

Integration by Substitution

Q.1. Solve : $\int 1/[x \cos^2 (1 + \log x)] dx$.

Solution : 1

We have $\int 1/[x \cos^2 (1 + \log x)] dx$,

Put $1 + \log x = t$

Then $1/x dx = dt$

Therefore, $\int (1/x) \{1/(1 + \log x)\} dx$

$$= \int dt/\cos^2 t = \int \sec^2 t dt$$

$$= \tan t + c$$

$$= \tan (1 + \log x) + c .$$

Q.2. Evaluate : $\int x^2(e^{x^3})\cos(2e^{x^3})dx$.

Solution : 2

We have $\int x^2(e^{x^3})\cos(2e^{x^3})dx$,

Put $2e^{x^3} = t$ then $2e^{x^3} \times 3x^2 = dt/dx$

$$\text{Or, } dx = dt/[2e^{x^3} \times 3x^2]$$

$$\text{Then } I = \int x^2(e^{x^3})\cos(t)[dt/(2e^{x^3} \times 3x^2)]$$

$$= 1/6 \int \cos t dt = 1/6 \sin t + c = 1/6 \sin(2e^{x^3}) + c .$$

Q.3. Evaluate : $\int \cos x/[\sin x + \sqrt{(\sin x)}] dx$.

Solution : 3

$$I = \int \cos x / [\sin x + \sqrt{(\sin x)}] dx, \text{ put } \sin x = t^2, \cos x dx = 2t dx.$$

$$\text{Therefore, } I = 2 \int t dt / (t^2 + t) = 2 \int t dx / [t(t + 1)]$$

$$= 2 \int dt / (t + 1) = 2 \log |t + 1| + c$$

$$= 2 \log |\sqrt{(\sin x)} + 1| + c.$$

$$\text{Q.4. Integrate : } \int (1 + \tan^2 x) / [\sqrt{(1 - \tan^2 x)}] dx.$$

Solution : 4

$$\text{Let } I = \int (1 + \tan^2 x) / [\sqrt{(1 - \tan^2 x)}] dx$$

$$\text{Putting } \tan x = t, \text{ then } \sec^2 x dx = dt \Rightarrow (1 + \tan^2 x) dx = dt$$

$$\text{Therefore, } I = \int dt / \sqrt{(1 - t^2)}$$

$$= \sin^{-1} t$$

$$= \sin^{-1} (\tan x) + c.$$

$$\text{Q.5. Evaluate : } \int [x^2 / \sqrt{(x + 1)}] dx.$$

Solution : 5

$$\text{Let } I = \int [x^2 / \sqrt{(x + 1)}] dx \text{ and put } x + 1 = t^2 \Rightarrow dx = 2t dt$$

$$\text{Therefore, } I = \int [(t^2 - 1)^2 / t] \cdot 2t dt$$

$$= 2 \int (t^4 - 2t^2 + 1) \cdot dt$$

$$= 2[t^5/5 - 2 \times t^3/3 + t] + c$$

$$= 2t[(1/5) t^4 - (2/3) t^2 + 1] + c$$

$$= 2\sqrt{x+1}[(1/5)(x+1)^2 - (2/3)(x+1) + 1] + c.$$

Q.6. Evaluate : $\int \sqrt{1 + \sin 1/2 x} dx$

Solution : 6

Let $I = \int \sqrt{1 + \sin 1/2 x} dx$ and put $x = 2t \Rightarrow dx = 2dt$

Therefore, $I = \int \sqrt{1 + \sin t} \cdot 2dt$

$$= 2 \int \sqrt{(\sin^2 t/2 + \cos^2 t/2 + 2 \sin t/2 \cdot \cos t/2)} dt$$

$$= 2 \int \sqrt{(\sin t/2 + \cos t/2)^2} dt$$

$$= 2 \int (\sin t/2 + \cos t/2) \cdot dt$$

$$= 2 [-2 \cos t/2 + 2 \sin t/2] + c$$

$$= 4[\sin x/4 - \cos x/4] + c.$$

Q.7. Evaluate : $\int [\{\sin^2(\log x)\}/x] dx$.

Solution : 7

Let $I = \int [\{\sin^2(\log x)\}/x] \cdot dx$ and put $\log x = t \Rightarrow (1/x)dx = dt$

Therefore, $I = \int \sin^2 t \cdot dt = 1/2 \int (1 - \cos 2t) \cdot dt$

$$= 1/2 [t - (1/2) \sin 2t] + c$$

$$= 1/2 [\log x - (1/2) \sin (2 \log x)] + c.$$

Q.8. Evaluate : $\int [x^2/\{(a + bx)^2\}] \cdot dx$.

Solution : 8

Let $I = \int [x^2 / \{(a + bx)^2\}] . dx$ and put $a + bx = t \Rightarrow bdx = dt \Rightarrow dx = dt/b$

Therefore, $I = 1/b^2 \int [\{(bx)^2\} / \{(a + bx)^2\}] . dx$

$$= 1/b^2 \int [\{(t - a)^2\} / \{t^2\}] . (dt/b)$$

$$= 1/b^3 \int [(t^2 - 2at + a^2) / (t^2)] . dt$$

$$= 1/b^3 \int [1 - (2a/t) + a^2 t^{-2}] . dt$$

$$= 1/b^3 [t - 2a \log t - a^2/t] + c$$

$$= 1/b^3 [(a + bx) - 2a \log |(a + bx)| - a^2/(a + bx)] + c.$$

Q.9. Evaluate : $\int \sqrt{2 + \sin 3x} . \cos 3x . dx$.

Solution : 9

Let $I = \int \sqrt{2 + \sin 3x} . \cos 3x . dx$ and put $2 + \sin 3x = t \Rightarrow (1/3) \cos 3x dx = dt$

Therefore, $I = \int \sqrt{t} . dt / 3 = (1/3) \int \sqrt{t} . dt$

$$= (1/3)(2/3)(t)^{3/2} + c$$

$$= (2/9)(2 + \sin 3x)^{3/2} + c.$$

Q.10. Evaluate : $\int [x^{1/2} / (x^{1/2} - x^{1/3})] . dx$.

Solution : 10

Let $I = \int [x^{1/2} / (x^{1/2} - x^{1/3})] . dx$ and put $x = t^6 \Rightarrow dx = 6t^5 dt$

[LCM of 2 & 3 is 6]

Therefore, $I = \int [t^3/(t^3 - t^2)] \cdot 6t^5 dt$

$$= 6 \int [t^8/t^2(t-1)] dt = 6 \int [t^6/(t-1)] dt$$

$$= 6 \int [t^5 + t^4 + t^3 + t^2 + t + 1 + 1/(t-1)] dt \text{ [dividing } t^6 \text{ by } t-1]$$

$$= 6[(1/6)t^6 + (1/5)t^5 + (1/4)t^4 + (1/3)t^3 + (1/2)t^2 + t + \log |t-1|] + c$$

$$= 6[(1/6)x + (1/5)x^{5/6} + (1/4)x^{4/6} + (1/3)x^{3/6} + (1/2)x^{2/6} + x^{1/6} + \log |x^{1/6} - 1|] + c.$$

Q.11. Evaluate : $\int [(10x^9 + 10^x \log_e 10)/(10^x + x^{10})] dx$.

Solution : 11

$$\text{Let } I = \int (10x^9 + 10^x \log_e 10)/(10^x + x^{10}) dx$$

$$\text{Put } 10^x + x^{10} = t, \text{ then } (10x^9 + 10^x \log_e 10) dx = dt$$

$$\text{Therefore, } I = \int dt/t$$

$$= \log_e |t| + c = \log_e |10^x + x^{10}| + c.$$

Q.12. Evaluate : $\int [1/(1 + \cot x)] \cdot dx$.

Solution : 12

$$\text{Let } I = \int [1/(1 + \cot x)] dx = \int [\sin x dx / (\sin x + \cos x)] dx$$

$$= \int [\{(\sin x + \cos x) - (\cos x - \sin x)\} / \{2(\sin x + \cos x)\}] dx$$

$$= (1/2) \int dx - (1/2) \int [(\cos x - \sin x) / (\cos x + \sin x)] dx \text{ Put } \cos x + \sin x$$

$$= t \Rightarrow (-\sin x + \cos x) dx = dt$$

$$\text{Therefore, } I = (1/2)x - (1/2) \int dt/t$$

$$= (1/2) - (1/2) \log |t| + c$$

$$= (1/2)x - (1/2) \log |\cos x + \sin x| + c.$$

Q.13. Evaluate : $\int [\sin^2 x / (a^2 + b^2 \sin^2 x)] . dx$

Solution : 13

$$\text{Let } I = \int [\sin^2 x / (a^2 + b^2 \sin^2 x)] . dx.$$

$$\text{Put } a^2 + b^2 \sin^2 x = t \Rightarrow b^2 . (2 \sin x \cos x) dx = dt \Rightarrow \sin^2 x dx = dt / b^2$$

$$\text{Therefore, } I = 1/b^2 \int dt/t = 1/b^2 \log |t| + c$$

$$= 1/b^2 \log |a^2 + b^2 \sin^2 x| + c.$$

Q.14. Evaluate : $\int a/(b + ce^x) . dx.$

Solution : 14

$$\text{Let } I = \int a/(b + ce^x) . dx = \int ae^{-x}/(be^{-x} + c) . dx$$

$$\text{Put } be^{-x} + c = t \Rightarrow -be^{-x} dx = dt$$

$$\text{Therefore, } I = (-a/b) \int dt/t = -a/b \log |t| + k = -a/b \log |be^{-x} + c| + k$$

$$= -a/b \log |b.(1/e^x) + c| + k$$

$$= -a/b \log [|b + ce^x| - \log |e^x|] + k$$

$$= -a/b [\log |b + ce^x| - x] + k.$$

Q.15. Evaluate : $\int [1/(e^x + e^{-x})] . dx$

Solution : 15

$$\text{Let } I = \int [1/(e^x + e^{-x})].dx = \int [e^x/(e^{2x} + 1)].dx$$

$$\text{Put } e^x = t \Rightarrow e^x dx = dt$$

$$\text{Therefore, } I = \int dt/(t^2 + 1) = \tan^{-1}(t) + c = \tan^{-1}(e^x) + c.$$

Q.16. Evaluate : $\int [e^{2x}/(e^x + 1)].dx.$

Solution : 16

$$\text{Let } I = \int [e^{2x}/(e^x + 1)].dx = \int [(e^x \cdot e^x)/(e^x + 1)].dx$$

$$\text{Put } e^x = t \Rightarrow e^x dx = dt$$

$$\text{Therefore, } I = \int [t/(t + 1)]dt = \int [\{(t + 1) - 1\}/(t + 1)]dt$$

$$= \int [1 - \{1/(t + 1)\}]dt = t - \log |t + 1| + c$$

$$= e^x - \log |e^x + 1| + c.$$

Q.17. Evaluate : $\int [(e^x + e^{-x})/(e^x - e^{-x})].dx.$

Solution : 17

$$\text{Let } I = \int [(e^x + e^{-x})/(e^x - e^{-x})].dx$$

$$\text{Put } e^x - e^{-x} = t \Rightarrow (e^x + e^{-x})dx = dt.$$

$$\text{Therefore, } I = \int (1/t).dt = \log |t| + c = \log |e^x - e^{-x}| + c.$$

Q.18. Evaluate : $\int \{\cot(\log x)/x\}.dx.$

Solution : 18

Let $I = \int [\cot(\log x)].dx$. And put $\log x = z \Rightarrow (1/x)dx = dz$

Therefore, $I = \int \cot z.dz = \log |\sin z| + c$

$$= \log |\sin (\log x)| + c.$$

Q.19. Evaluate : $\int [\cos (x - a)/\cos x].dx$.

Solution : 19

Let $I = \int [\cos (x - a)/\cos x].dx$

$$= \int [(\cos x \cos a + \sin x \sin a)/\cos x]dx$$

$$= \int [\cos a + \tan x \sin a].dx$$

$$= \cos a \int dx + \sin a \int \tan x dx$$

$$= (\cos a) x + \sin a \log |\sec x| + c.$$

Q.20. Evaluate : $\int dx/\sqrt{(1 + \sin x)}$.

Solution : 20

Let $I = \int dx/\sqrt{(1 + \sin x)}$

$$= \int dx/\sqrt{\{1 - \cos (\pi/2 + x)\}}$$

$$= \int dx/\sqrt{\{2 \sin^2 (\pi/4 + x/2)\}}$$

$$= (1/\sqrt{2}) \int \operatorname{cosec} (\pi/4 + x/2).dx$$

Put $\pi/4 + x/2 = t \Rightarrow dx = 2dt$

Therefore, $I = (2/\sqrt{2}) \int \operatorname{cosec} t dt$

$$= \sqrt{2} \log |\tan (t/2)| + c$$

$$= \sqrt{2} \log |\tan (\pi/8 + x/4)| + c.$$