

Linear Differential Equation

Q.1. Solve the differential equation : $(x + 1) \frac{dy}{dx} - y = e^{3x} (x + 1)^2$.

Solution : 1

We have $(x + 1) \frac{dy}{dx} - y = e^{3x} (x + 1)^2$.

Dividing both sides by $(x + 1)$, we get

$$\frac{dy}{dx} - \frac{y}{(x + 1)} = e^{3x} (x + 1).$$

Therefore, I. F. = $e^{-\int 1/(x + 1)dx} = e^{-\log(x + 1)} = 1/(x + 1)$. Multiplying by I. F. we get $1/(x + 1) [dy/dx - 1/(x + 1)] = e^{3x} \cdot (x + 1) \cdot 1/(x + 1)$.

Integrating both sides , we get $y \cdot 1/(x + 1) = \int e^{3x} dx$ Or, $y/(x + 1) = (e^{3x})/3 + c$ Or, $y = \{(e^{3x})/3\}(x + 1) + c(x + 1)$.

Q.2. Solve the differential equation : $ydx - (x + 2y^2)dy = 0$.

Solution : 2

We have $ydx - (x + 2y^2)dy = 0$

$$\text{Or, } ydx/dy - x - 2y^2 = 0$$

$$\text{Or, } dx/dy - x/y - 2y = 0$$

$$\text{Or, } dx/dy - x/y = 2y$$

$$\text{I.F.} = e^{\int p dy} = e^{-\int (1/y)dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = 1/y.$$

Multiplying both sides by $1/y$, we get

$$1/y[dx/dy - x/y] = 1/y \times 2y$$

Integrating , we get

$$x \cdot 1/y = \int 2dy$$

$$\text{Or, } x/y = 2y + c \Rightarrow x = 2y^2 + cy .$$

Q.3. Solve the differential equation : $\tan x \, dy/dx + 2y = \sec x$.

Solution : 3

We have $\tan x \, dy/dx + 2y = \sec x$

Or, $dy/dx + (2/\tan x)y = 1/\sin x$ [$dy/dx + Py = Q$]

Here we have $P = 2/\tan x$, $Q = 1/\sin x$

I . F . = $e^{\int P dx} = e^{2 \int \cot x dx} = e^{2 \log \sin x} = e^{\log \sin^2 x} = \sin^2 x$

Integrating, we get $y \cdot \sin^2 x = \int (1/\sin x) \cdot \sin^2 x \, dx + c$

Or, $(\sin^2 x)y = \int \sin x \, dx + c = -\cos x + c$

Or, $(\sin^2 x) y + \cos x = c$.

Q.4. Solve the differential equation : $(1 - x^2) \, dy/dx - xy = x$, given $y = 2$ when $x = 0$.

Solution : 4

We have, $(1 - x^2) \, dy/dx - xy = x$

Or, $dy/dx - [x/(1 - x^2)]y = x/(1 - x^2)$

The I.F. = $P = e^{\int -x/(1 - x^2) dx}$

Then in $\int -x/(1 - x^2) dx$ [put $1 - x^2 = t \Rightarrow dt = -2x dx$

$= 1/2 \int dt/t = 1/2 \log t$

Then $P = e^{1/2 \log t} = e^{\log \sqrt{t}} = \sqrt{t} = \sqrt{1 - x^2}$

Therefore, $dy/dx \cdot \sqrt{1 - x^2} - [x/(1 - x^2)]y \sqrt{1 - x^2} = x/(1 - x^2) \times \sqrt{1 - x^2}$

Or, $d/dx [y \cdot \sqrt{1 - x^2}] = x/\sqrt{1 - x^2}$

Integrating we get $\int d/dx [y \cdot \sqrt{1 - x^2}] \cdot dx = \int [x/\sqrt{1 - x^2}] dx + c$

Or, $y \cdot \sqrt{1 - x^2} = 1/2 \int [2x/\sqrt{1 - x^2}] dx + c$

$$\begin{aligned}
&= -1/2 \int [-2x/(1 - x^2)]dx + c \\
&= -1/2 \int dt/\sqrt{t} + c \text{ [where, } t = 1 - x^2 \Rightarrow dt = -2xdx] \\
&= -1/2 [t^{-1/2}] + c \\
&= -\sqrt{(1 - x^2)} + c \\
\text{Or, } y\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)} &= c \\
\text{Putting } x = 0 \text{ and } y = 2 \text{ we get } c &= 3 \\
\text{Hence, } y\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)} &= 3.
\end{aligned}$$

Q.5. Solve : $(1 + y + x^2y)dx + (x + x^3)dy = 0$.

Solution : 5

$$\begin{aligned}
&\text{We have } (1 + y + x^2y)dx + (x + x^3)dy = 0 \\
&\text{Or, } 1 + y + x^2y + (x + x^3).dy/dx = 0 \\
&\text{Or, } 1 + y(1 + x^2) + x(1 + x^2).dy/dx = 0 \\
&\text{Or, } dy/dx + [y(1 + x^2)]/[x(1 + x^2)] + 1/[x(1 + x^2)] = 0 \\
&\text{Or, } dy/dx + y/x = -1/[x(1 + x^2)] \\
&\text{Which is of the form : } dy/dx + P.y = Q; \text{ P \& Q are function of } x. \\
&\text{Integrating factor} = e^{\int Pdx} = e^{\int (1/x)dx} \text{ [P = } 1/x] \\
&= e^{\log x} = x. \\
&\text{Multiplying both sides by I.F., we get} \\
&x.[dy/dx + y/x] = x \cdot \{-1/[x(1 + x^2)]\} \\
&\text{Or, } x.dy/dx + y = -1/(1 + x^2) \\
&\text{Or, } d(x.y)/dx = -1/(1 + x^2) \\
&\text{Integrating both sides, we get} \\
&\int d(x.y).dx = \int [(-1)/(1 + x^2)]dx
\end{aligned}$$

Or, $x.y = \tan^{-1} x + c$

Q.6. Solve : $\cos^2 x \, dy/dx + y = \tan x$.

Solution : 6

We are given, $\cos^2 x \, dy/dx + y = \tan x$

Or, $dy/dx + y \cdot \sec^2 x = \tan x \cdot \sec^2 x$ [dividing by $\cos^2 x$]

This is of the form , $dy/dx + Py = Q$, here $P = \sec^2 x$

$\int P \cdot dx = \int \sec^2 x \, dx = \tan x$.

Integrating factor = $e^{\int P \cdot dx} = e^{\tan x}$

Multiplying both sides by I.F., we get

$e^{\tan x} \cdot dy/dx + y \cdot \tan x \cdot \sec^2 x = e^{\tan x} \cdot \tan x \cdot \sec^2 x$

Or, $d/dx(y \cdot e^{\tan x}) = e^{\tan x} \cdot \tan x \cdot \sec^2 x$.

Integrating both sides we get,

$\int d/dx(y \cdot e^{\tan x}) \cdot dx = \int e^{\tan x} \cdot \tan x \cdot \sec^2 x \cdot dx$

Or, $y \cdot e^{\tan x} = \int e^{\tan x} \cdot \tan x \cdot \sec^2 x \cdot dx$

$= \int e^t \cdot t \cdot dt$ [put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$]

$= t \int e^t \cdot dt - \int e^t \cdot dt$ [Integrating by parts]

$= t \cdot e^t - e^t + c$

$= e^{\tan x}(\tan x - 1)$

Therefore, the solution is $y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + c$.

Q.7. Solve : $dy/dx + \{2x/(1 + x^2)\}y = 1/(1 + x^2)^2$.

Solution : 7

We are given,

$$dy/dx + \{2x/(1 + x^2)\}y = 1/(1 + x^2)^2$$

This is of the form

$dy/dx + Py + Q$, where P and Q are functions of x.

$$P = 2x/(1 + x^2), \text{ then } \int Pdx = \int 2x/(1 + x^2)dx = \log (1 + x^2)$$

$$\text{Therefore, I.F.} = e^{\int Pdx} = e^{\log (1 + x^2)} = 1 + x^2$$

Multiplying both sides by $(1 + x^2)$, we get

$$(1 + x^2).dy/dx + 2x.y = 1/(1 + x^2)$$

$$\text{Or, } d/dx[y.(1 + x^2)] = 1/(1 + x^2)$$

Integrating both sides we get

$$\int d/dx[y.(1 + x^2)].dx = \int dx/(1 + x^2) + c$$

$$\text{Or, } y.(1 + x^2) = \tan^{-1} (x) + c.$$

Q.8. Solve : $(1 + y^2) dx/dy = \tan^{-1} y - x$.

Solution : 8

We are given , $(1 + y^2) dx/dy = \tan^{-1} y - x$

$$\text{Or, } dx/dy = (\tan^{-1} y)/(1 + y^2) - x/(1 + y^2)$$

$$\text{Or, } dx/dy + x.\{1/(1 + y^2)\} = \tan^{-1} y / (1 + y^2)$$

This is of the form :

$$dx/dy + Px = Q$$

$$\int P.dy = \int dy/(1 + y^2) = \tan^{-1} y$$

$$\text{I.F.} = e^{\int Pdy} = e^{\tan^{-1} y}$$

Multiplying by I.F.,

$$\text{we get } e^{\tan^{-1} y} [dx/dy + \{1/(1 + y^2)\}.x] = e^{\tan^{-1} y} [\tan^{-1} y / (1 + y^2)]$$

$$\text{Or, } d/dy[e^{\tan^{-1} y} \cdot x] = e^{\tan^{-1} y} [\tan^{-1} y / (1 + y^2)]$$

Integrating ,

$$\text{we get } e^{\tan^{-1} y} \cdot x = \int e^{\tan^{-1} y} [\tan^{-1} y / (1 + y^2)] dy$$

$$\text{Now } \int e^{\tan^{-1} y} [\tan^{-1} y / (1 + y^2)] dy$$

$$[\text{put } y = \tan \theta \Rightarrow dy = \sec^2 \theta d\theta \text{ and } \tan^{-1} y = \theta]$$

$$= \int e^{\theta} \cdot \theta \cdot (1/\sec^2 \theta) \cdot \sec^2 \theta \cdot d\theta$$

$$= \int \theta \cdot e^{\theta} d\theta = \theta \cdot e^{\theta} - \int 1 \cdot e^{\theta} \cdot d\theta$$

$$= \theta \cdot e^{\theta} - e^{\theta} = e^{\theta} \cdot (\theta - 1)$$

$$= e^{\tan^{-1} y} \cdot (\tan^{-1} y - 1)$$

$$\text{Therefore, } x \cdot e^{\tan^{-1} y} = e^{\tan^{-1} y} \cdot (\tan^{-1} y - 1) + c .$$

Q.9. Solve the following differential equation : $\sin x \, dy/dx - y = \cos^2 x \sin x \tan (x/2)$.

Solution : 9

We are given,

$$\sin x \, dy/dx - y = \cos^2 x \sin x \tan (x/2)$$

$$\text{Or, } dy/dx - y/\sin x = \cos^2 x \tan (x/2) \text{ [Dividing by } \sin x]$$

It is of the form, $dy/dx + Py = Q$,

$$\int P dx = \int (-1/\sin x) dx = \int (-\operatorname{cosec} x) dx = -\log \tan (x/2) = \log \cot (x/2)$$

$$\text{I.F.} = e^{\int P dx} = e^{\log \cot (x/2)} = \cot (x/2)$$

Multiplying by $\cot(x/2)$, we get

$$(dy/dx - y/\sin x) \cot(x/2) = \cos^2 x \tan (x/2) \cot (x/2)$$

$$\text{integrating , we get } y \cot(x/2) = \int \cos^2 x \, dx + c$$

$$= \int [(1 + \cos 2x)/2] dx + c$$

$$y \cot(x/2) = (1/2)x + (\sin 2x)/4 + c .$$

Q.10. Solve the following differential equation for a particular solution : $dy = (5x - 4y)dx$; when $y = 0$ and $x = 0$.

Solution : 10

We have, $dy = (5x - 4y)dx$

Or, $dy/dx + 4y = 5x$

I.F. = $e^{\int 4dx} = e^{4x}$

Multiplying by e^{4x} and integrating we get,

$$y.e^{4x} = \int 5x.e^{4x} dx$$

$$= 5x.e^{4x}/4 - 5/4 \int e^{4x} dx + c$$

$$= 5/4x.e^{4x} - 5/16 e^{4x} + c.$$