

Complex Number

Q.1. Find the value of x and y , given that $(x + iy)(2 - 3i) = 4 + i$.

Solution : 1

We have $(x + iy)(2 - 3i) = 4 + i$

Or, $(2x + 3y) + i(2y - 3x) = 4 + i$

Comparing real and imaginary parts we have ,

$$2x + 3y = 4 \text{ ----- (1)}$$

$$- 3x + 2y = 1 \text{ ----- (2)}$$

solving these two equations , we get $x = 5/13$ and $y = 14/13$.

Q.2. If the ratio $(z - i)/(z - 1)$ is purely imaginary , prove that the point z lies on the circle whose centre is the point $1/2 (1 + i)$ and radius is $1/\sqrt{2}$.

Solution : 2

$$(z - i)/(z - 1) = (x + iy - i)/(x + iy - 1)$$

$$= \{x + i(y - 1)\}/\{(x - 1) + iy\}$$

$$= [\{x + i(y - 1)\}\{(x - 1) - iy\}]/[\{(x - 1) + iy\}\{(x - 1) - iy\}]$$

$$= [\{x(x - 1) + y(y - 1)\} + i\{(y - 1)(x - 1) - xy\}] / [(x - 1)^2 + y^2]$$

As the given ratio is purely imaginary ,

$$\text{Hence , } [x(x - 1) + y(y - 1)]/[(x - 1)^2 + y^2] = 0$$

$$\text{Or, } x^2 + y^2 - x - y = 0 .$$

This is the equation of the circle with centre $(1/2 , 1/2)$ i.e. $1/2 (1 + i)$ and radius is $\sqrt{\{(1/2)^2 + (1/2)^2 - 0\}}$ i.e. $1/\sqrt{2}$.

The centre = $1/2 (1 + i)$ and radius = $1/\sqrt{2}$.

Q.3. Find the modulus and amplitude of the complex number $(2 + 3i)/(3 + 2i)$.

Solution : 3

$$\begin{aligned}\text{We have , } (2 + 3i)/(3 + 2i) &= (2 + 3i)(3 - 2i)/(3 + 2i)(3 - 2i) \\ &= (12 + 5i)/13 \\ &= (12/13) + (5/13)i\end{aligned}$$

$$\text{Therefore , Modulus} = \sqrt{[(12/13)^2 + (5/13)^2]} = 1.$$

$$\text{And Amplitude} = \tan^{-1} [(5/13)/(12/13)] = \tan^{-1} (5/12) .$$

Q.4. Express $(1 - 2i)/(2 + i) + (3 + i)/(2 - i)$ in the form of $a + ib$.

Solution : 4

$$\begin{aligned}(1 - 2i)/(2 + i) + (3 + i)/(2 - i) &= [(1 - 2i)(2 - i) + (3 + i)(2 + i)]/(2 + i)(2 - i) \\ &= (2 - 4i - i + 2i^2 + 6 + i^2 + 2i + 3i)/(2^2 - i^2) \\ &= (2 - 5i - 2 + 6 - 1 + 5i)/(4 + 1) \\ &= 5/5 = 1 = 1 + 0i.\end{aligned}$$

Q.5. Express $13i/(2 - 3i)$ in the form of $A + iB$.

Solution : 5

$$\begin{aligned}\text{We have, } 13i/(2 - 3i) &= [13i (2 + 3i)]/[(2 - 3i)(2 + 3i)] \\ &= (26i + 39i^2)/(4 - 9i^2) \\ &= (26i - 39)/(4 + 9) \\ &= (26i - 39)/13 \\ &= 2i - 3 \\ &= -3 + 2i \text{ in the form of } A + iB.\end{aligned}$$

Q.6. Find the cube root of -27 and show that the sum of the cube roots is equal to zero.

Solution : 6

Let the cube root of -27 be x .

$$\text{Therefore, } x = (-27)^{1/3}$$

$$\text{Or, } x^3 = -27 = 27(\cos \pi + i \sin \pi)$$

$$\begin{aligned}\text{Or, } x &= 3(\cos \pi + i \sin \pi)^{1/3} \\ &= 3[\cos (2k\pi + \pi)/3 + i \sin(2k\pi + \pi)/3]\end{aligned}$$

[Where $k = 0, 1, 2$]

$$\text{At } k = 0, x = 3\{\cos \pi/3 + i \sin \pi/3\};$$

$$k = 1, x = 3\{\cos 3\pi/3 + i \sin 3\pi/3\};$$

$$k = 2, x = 3\{\cos 5\pi/3 + i \sin 5\pi/3\}.$$

Therefore the roots are : $x_1 = 3(1/2 + i\sqrt{3}/2)$; $x_2 = 3(-1 + 0) = -3$ and $x_3 = 3(1/2 - i\sqrt{3}/2)$.

$$\text{Their sum} = x_1 + x_2 + x_3 = 3\{1/2 + i\sqrt{3}/2 - 1 + 1/2 - i\sqrt{3}/2\} = 0.$$

Q.7. Find the amplitude of the complex number : $\sin 6\pi/5 + i(1 - \cos 6\pi/5)$.

Solution : 7

$$\text{We have } \sin 6\pi/5 + i(1 - \cos 6\pi/5) = 2 \sin 3\pi/5 \cos 3\pi/5 + i(2 \sin^2 3\pi/5)$$

$$= 2 \sin 3\pi/5 (\cos 3\pi/5 + i \sin 3\pi/5)$$

Comparing with $r(\cos \theta + i \sin \theta)$,

$$\text{Amplitude} = \theta = 3\pi/5 = 108^\circ.$$

Q.8. If $i = \sqrt{-1}$, prove the following :

$$(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i) = x^4 + 4.$$

Solution : 8

$$\begin{aligned} \text{L.H.S.} &= (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i) \\ &= [(x + 1)^2 - i^2][(x - 1)^2 - i^2] \\ &= [(x + 1)^2 + 1][(x - 1)^2 + 1] \\ &= [x^2 + 2 + 2x][x^2 + 2 - 2x] \\ &= (x^2 + 2)^2 - (2x)^2 \\ &= x^4 + 4 + 4x^2 - 4x^2 \\ &= x^4 + 4 = \text{R.H.S.} \end{aligned}$$

Q.9. If $z = x + iy$ and $|2z + 1| = |z - 2i|$, show that $3(x^2 + y^2) + 4(x + y) = 3$.

Solution : 9

$$\begin{aligned} \text{We have, } |2z + 1| &= |z - 2i| \\ \text{Or, } |2(x + iy) + 1| &= |x + iy - 2i| \\ \text{Or, } |2x + 1 + 2iy| &= |x + i(y - 2)| \\ \text{Or, } \sqrt{[(2x + 1)^2 + (2y)^2]} &= \sqrt{[x^2 + (y - 2)^2]} \\ \text{Or, } 4x^2 + 1 + 4x + 4y^2 &= x^2 + y^2 + 4 - 4y \\ \text{Or, } 3x^2 + 3y^2 + 4x + 4y - 3 &= 0 \\ \text{Or, } 3(x^2 + y^2) + 4(x + y) &= 3. \end{aligned}$$

Q.10. Find the locus of the complex number $z = x + iy$ satisfying the relation $|2z + 3i| \geq |2z + 5|$. Illustrate the locus in the Argand plane.

Solution : 10

We have $|2z + 3i| \geq |2z + 5|$

Substituting $z = x + iy$, we get

$$|2x + 2iy + 3i| \geq |2x + 2iy + 5|$$

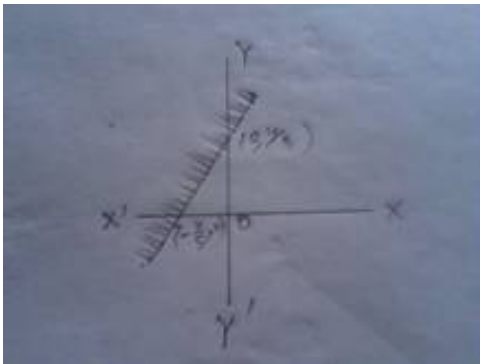
$$\text{Or, } |2x + i(2y + 3)| \geq |(2x + 5) + i2y|$$

$$\text{Or, } \sqrt{\{(2x)^2 + (2y + 3)^2\}} \geq \sqrt{\{(2x + 5)^2 + (2y)^2\}}$$

$$\text{Or, } 4x^2 + 4y^2 + 12y + 9 \geq 4x^2 + 4y^2 + 20x + 25$$

$$\text{Or, } 12y - 20x \geq 16$$

$$\text{Or, } 3y - 5x \geq 4.$$



Q.11. Find the real values of x and y satisfying the equality : $[(x - 2) + (y - 3)i] / (1 + i) = 1 - 3i$.

Solution : 11

We have $[(x - 2) + (y - 3)i] / (1 + i) = 1 - 3i$.

$$\text{Or, } (x - 2) + (y - 3)i = (1 + i)(1 - 3i)$$

$$= 1 + i - 3i - 3i^2 = 1 - 2i + 3$$

$$\text{Or, } (x - 2) + (y - 3)i = 4 - 2i.$$

Comparing real and imaginary parts, we get

$$x - 2 = 4 \text{ and } y - 3 = -2$$

$$\text{Or, } x = 6 \text{ and } y = 1.$$

Q.12. If $z = x + iy$ and $[|z - 1 - i| + 4]/[3|z - 1 - i| - 2] = 1$, show that $x^2 + y^2 - 2x - 2y - 7 = 0$.

Solution : 12

We are given that $z = x + iy$ and $[|z - 1 - i| + 4]/[3|z - 1 - i| - 2] = 1$

$$\text{Or, } |z - 1 - i| + 4 = 3|z - 1 - i| - 2$$

$$\text{Or, } 6 = 2|z - 1 - i|$$

$$\text{Or, } |z - 1 - i| = 3$$

$$\text{Or, } |x + iy - 1 - i| = 3$$

$$\text{Or, } \sqrt{(x - 1)^2 + (y - 1)^2} = 3$$

$$\text{Or, } x^2 - 2x + 1 + y^2 - 2y + 1 = 9$$

$$\text{Or, } x^2 + y^2 - 2x - 2y - 7 = 0.$$

Q.13. If $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = a + ib$, find the real numbers a and b . With these values of a and b find the modulus of $a + ib$.

Solution : 13

We have,

$$(-2 + \sqrt{3}i)(-3 + 2\sqrt{3}i) = a + ib$$

$$\text{Or, } 6 - 6 - 7\sqrt{3}i = a + ib$$

$$\text{Or, } a = 0 \text{ and } b = -7\sqrt{3}.$$

Q.14. Given $z = x + iy$ and $z_1 = 1 + 2i$, determine the region in the complex plane represented by $1 < |z - z_1| \leq 3$. Represent it on the Argand plane.

Solution : 14

We have, $z = x + iy$ and $z^1 = 1 + 2i$,

Then $z - z^1 = (x + iy) - (1 + 2i) = (x - 1) + i(y - 2)$

Therefore, $|z - z^1| = \sqrt{(x - 1)^2 + (y - 2)^2}$

But $1 < |z - z^1| \leq 3$

Or, $1 < \sqrt{(x - 1)^2 + (y - 2)^2} \leq 3$

Or, $1 < (x - 1)^2 + (y - 2)^2 \leq 9$

Now, $(x - 1)^2 + (y - 2)^2 > 1$

The points satisfying this inequality will lie outside the circle drawn on Argand plane with centre (1, 2) and radius unity.

And in $(x - 1)^2 + (y - 2)^2 \leq 9$

Points satisfying it will lie outside/on the boundary of circle with centre (1, 2) and radius thrice the unity.

Q.15. Sketch in the complex plane the set of points z satisfying : $|z - 3|/|z + 1| = 3$.

Solution : 15

Fig. soln 14(a)/page - 453 [10 yrs]

Let $z = x + iy$,

Then, $z - 3 = x + iy - 3 = (x - 3) + iy$ and $z + 1 = x + iy + 1 = (x + 1) + iy$.

Now $|z - 3| = \sqrt{(x - 3)^2 + y^2}$ and $|z + 1| = \sqrt{(x + 1)^2 + y^2}$

Therefore, $|z - 3|/|z + 1| = \sqrt{(x - 3)^2 + y^2}/\sqrt{(x + 1)^2 + y^2} = 3$

Squaring we get,

$$[(x - 3)^2 + y^2]/[(x + 1)^2 + y^2] = 9$$

$$\text{Or, } (x - 3)^2 + y^2 = 9 [(x + 1)^2 + y^2]$$

$$\text{Or, } x^2 + y^2 - 6x + 9 = 9 [x^2 + y^2 + 2x + 1]$$

$$\text{Or, } 8x^2 + 8y^2 + 24x = 0$$

$$\text{Or, } x^2 + y^2 + 3x = 0$$

$$\text{Or, } [x^2 + 2 \cdot (3/2) \cdot x + 9/4] + y^2 = 9/4$$

$$\text{Or, } (x + 3/2)^2 + y^2 = (3/2)^2$$

Which is a circle with centre $(-3/2, 0)$ and radius $3/2$ units.

Q.16. Given that : $[2\sqrt{3}(\cos 30^\circ) - 2i(\sin 30^\circ)] / [\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)] = A + i B$.
Find the value of A and B.

Solution : 16

We have,

$$[2\sqrt{3}(\cos 30^\circ) - 2i(\sin 30^\circ)] / [\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)] = A + i B.$$

$$\text{Or, } [2\sqrt{3} \times \sqrt{3}/2 - 2i \times 1/2] / [\sqrt{2}(1/\sqrt{2} + i \times 1/\sqrt{2})] = A + i B.$$

$$\text{Or, } (3 - i) / (1 + i) = A + i B.$$

Rationalizing the denominator we get,

$$[(3 - i)(1 - i)] / [(1 + i)(1 - i)] = A + i B.$$

$$\text{Or, } (3 + i^2 - 3i - i) / (1 - i^2) = A + i B.$$

$$\text{Or, } [3 + (-1) - 4i] / [1 - (-1)] = A + i B.$$

$$\text{Or, } (2 - 4i) / 2 = A + i B.$$

$$\text{Or, } 1 - 2i = A + i B.$$

Equating real and imaginary parts we get,

$$A = 1 \text{ and } B = -2.$$

Q.17. If $z_1, z_2 \in \mathbb{C}$, prove that : $|z_1 - z_2| \leq |z_1| + |z_2|$.

Solution : 17

$$\text{Let } z_1 = x_1 + i y_1 \text{ and } z_2 = x_2 + i y_2,$$

$$\text{Then, } z_1 - z_2 = (x_1 + i y_1) - (x_2 + i y_2) = (x_1 - x_2) + i (y_1 - y_2)$$

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2}$$

$$|z_1| = \sqrt{x_1^2 + y_1^2}, |z_2| = \sqrt{x_2^2 + y_2^2} \text{ and } |z_1| + |z_2| = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

$$\text{Therefore, } [|z_1| + |z_2|]^2 = \{\sqrt{x_1^2 + y_1^2}\}^2 + \{\sqrt{x_2^2 + y_2^2}\}^2 + 2\sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{x_1^2 x_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + y_1^2 y_2^2}$$

$$= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1x_2 + y_1y_2)^2 + (x_1y_2 - x_2y_1)^2}$$

$$\text{If } x_1y_2 - x_2y_1 = 0 \Rightarrow x_1y_2 = x_2y_1 \Rightarrow x_1/x_2 = y_1/y_2,$$

$$\text{Then, } [|z_1| + |z_2|]^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1x_2 + y_1y_2)^2}$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1x_2 + y_1y_2)$$

$$= (x_1 + x_2)^2 + (y_1 + y_2)^2 = |z_1 + z_2|^2$$

$$\text{Therefore, } |z_1| + |z_2| = |z_1 + z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

$$[\text{When } x_1y_2 - x_2y_1 = 0]$$

$$\text{But, } |z_1 - z_2| < |z_1 + z_2|$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\text{Therefore, } |z_1 - z_2| \leq |z_1| + |z_2|$$

Q.18. If $z = x + iy$, $\omega = (2 - iz)/(2z - i)$, and $|\omega| = 1$, find the locus and illustrate it in the complex plane.

Solution : 18



Fig

We have, $z = x + i y$;

$$\omega = (2 - i z)/(2z - i)$$

$$= [2 - i (x + i y)]/[2(x + i y) - i]$$

$$= [(2 + y) - i x]/[2x + i (2y - 1)]$$

Now, $|\omega| = 1$; It means

$$\sqrt{[(2 + y)^2 + x^2]} = \sqrt{[(2x)^2 + (2y - 1)^2]}$$

$$\text{Or, } 4 + y^2 + 4y + x^2 = 4x^2 + 4y^2 + 1 - 4y$$

$$\text{Or, } 3x^2 + 3y^2 - 8y - 3 = 0$$

$$\text{Or, } x^2 + y^2 - 8/3 y - 1 = 0$$

Which is a circle, $x^2 + (y - 4/3)^2 = (5/3)^2$

Whose centre is $(0, 4/3)$ and radius = $5/3$.

Q.19. Illustrate in the complex plane the set of points z satisfying : $|z + i - 2| \leq 2$.

Solution : 19

Let $z = x + iy$, then

$$|z + i - 2| \leq 2. \Rightarrow |x + iy + i - 2| \leq 2.$$

$$\text{Or, } \sqrt{[(x - 2)^2 + (y + 1)^2]} \leq 2.$$

$$\text{Or, } (x - 2)^2 + (y + 1)^2 \leq 4.$$

$$\text{Or, } x^2 + 4 - 4x + y^2 + 1 + 2y \leq 4$$

$$\text{Or, } x^2 + y^2 - 4x + 2y + 1 \leq 0$$

$$\text{Now, } x^2 + y^2 - 4x + 2y + 1 = 0$$

Or, $(x - 2)^2 + (y + 1)^2 = 4$, is a circle with centre $(2, -1)$ and radius = $\sqrt{4} = 2$. **[Ans.]**

The circle cuts x-axis at $y = 0$,

Therefore, $x^2 - 4x + 1 = 0$ & $x = [4 \pm \sqrt{(16 - 4)}]/2$

$$= [4 \pm 2\sqrt{3}]/2$$

$$= 2 \pm \sqrt{3} = 2 \pm 1.732 = 3.732, 0.276.$$

It cuts x-axis at $x = 0$,

Therefore, $y^2 + 2y + 1 = 0 \Rightarrow (y + 1)^2 = 0 \Rightarrow y = -1, -1.$

Thus the circle cuts y-axis at $(0, -1)$.

Q.20. Solve the equation : $2z = |z| + 2i$.

Solution : 20

We have, $2z = |z| + 2i$.

Let $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$

Therefore, $|z| + 2i = \sqrt{x^2 + y^2} + 2i$, and $2z = 2x + 2iy$.

Comparing real and imaginary parts on both sides, we get

$$2x = \sqrt{x^2 + y^2},$$

$$2y = 2 \Rightarrow y = 1.$$

$$\text{Hence, } 2x = \sqrt{x^2 + 1} \Rightarrow 4x^2 = x^2 + 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm 1/\sqrt{3}.$$

Thus, $z = \pm 1/\sqrt{3} + i$.