

Homogeneous Differential Equation

Q.1. Solve the differential equation : $(x^2 + y^2) dx - 2xy dy = 0$, given that $y = 0$, when $x = 1$.

Solution : 1

We have $(x^2 + y^2) dx - 2xy dy = 0$ & $y = 0$ when $x = 1$.

The equation may be written as –

$$dy/dx = (x^2 + y^2)/2xy .$$

This is homogeneous equation . Put $y = vx$

$$\text{Then } dy/dx = v + x dv/dx$$

The equation reduces to ,

$$v + x dv/dx = (x^2 + v^2 x^2)/2x^2v = (1 + v^2)/2v$$

$$\text{Or, } x dv/dx = (1 + v^2)/2v - v = (1 - v^2)/2v$$

$$\text{Or, } 2v/(1 - v^2) dv = dx/x$$

Integrating both sides , we get

$$\int [2v/(1 - v^2)] dv = \int dx/x$$

$$\text{Or, } -\log [1 - v^2] = \log x + c$$

$$\text{Or, } -\log [1 - (y/x)^2] = \log x + c$$

Putting $x = 1$ and $y = 0$, we get $c = 0$.

$$\text{Then } 1/[1 - y^2/x^2] = x$$

$$\text{Or, } x = x^2 - y^2 .$$

Q.2. Solve the differential equation : $x dy/dx - y = \sqrt{(x^2 + y^2)}$.

Solution : 2

We have $x \frac{dy}{dx} - y = \sqrt{x^2 + y^2}$

$$\text{Or, } \frac{dy}{dx} - \frac{y}{x} = \sqrt{\frac{x^2 + y^2}{x^2}} \text{ ----- (1)}$$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + \frac{dv}{dx}$

Putting in (1) we get $v + x \frac{dv}{dx} - v = \sqrt{x^2 + v^2 x^2}/x$

$$\text{Or, } x \frac{dv}{dx} = \sqrt{1 + v^2}$$

Integrating we get , $\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} + \text{const.}$

$$\text{Or, } \log [v + \sqrt{1 + v^2}] = \log x + \log c$$

$$\text{Or, } \log [\frac{y}{x} + \sqrt{\frac{x^2 + y^2}{x^2}}] = \log x + \log c$$

$$\text{Or, } \log [\{y + \sqrt{x^2 + y^2}\}/x^2] = \log c$$

$$\text{Or, } [y + \sqrt{x^2 + y^2}] = cx^2 .$$

Q.3. Solve the differential equation : $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ by substituting $y = vx$.

Solution : 3

$$\text{We have, } 2 \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2} \text{ ----- (1)}$$

Putting $y = vx$, we get $\frac{dy}{dx} = v + \frac{dv}{dx}$

Putting in (1), we get

$$2(v + \frac{dv}{dx}) = v + v^2$$

$$\text{Or, } 2\frac{dv}{dx} = v + v^2 - 2v = v^2 - v$$

Integrating both sides, we get

$$\int \frac{1}{(v^2 - v)} dv = \frac{1}{2} \int \frac{dx}{x}$$

$$\text{Or, } \int \frac{dv}{(v^2 - v + 1/4 - 1/4)} = \frac{1}{2} \int \frac{dx}{x}$$

$$\text{Or, } \int \frac{dv}{[(v - 1/2)^2 - (1/2)^2]} = \frac{1}{2} \int \frac{dx}{x}$$

$$\text{Or, } \frac{1}{(2 \times 1/2)} \log \left[\frac{v - 1/2 - 1/2}{v - 1/2 + 1/2} \right] = \frac{1}{2} \log x$$

$$\text{Or, } \log [(v - 1)/v] = 1/2 \log x$$

$$\text{Or, } \log [(y - x)/y] = 1/2 \log x + c.$$

Q.4. Solve : $x \, dy/dx - y = \sqrt{(x^2 + y^2)}$.

Solution : 4

We are given $x \, dy/dx - y = \sqrt{(x^2 + y^2)}$

$$\text{Or, } dy/dx = y/x + \{\sqrt{(x^2 + y^2)}\}/x$$

$$\text{Or, } dy/dx = y/x + \sqrt{\{(x^2 + y^2)/x^2\}}$$

$$\text{Or, } dy/dx = y/x + \sqrt{\{1 + (y/x)^2\}}$$

Putting $y = vx$ we get

$$dy/dx = v + x \, dv/dx$$

and our equation reduces to

$$v + x \, dv/dx = v + \sqrt{(1 + v^2)}$$

$$\text{Or, } x \, dv/dx = \sqrt{(1 + v^2)}$$

$$\text{Or, } dv/\sqrt{(1 + v^2)} = dx/x$$

Integrating both sides we get

$$\int dv/\sqrt{(1 + v^2)} = \int dx/x$$

$$\text{Or, } \log [v + \sqrt{(1 + v^2)}] = \log x + \log c$$

$$\text{Or, } \log[y/x + \sqrt{\{1 + (y/x)^2\}}] = \log x + \log c$$

$$\text{Or, } \log[y + \sqrt{(x^2 + y^2)}] - \log x = \log x + \log c$$

$$\text{Or, } \log [y + \sqrt{(x^2 + y^2)}] = 2 \log x + \log c$$

$$\text{Or, } \log [y + \sqrt{(x^2 + y^2)}] = \log x^2 + \log c$$

$$\text{Or, } \log [y + \sqrt{(x^2 + y^2)}] = \log cx^2$$

$$\text{Therefore, } y + \sqrt{(x^2 + y^2)} = cx^2.$$

Q.5. Solve : $x(x - y) dy + y^2 dx = 0$.

Solution : 5

We are given,

$$x(x - y) dy + y^2 dx = 0$$

$$\text{Or, } dy/dx = -y^2/(x^2 - xy) = y^2/(xy - x^2)$$

Putting $y = v x$,

we get $dy/dx = v + xdv/dx$

$$\text{Or, } v + xdv/dx = v^2 x^2/(x.vx - x^2) = v^2 x^2/(vx^2 - x^2)$$

$$\text{Or, } v + xdv/dx = v^2/(v - 1)$$

$$\text{Or, } x dv/dx = v^2/(v - 1) - v = (v^2 - v^2 + v)/(v - 1)$$

$$\text{Or, } x dv/dx = v/(v - 1)$$

$$\text{Or, } [(v - 1)/v]dv = (1/x) dx$$

Integrating, we get

$$\int (1 - 1/v) dv = \int (1/x) dx + c$$

$$\text{Or, } v - \log v = \log x + c$$

$$\text{Or, } y/x - \log (y/x) = \log x + c$$

$$\text{Or, } y/x - [\log y/x + \log x] = c$$

$$\text{Or, } y/x - [\log y - \log x + \log x] = c$$

$$\text{Or, } y/x - \log y = c$$

$$\text{Or, } y = x \log y = c x.$$