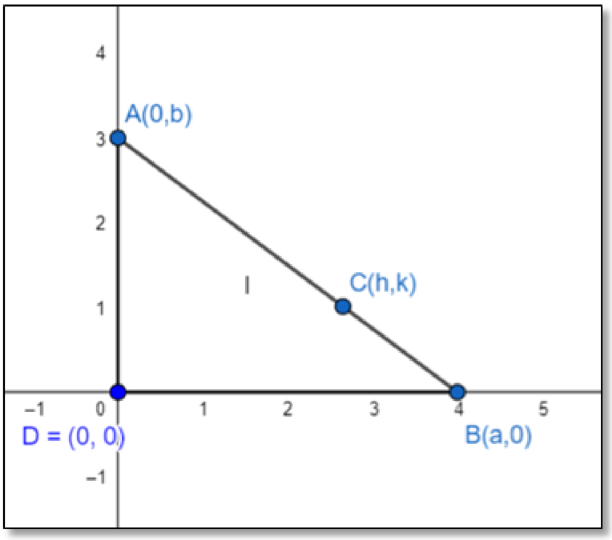
• **Idea of section formula-** Let two points A(x_1,y_1) and B(x_2,y_2) forms a line segment. If a point C(x,y) divides line segment AB in the ratio of m:n internally, then coordinates of C is given as:

C = $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k). Name the moving point to be C

Assume the two perpendicular lines on which rod slides are x and y-axis respectively.



Here line segment AB represents the rod of length l also ΔADB formed is a right triangle. Coordinates of A and B are assumed to be (0,b) and (a,0) respectively.

∴ $a^2 + b^2 = l^2$**...eqn 1**

As, (h,k) divides AB in ratio of 1:2

∴ from section formula we have coordinate of point C as-

C = $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$ **=** $\left(\frac{1\times0+2\times a}{2+1},\frac{1\times b+2\times0}{2+1}\right)=\left(\frac{2a}{3},\frac{b}{3}\right)$

As, a and b are assumed parameters so we have to remove it.

∵ h = 2a/3 ⇒ a = 3h/2

And k = b/3 ⇒ b = 3k

From eqn 1:

$a^2 + b^2 = l^2$

∴ $\left(\frac{3h}{2}\right)^2+(3k)^2=l^2$

⇒ $\frac{9h^2}{4}+9k^2=l^2 \;\Rightarrow \frac{h^2}{4}+k^2=\frac{l^2}{9}$

Replace (h,k) with (x,y)

Thus, the locus of a point on the rod is: $\frac{x^2}{4}+y^2=\frac{l^2}{9}$ **....ans**

11. Question

Find the locus of the mid-point of the portion of the x cosα + y sin α = p which is intercepted between the axes.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- **AB =** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

• **Idea of section formula-** Let two points A(x_1,y_1) and B(x_2,y_2) forms a line segment. If a point C(x,y) divides line segment AB in the ratio of m:n internally, then coordinates of C is given as:

C = $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ when m = n =1 , C becomes the midpoint of AB and C is given as **C =** $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k). Name the moving point to be C

Given that (h,k) is the midpoint of line x cos α + y sin α = p intercepted between axes.

So we need to find the points at which x cos α + y sin α = p cuts the axes after which we will apply the section formula to get the locus.

Put y = 0

∴ x = p/cos α ⇒ coordinates on x-axis is (p/cos α , 0). Name the point A

Similarly, Put x = 0

∴ y = p/sin α ⇒ coordinates on y-axis is (0, p/sin α). Name this point B

As C(h,k) is the midpoint of AB

∴ coordinate of C is given by:

C = $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right) = \left(\frac{\frac{p}{\cos\alpha}+0}{2}, \frac{0+\frac{p}{\sin\alpha}}{2}\right) = \left(\frac{p}{2\cos\alpha}, \frac{p}{2\sin\alpha}\right)$

Thus,

$h = \frac{p}{2\cos\alpha} \Rightarrow \frac{p}{2h} = \cos\alpha$ **...equation 1**

and $k = \frac{p}{2\sin\alpha} \Rightarrow \frac{p}{2k} = \sin\alpha$ **...equation 2**

Squaring and adding equation 1 and 2:

$$\frac{p^2}{4h^2} + \frac{p^2}{4k^2} = \cos^2\alpha + \sin^2\alpha$$

$$\Rightarrow \frac{p^2}{4h^2} + \frac{p^2}{4k^2} = 1$$

Replace (h,k) with (x,y)

Thus, the locus of a point on the rod is: $\frac{p^2}{4x^2} + \frac{p^2}{4y^2} = 1$ **....ans**

12. Question

If O is the origin and Q is a variable point on y^2 = x, Find the locus of the mid-point of OQ.

Answer

Key points to solve the problem:

• **Idea of section formula-** Let two points A(x_1,y_1) and B(x_2,y_2) forms a line segment. If a point C(x,y) divides line segment AB in the ratio of m:n internally, then coordinates of C is given as:

C = $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ when m = n =1 , C becomes the midpoint of AB and C is given as **C =** $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k). Name the moving point to be C

As, coordinate of mid point is (h,k) {by our assumption},

Let Q(a,b) be the point such that Q lies on curve $y^2 = x$

$b^2 = a$ **......equation 1**

According to question C is the midpoint of OQ

∵ C = $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$ ⇒ **C =** $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

$\therefore h = \frac{a}{2} \text{ or } a = 2h$

Similarly, $k = \frac{b}{2} \text{ or } b = 2k$

Putting values of a and b in equation 1,we have:

$(2k)^2 = 2h \Rightarrow 4k^2 = 2h \Rightarrow 2k^2 = h$

Replace (h,k) with (x,y)

Thus, the locus of a point is: $2y^2 = x$ **....ans**

Very Short Answer

1. Question

Write the sum of the series : 2 + 4 + 6 + 8 + ... + 2n

Answer

Let S = 2 + 4 + 6 + 8 + ... + 2n

S = 2(1 + 2 + 3 + 4 + ... + n)

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

Substituting the above value

$$S = 2[\frac{n(n+1)}{2}]$$

S = n(n + 1)

2. Question

Write the sum of the series : $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots\ldots + (2n-1)^2 - (2n)^2$

Answer

$S=1^2 + 3^2 + 5^2 \ldots\ldots\ldots + (2n-1)^2 - \{2^2 + 4^2 + 6^2 + \ldots\ldots + (2n)^2\}$

$S=1^2 + 3^2 + 5^2 \ldots\ldots\ldots + (2n-1)^2 - 2^2\{1^2 + 2^2 + 3^2 + \ldots\ldots + (n)^2\}$ ....(1)

We know

$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$ ....(2)

$$\sum_{r=1}^{2n} r^2 = 1^2 + 2^2 + 3^2 \ldots (2n-1)^2 + 4n^2 = [\frac{2n(2n+1)(4n+1)}{6}]$$

$$1^2 + 3^2 \ldots (2n-1)^2 + 2^2 + 4^2 \ldots + (2n)^2 = [\frac{2n(2n+1)(4n+1)}{6}]$$

$$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{2n(2n+1)(4n+1)}{6}] - 2^2 - 4^2 \ldots - (2n)^2$$

$$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{2n(2n+1)(4n+1)}{6}] - 2^2\{1^2 + 2^2 \ldots + (n)^2\} \ldots(3)$$

Substituting (3) in (1), we get,

$$s = [\frac{2n(2n+1)(4n+1)}{6}] - 2^2\{1^2 + 2^2 \ldots + (n)^2\} - 2^2\{1^2 + 2^2 + \ldots \ldots . + (n)^2\}$$

$$s = [\frac{2n(2n+1)(4n+1)}{6}] - (2)2^2\{1^2 + 2^2 \ldots + (n)^2\}$$

Substituting (2) in the above equation

$$s = [\frac{2n(2n+1)(4n+1)}{6}] - (2)2^2\{[\frac{n(n+1)(2n+1)}{6}]\}$$

$$s = [\frac{2n(2n+1)(4n+1) - 8n(n+1)(2n+1)}{6}]$$

3. Question

Write the sum to n terms of a series whose rth term is: $r + 2^r$

Answer

The general term be = $i + 2^i$

$\sum_{i=1}^{n} i + 2^i$ = Sum of series

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2} \ldots\ldots d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2} \ldots\ldots + d_0\sum 1$

Thus

$$\sum_{i=1}^{n} i + 2^i = \sum_{i=1}^{n} i + \sum_{i=1}^{n} 2^i$$

$$\sum_{i=1}^{n} i + 2^i = \sum_{i=1}^{n} i + 2(2^n - 1)$$

$$\sum_{n=1}^{n} n = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

Substituting the above value

Thus

$$\sum_{i=1}^{n} i + 2i = \left[\frac{n(n+1)}{2}\right] + 2(2^n - 1)$$

4. Question

If $\sum_{r=1}^{n} r = 55$, find $\sum_{r=1}^{n} r^3$.

Answer

$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots . n = \frac{n(n+1)}{2}$ .... (1)

Given

$$\sum_{r=1}^{n} r = 55$$

From (1) we have

$$\frac{n(n + 1)}{2} = 55$$

Solving the above equation

n = 10

We know

$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \ldots \ldots \ldots + n^3 = [\frac{n(n+1)}{2}]^2$ ....(2)

Thus

Putting n = 10 in eq(2)

$$= [\frac{10(10 + 1)}{2}]^2$$

$= 55^2$

= 3025

5. Question

If the sum of first n even natural numbers is equal to k times the sum of first n odd natural numbers, then write the value of k.

Answer

we know

$$\sum_{r=1}^{2n} r = 1 + 2 + 3 \ldots .2n = \frac{2n(2n + 1)}{2}$$

$\sum_{r=1}^{n} 2r = 2 + 4 + 6 \ldots .2n = \frac{2n(n+1)}{2}$ ......(1)

Given 2 + 4 + 6 + 8.......2n=k(1 + 3 + 5.......2n - 1)

From (1) n(n + 1)=k(1 + 3 + 5...2n - 1) ....(2)

$1 + 3 + 5 \ldots . (2n - 1) + 2 + 4 + \ldots 2n = \frac{2n(2n+1)}{2}$ .... (3)

Thus substituting the values from (1) in (3), we get,

$$1 + 3 + 5 \ldots . (2n - 1) + \frac{2n(n + 1)}{2} = \frac{2n(2n + 1)}{2}$$

1 + 3 + 5...(2n - 1) = n(2n + 1) - n(n + 1)

1 + 3 + 5...(2n - 1) = n^2 ....(4)

Substituting (4) in (2), we get,

$n(n + 1) = k(n^2)$

$k = (n + 1)/n$

6. Question

Write the sum of 20 terms of the series :

$$1+\frac{1}{2}(1+2)+\frac{1}{3})1+2+3)+\ldots\ldots.$$

Answer

The general term would be

$$\frac{r + (r-1) + (r-2) \ldots..(r-(r-1))}{r}$$

$$=\frac{r\times r-(1+2+3\ldots r-1)}{r} \ldots\ldots\ldots..(1)$$

Since, $\sum_{i=1}^{r} i = 1 + 2 + 3 \ldots\ldots\ldots + r = [\frac{r(r+1)}{2}]$

$$\sum_{i=1}^{r-1} i = 1 + 2 + 3 \ldots\ldots\ldots + r-1 = [\frac{r(r-1)}{2}]$$

From equation (1),we get,

$$=\frac{r\times r-\frac{r(r-1)}{2}}{r}$$

$$r-\frac{(r-1)}{2}=\frac{(r+1)}{2}$$

Thus the general term would be, $\frac{(r+1)}{2}$

To find $\sum_{r=1}^{20}\frac{(r+1)}{2}$ (2)

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}\ldots\ldots.d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}\ldots\ldots.. + d_0\sum 1$

Since ,

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + 1 \ldots\ldots\ldots + 1 = [n]$$

Thus equation (2) becomes

$$\sum_{r=1}^{20}\frac{(r+1)}{2}=\frac{1}{2}\sum_{r=1}^{20} r + \frac{1}{2}\sum_{r=1}^{20} 1$$

$$\sum_{r=1}^{20}\frac{(r+1)}{2}=\frac{1}{2}\left[\frac{n(n+1)}{2}\right] + \frac{n}{2} = 95$$

7. Question

Write the 50^{th} term of the series 2 + 3 + 6 + 11 + 18 + ...

Answer

Let s=2 + 3 + 6 + 11 + 18 + ... + n

By shifting each term by one

S =2 + 3 + 6 + 11 + 18 + ... + nth .... (1)

S = 2 + 3 + 6 + 11 + 18 + ... + (n - 1)th + nth ...(2)

by (1) - (2) we get

0 = 2 + 1 + 3 + 5 + 7 + ...nth - (n - 1)th - nth

Nth = 2 + (1 + 3 + 5 + 7 + 9 + ...2r - 1) ....(3)

Nth = 2 + (summation of first (n - 1)th term)

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2} ....d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2} .... + d_0\sum 1$

Therefore,

$$\sum_{r=1}^{n-1} 2r - 1 = 2\sum_{r=1}^{n-1} r - \sum_{r=1}^{n-1} 1 = 1 + 3 + 5 \ldots\ldots = [n-1]^2$$

Since,

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n+1)}{2}]$$

$$\sum_{r=1}^{n-1} r = 1 + 2 + 3 \ldots\ldots\ldots + n = [\frac{n(n-1)}{2}]$$

$$\sum_{r=1}^{n} 1 = 1 + 1 + 1 \ldots\ldots\ldots + 1 = [n]$$

$$\sum_{r=1}^{n-1} 1 = 1 + 1 + 1 \ldots\ldots\ldots + 1 = [n-1]$$

Thus from (3)

Nth = $2 + (n - 1)^2$

Hence 50^{th} term be

$50^{th} = 2 + (50 - 1)^2$

$50^{th} = 2 + (49)^2$

8. Question

Let S_n denote the sum of the cubes of first n natural numbers, and s_n denote the sum of first n natural numbers. Then write the value of $\sum_{r=1}^{n} \frac{S_r}{s_r}$

Answer

To find

Let I = $\sum_{r=1}^{n} \frac{S_r}{s_r}$ ...... (1)

Given,

$$S_r = \sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 \dots\dots\dots + n^3 = [\frac{n(n+1)}{2}]^2$$

$$s_r = \sum_{r=1}^{n} r = 1 + 2 + 3 \dots\dots\dots + n = [\frac{n(n+1)}{2}]$$

Substituting in equation (1)

$$I = \sum_{r=1}^{n} \frac{[\frac{n(n+1)}{2}]^2}{[\frac{n(n+1)}{2}]}$$

$$I = \sum_{r=1}^{n} [\frac{n(n+1)}{2}]$$

$$I = \sum_{r=1}^{n} \frac{[n^2+n]}{2}$$

We know by property that:

$\sum ax^n + bx^{n-1} + cx^{n-2}.......d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}........ + d_0\sum 1$

Thus

$$I = \frac{1}{2}\sum_{r=1}^{n} n^2 + \frac{1}{2}\sum_{r=1}^{n} n$$

And We know

$$\sum_{r=1}^{n} r = 1 + 2 + 3 \dots\dots\dots + n = [\frac{n(n+1)}{2}]$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \dots\dots\dots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$$

Substituting the values

$$I = \frac{1}{2}[\frac{n(n+1)(2n+1)}{6}] + \frac{1}{2}[\frac{n(n+1)}{2}]$$

$$I = \frac{n(n+1)}{4}\{[\frac{(2n+1)}{3}] + [1]\}$$

$$I = \frac{n(n+1)}{4}\{[\frac{(2n+4)}{3}]\}$$

$$I = \frac{n(n+1)}{2}\{[\frac{(n+2)}{3}]\}$$

MCQ

1. Question

The sum to n terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + ...\text{is}$

A. $\sqrt{2n+1}$

B. $\frac{1}{2}\sqrt{2n+1}$

C. $\sqrt{2n+1}-1$

D. $\frac{1}{2}\left\{\sqrt{2n+1}-1\right\}$

Answer

To find:

$$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}\dots\frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}$$

Rationalizing the above equation:

$$=\frac{1(-\sqrt{1}+\sqrt{3})}{(\sqrt{1}+\sqrt{3})(-\sqrt{1}+\sqrt{3})}+\frac{1(-\sqrt{3}+\sqrt{5})}{(\sqrt{3}+\sqrt{5})(-\sqrt{3}+\sqrt{5})}$$
$$+\frac{1(-\sqrt{5}+\sqrt{7})}{(\sqrt{5}+\sqrt{7})(-\sqrt{5}+\sqrt{7})}\dots\frac{1}{(\sqrt{2n-1}+\sqrt{2n+1})}\left\{\frac{(-\sqrt{2n-1}+\sqrt{2n+1})}{(-\sqrt{2n-1}+\sqrt{2n+1})}\right\}$$

$$=\frac{1(-\sqrt{1}+\sqrt{3})}{3-1}+\frac{1(-\sqrt{3}+\sqrt{5})}{5-3}$$
$$+\frac{1(-\sqrt{5}+\sqrt{7})}{7-5}\dots\left\{\frac{(-\sqrt{2n-1}+\sqrt{2n+1})}{2n+1-(2n-1)}\right\}$$

since $a^2 - b^2 = (a + b)(a - b)$

$$=\frac{1}{2}[(-\sqrt{1}+\sqrt{3})+(-\sqrt{3}+\sqrt{5}+(-\sqrt{5}+\sqrt{7})+\dots(-\sqrt{2n-1}$$
$$+\sqrt{2n+1})]$$

$$=\frac{1}{2}[-1+\sqrt{2n+1})]$$

2. Question

The sum of the series : $\frac{1}{\log_2 4}+\frac{1}{\log_4 4}+\frac{1}{\log_8 4}+\dots+\frac{1}{\log_{2^n} 4}$ is

A. $\frac{n(n+1)}{2}$

B. $\frac{n(n+1)(2n+1)}{12}$

C. $\frac{n(n+1)}{4}$

D. None of these

Answer

$$\frac{1}{\log_2 4}+\frac{1}{\log_4 4}+\frac{1}{\log_8 4}\dots\dots\frac{1}{\log_{2^n} 4}$$

We know $\log_a b = \log b/\log a$

$$\frac{\log 2}{\log 4}+\frac{\log 4}{\log 4}+\frac{\log 8}{\log 4}\ldots\ldots\ldots\frac{\log 2^n}{\log 4}$$

We know $\log m^n = n\log m$

$$=\frac{\log 2}{\log 4}+\frac{\log 2^2}{\log 4}+\frac{\log 2^3}{\log 4}\ldots\ldots\ldots\frac{\log 2^n}{\log 4}$$

$$=\frac{\log 2}{\log 4}+\frac{2\log 2}{\log 4}+\frac{3\log 2}{\log 4}\ldots\ldots\ldots\frac{n\log 2}{\log 4}$$

$$=\frac{\log 2[1+2+3\ldots.n]}{\log 4}$$

$$=\frac{\log 2[1+2+3\ldots.n]}{2\log 2}$$

= [1 + 2 + 3.......n]/2

= n(n + 1)/4

3. Question

The value of $\sum_{r=1}^{n}\left\{(2r-1)a+\frac{1}{b^r}\right\}$ is equal to

A. $an^2+\frac{b^{n-1}-1}{b^{n-1}(b-1)}$

B. $an^2+\frac{b^n-1}{b^n(b-1)}$

C. $an^3+\frac{b^{n-1}-1}{b^n(b-1)}$

D. none of these

Answer

We know by property that $\sum ax^n + bx^{n-1} + cx^{n-2}\ldots\ldots.d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2}\ldots\ldots. + d_0\sum 1$

$$\sum_{r=1}^{n}(2r-1)a+\frac{1}{b^r}=2a\sum_{r=1}^{n}r-a\sum_{r=1}^{n}1+\sum_{r=1}^{n}\frac{1}{b^r}$$

We know

$$\sum_{r=1}^{n}r=1+2+3\ldots\ldots\ldots+n=[\frac{n(n+1)}{2}]$$

$$\sum_{r=1}^{n}r^2=1^2+2^2+3^2\ldots\ldots\ldots+n^2=[\frac{n(n+1)(2n+1)}{6}]$$

$$\sum_{r=1}^{n}1=1+1+\ldots\ldots.n\text{ times}=n$$

$2a\sum_{r=1}^{n} r = \frac{2an(n+1)}{2}$ (1)

$a\sum_{r=1}^{n} 1 = an$ (2)

$$\sum_{r=1}^{n} \frac{1}{b^r} = \frac{1/b\left[1-\left(\frac{1}{b}\right)^n\right]}{\left[1-\frac{1}{b}\right]}$$

$$\sum_{r=1}^{n} \frac{1}{b^r} = \frac{1\left[1-\left(\frac{1}{b}\right)^n\right]b}{b[b-1]}$$

$$\sum_{r=1}^{n} \frac{1}{b^r} = \frac{\left[1-\left(\frac{1}{b}\right)^n\right]}{[b-1]}$$

$$\sum_{r=1}^{n} \frac{1}{b^r} = \frac{\left[1-\left(\frac{1}{b}\right)^n\right]}{[b-1]}$$

$$\sum_{r=1}^{n} \frac{1}{b^r} = \frac{\left[1-\left(\frac{1}{b}\right)^n\right]}{[b-1]}$$

$$\sum_{r=1}^{n} \frac{1}{b^r} = \frac{1-\left(\frac{1}{b}\right)^n}{[b-1]}$$

$\sum_{r=1}^{n} \frac{1}{b^r} = \frac{b^n-1}{b^n[b-1]}$ (3)

Adding (1) (2) and (3)

$$\sum_{r=1}^{n} (2r-1)a + \frac{1}{b^r} = 2a\sum_{r=1}^{n} r - a\sum_{r=1}^{n} 1 + \sum_{r=1}^{n} \frac{1}{b^r}$$

$$= \frac{2an(n+1)}{2} - an + \frac{b^n-1}{b^n[b-1]}$$

$$= \frac{b^n-1}{b^n[b-1]} + an^2$$

4. Question

If $\sum n = 210$, then $\sum n^2 =$

A. 2870

B. 2160

C. 2970

D. none of these

Answer

$$\sum n = \frac{n(n+1)}{2} = 210$$

Solving we get n=20

$$\sum n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Substituting the value

$$\frac{20(20 + 1)(40 + 1)}{6}$$

10(7)(41)

2870

5. Question

If $S_n = \sum_{r=1}^{n} \frac{1+2+2^2+\ldots\text{Sum to r terms}}{2^r}$, then S_n is equal to

A. $2^n - n - 1$

B. $1 - \frac{1}{2^n}$

C. $n - 1 + \frac{1}{2^n}$

D. $2^n - 1$

Answer

$1 + 2 + 2^2 + \ldots\ldots 2^{r-1} = 1(2^r - 1)/2 - 1$

$1 + 2 + 2^2 + \ldots\ldots 2^{r-1} = 1(2^r - 1)$

$$\sum_{r=1}^{n} \frac{(2^r - 1)}{2^r}$$

$$\sum_{r=1}^{n} 1 - \frac{1}{2^r}$$

$$\sum_{r=1}^{n} 1 - \sum_{r=1}^{n} \frac{1}{2^r}$$

$$n - \frac{1}{2}\left[\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right]$$

$$n - 1 + \frac{1}{2^n}$$

6. Question

If $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \ldots\ldots$ to n terms is S. Then, S is equal to

A. $\frac{n(n+3)}{4}$

B. $\frac{n(n+2)}{4}$

C. $\frac{n(n+1)(n+2)}{6}$

D. n^2

Answer

$$1+\frac{1+2}{2}+\frac{1+2+3}{3}\ldots\frac{1+2+3\ldots n}{n}$$

$$\frac{1+2+3\ldots n}{n}=\frac{n(n+1)}{2n}$$

$$\frac{1+2+3\ldots n}{n}=\frac{(n+1)}{2}$$

Thus the nth term would be

nth=(n + 1)/2

the general term would be

rth=(r + 1)/2

$$\sum_{r=1}^{n}\frac{r+1}{2}$$

We know by property that $\sum ax^n + bx^{n-1} + cx^{n-2} \ldots\ldots d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2} \ldots\ldots + d_0\sum 1$

$$\sum_{r=1}^{n}\frac{r+1}{2}=1/2\left[\sum_{r=1}^{n} r+\sum_{r=1}^{n} 1\right]$$

$$=\frac{1}{2}\left[\frac{n(n+1)}{2}+n\right]$$

$$=\frac{1}{2}\left[\frac{n(n+1)+2n}{2}\right]$$

$$=\frac{1}{2}\left[\frac{n^2+3n}{2}\right]$$

$$=\left[\frac{n(n+3)}{4}\right]$$

7. Question

Sum of n terms of the series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+\ldots$ is

A. $\frac{n(n+1)}{2}$

B. 2n(n + 1)

C. $\frac{n(n+1)}{\sqrt{2}}$

D. 1

Answer

Let S = √2 + √8 + √18 + √32 + ...

It can be written as,

S = √2(1 + 2 + 3 + ...n)

We know that,

$1 + 2 + 3 + 4 + \cdots ... + n = \frac{n(n+1)}{2}$

$S = \frac{\sqrt{2}(n(n+1))}{2}$

$S = \frac{n(n+1)}{\sqrt{2}}$

8. Question

The sum of 10 terms of the series $\sqrt{2} + \sqrt{6} + \sqrt{18} + ...$ is

A. $121(\sqrt{6} + \sqrt{2})$

B. $243(\sqrt{3} + 1)$

C. $\frac{121}{\sqrt{3} - 1}$

D. $242(\sqrt{3} - 1)$

Answer

Let S = √2 + √6 + √18 + ...

It can also be written as,

S = √2(1 + √3 + 3 + 3√3 + ...)

Now,

1 + √3 + 3 + 3√3 + ... is a G.P with common ratio √3

Sum of the 10 terms will be given by,

$S_{10} = \sqrt{2}\left(1 \frac{(\sqrt{3})^{10} - 1}{\sqrt{3} - 1}\right)$

$S_{10} = \sqrt{2}\left(\frac{3^5 - 1}{\sqrt{3} - 1}\right)$

$S_{10} = \sqrt{2}\left(\frac{243 - 1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}\right)$

$S_{10} = \sqrt{2}(121)(\sqrt{3} + 1)$

S_{10} = 121(√6 + √2)

9. Question

The sum of the series $1^2 + 3^2 + 5^2 + ...$ to n terms is

A. $\frac{n(n+1)(2n+1)}{2}$

B. $\frac{n(2n-1)(2n+1)}{3}$

C. $\frac{(n-1)^2(2n-1)}{6}$

D. $\frac{(2n+1)^3}{3}$

Answer

We know

$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = [\frac{n(n+1)(2n+1)}{6}]$ ....(1)

$$\sum_{r=1}^{2n} r^2 = 1^2 + 2^2 + 3^2 \ldots (2n-1)^2 + 4n^2 = [\frac{2n(2n+1)(4n+1)}{6}]$$

$$S = 1^2 + 3^2 \ldots (2n-1)^2 + 2^2 + 4^2 \ldots + (2n)^2 = [\frac{2n(2n+1)(4n+1)}{6}]$$

$$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{2n(2n+1)(4n+1)}{6}] - 2^2 - 4^2 \ldots - (2n)^2$$

$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{2n(2n+1)(4n+1)}{6}] - 2^2\{1^2 + 2^2 \ldots + (n)^2\}$ ....(2)

Substituting (2) in (1)

$$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{2n(2n+1)(4n+1)}{6}] - 2^2\{1^2 + 2^2 \ldots + (n)^2\}$$

$$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{2n(2n+1)(4n+1)}{6}] - 2^2\left\{\frac{n(n+1)(2n+1)}{6}\right\}$$

$$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{2n(2n+1)(4n+1)}{6}] - 2^2\{[\frac{n(n+1)(2n+1)}{6}]\}$$

$$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{2n(2n+1)(4n+1) - 4n(n+1)(2n+1)}{6}]$$

$$1^2 + 3^2 \ldots (2n-1)^2 = [\frac{n(2n+1)(2n-1)}{3}]$$

10. Question

The sum of the series $\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + \ldots\ldots$ to n terms is

A. $n - \frac{1}{2}(3^{-n} - 1)$

B. $n - \frac{1}{2}(1 - 3^{-n})$

C. $n + \frac{1}{2}(3^{n} - 1)$

D. $n - \frac{1}{2}(3^{n} - 1)$

Answer

We can write

$$\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}\ldots.=\frac{3-1}{3}+\frac{9-1}{9}+\frac{27-1}{27}+\frac{81-1}{81}\ldots.$$

$$\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}\ldots.=1+\frac{-1}{3}+1+\frac{-1}{9}+1+\frac{-1}{27}+1+\frac{-1}{81}\ldots.$$

$$\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}\ldots.=n-(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}\ldots\frac{1}{3^n})$$

$$\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}\ldots.=n-(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}\ldots\frac{1}{3^n})$$

Since

$$\sum_{r=1}^{n}\frac{1}{b^r}=\frac{1/b\left[1-\left(\frac{1}{b}\right)^n\right]}{\left[1-\frac{1}{b}\right]}$$

$$\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}\ldots.=n-\frac{1}{3}[\frac{1-\frac{1}{3^n}}{1-\frac{1}{3}}]$$

$$\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}\ldots.=n-[\frac{1-\frac{1}{3^n}}{2}]$$