

Integration by parts

Q.1. Evaluate : $\int_0^{1/2} \sin^{-1} x / (1 - x^2)^{3/2} dx$.

Solution : 1

We have , $I = \int_0^{1/2} \sin^{-1} x / (1 - x^2)^{3/2} dx$,

Put $\sin^{-1} x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t dt$

[when $x = 0$, $\sin t = 0 \Rightarrow t = 0$ and when $x = 1/2$, $\sin t = 1/2 \Rightarrow t = \pi/6$]

Therefore, $I = \int_0^{1/2} \sin^{-1} x / (1 - x^2)^{3/2} dx = \int_0^{\pi/6} (t / \cos^3 t) \cdot \cos t dt$

$$= \int_0^{\pi/6} t \sec^2 t dt$$

$$= [t \int \sec^2 t dt - \int \{d/dt(t) \cdot \int \sec^2 t dt\}]_0^{\pi/6} \text{ [Integrating by parts]}$$

$$= [t \cdot \tan t - \int \tan t dt]_0^{\pi/6}$$

$$= [t \cdot \tan t - \log \sec t]_0^{\pi/6}$$

$$= [\pi/6 \tan \pi/6 - \log \sec \pi/6] - [0 - \log \sec 0]$$

$$= [\pi/6(1/\sqrt{3}) - \log (2/\sqrt{3})] - [0 - 0]$$

$$= [\pi/(6\sqrt{3}) - \log (2/\sqrt{3})]$$

Q.2. Evaluate : $\int e^{-2x} \sin x dx$.

Solution : 2

$$\text{Let } I = \int e^{-2x} \sin x dx = \sin x \int e^{-2x} dx - \int \cos x (-1/2) e^{-2x} dx$$

$$= -1/2 e^{-2x} \sin x + 1/2 \int e^{-2x} \cos x dx$$

$$\int e^{-2x} \cos x dx = \cos x \int e^{-2x} dx - \int (-\sin x)(-1/2 e^{-2x}) dx$$

$$= -1/2 e^{-2x} \cos x - 1/2 \int e^{-2x} \sin x dx$$

$$\text{Therefore, } I = -1/2 e^{-2x} \sin x + 1/2 \{-1/2 e^{-2x} \cos x - 1/2 \int e^{-2x} \sin x dx\}$$

$$= -\frac{1}{2} e^{-2x} \sin x - \frac{1}{4} e^{-2x} \cos x - \frac{1}{4} I$$

$$\text{Or, } \frac{5}{4} I = -\frac{1}{2} e^{-2x} \sin x - \frac{1}{4} e^{-2x} \cos x$$

$$\text{Or, } I = -\frac{2}{5} e^{-2x} \sin x - \frac{1}{5} e^{-2x} \cos x + c.$$

Q.3. Evaluate : $\int e^x [(1 + \sin x)/(1 + \cos x)] dx$.

Solution : 3

We are given,

$$\int e^x [(1 + \sin x)/(1 + \cos x)] dx$$

$$= \int e^x [\{(1 + \sin x)(1 - \cos x)\} / \{(1 + \cos x)(1 - \cos x)\}] dx$$

$$= \int e^x [(1 - \cos x + \sin x - \sin x \cos x) / (1 - \cos^2 x)] dx$$

$$= \int e^x [(1 - \cos x + \sin x - \sin x \cos x) / \sin^2 x] dx$$

$$= \int e^x (\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x + \operatorname{cosec} x - \cot x) dx$$

$$= \int e^x \operatorname{cosec}^2 x dx - \int e^x \operatorname{cosec} x dx + \int e^x \operatorname{cosec} x dx - \int e^x \cot x dx$$

$$= e^x \int \operatorname{cosec}^2 x dx - \int e^x (-\cot x) dx + e^x \int (-\operatorname{cosec} x \cot x) dx - \int e^x \operatorname{cosec} x dx +$$

$$e^x \int \operatorname{cosec} x dx - \int e^x \cot x dx$$

$$= e^x (-\cot x) + \int e^x \cot x dx + e^x \operatorname{cosec} x - \int e^x \operatorname{cosec} x dx + \int e^x \operatorname{cosec} x dx - \int e^x \cot x dx$$

$$= e^x \operatorname{cosec} x - e^x \cot x = e^x (\operatorname{cosec} x - \cot x) + c.$$

Q.4. Evaluate : $\int x \cdot \cos x \cdot dx$.

Solution : 4

$$\text{Let } I = \int x \cdot \cos x \cdot dx = x \int \cos x dx - \int \{d/dx(x) \cdot \int \cos x dx\} \cdot dx$$

$$= x \sin x - \int 1 \cdot \sin x dx = x \sin x - \int \sin x dx$$

$$= x \sin x + \cos x + c.$$

Q.5. Evaluate : $\int x^2 \sin 2x \cdot dx$.

Solution : 5

$$\begin{aligned}\int x^2 \sin 2x \, dx &= x^2 \int \sin 2x \, dx - \int \{d/dx(x^2) \cdot \int \sin 2x \, dx\} \cdot dx \\&= x^2 \{(-1/2) \cos 2x\} - \int 2x \cdot \{(-1/2) \cos 2x\} \, dx \\&= (-1/2)x^2 \cos 2x + \int x \cos 2x \cdot dx \\&= (-1/2)x^2 \cos 2x + x \cdot \{(1/2) \sin 2x\} - \int 1 \cdot \{(1/2) \sin 2x\} \, dx \\&= (-1/2)x^2 \cos 2x + (1/2)x \sin 2x - 1/2 \{(-1/2) \cos 2x\} + c \\&= (-1/2)x^2 \cos 2x + (1/2)x \sin 2x + (1/4) \cos 2x + c.\end{aligned}$$

Q.6. Evaluate : $\int x^2 e^{3x} \cdot dx$.

Solution : 6

$$\begin{aligned}\int x^2 e^{3x} \, dx &= x^2 \int e^{3x} \cdot dx - \int [d/dx(x^2) \int e^{3x} \cdot dx] \cdot dx \\&= x^2 (1/3) e^{3x} - \int 2x (1/3) e^{3x} \cdot dx \\&= (1/3)x^2 e^{3x} - (2/3) \int x e^{3x} \cdot dx \\&= (1/3)x^2 e^{3x} - (2/3) [x (1/3) e^{3x} - \int 1 \cdot (1/3) e^{3x} \cdot dx] \\&= (1/3)x^2 e^{3x} - (2/9)x e^{3x} + (2/9)(1/3) e^{3x} + c \\&= [(1/3)x^2 - (2/9)x + 2/27] e^{3x} + c.\end{aligned}$$

Q.7. Evaluate : $\int [x^2/(a + bx)^2] \cdot dx$.

Solution : 7

$$\int [x^2/(a + bx)^2] \cdot dx = \int [x^2(a + bx)^{-2}] \cdot dx$$

$$\begin{aligned}
&= x^2 \int (a + bx)^{-2} \cdot dx - \int [d/dx(x^2) \int (a + bx)^{-2} \cdot dx] \cdot dx \\
&= x^2 [(a + bx)^{-1}/(-b)] - \int 2x [(a + bx)^{-1}/(-b)] \cdot dx \\
&= - [x^2/\{b(a + bx)\}] + (2/b^2) \int [bx/(a + bx)] \cdot dx \\
&= - [x^2/\{b(a + bx)\}] + (2/b^2) \int [\{(a + bx) - a\}/(a + bx)] \cdot dx \\
&= - [x^2/\{b(a + bx)\}] + (2/b^2) [\int dx - \int \{a/(a + bx)\} \cdot dx] \\
&= - [x^2/\{b(a + bx)\}] + (2/b^2) [x - (a/b) \log (a + bx)] + c \\
&= - [x^2/\{b(a + bx)\}] + 2x/b^2 - (2a/b^3) \log (a + bx) + c.
\end{aligned}$$

Q.8. Evaluate : $\int [\log x/x] \cdot dx$.

Solution : 8

$$\begin{aligned}
\text{Let } I &= \int [\log x/x] \cdot dx = \int (\log x)(1/x) \cdot dx \\
&= \log x \int (1/x) \cdot dx - \int [d/dx(\log x) \int (1/x) \cdot dx] \cdot dx \\
&= (\log x) \cdot (\log x) - \int (1/x) \cdot (\log x) \cdot dx = (\log x)^2 - I \\
\text{Or, } 2I &= (\log x)^2 \\
\text{Or, } I &= 1/2 (\log x)^2 + c.
\end{aligned}$$

Q.9. Evaluate : $\int (\log x)^2 \sqrt{x} \cdot dx$.

Solution : 9

$$\begin{aligned}
\text{Let } I &= \int (\log x)^2 \sqrt{x} \cdot dx = (\log x)^2 \int \sqrt{x} \cdot dx - \int [d/dx\{(\log x)^2\} \int (\sqrt{x}) \cdot dx] \cdot dx \\
&= (\log x)^2 \{(x^{3/2})/(3/2)\} - \int \{(2 \log x)(1/x)\} \{(x^{3/2})/(3/2)\} \cdot dx \\
&= (2/3)(x^{3/2})(\log x)^2 - 4/3 \int (\log x)x^{1/2} \cdot dx \\
&= (2/3)(x^{3/2})(\log x)^2 - 4/3 [\log x \int \sqrt{x} \cdot dx - \int \{d/dx(\log x) \int \sqrt{x} \cdot dx\} \cdot dx]
\end{aligned}$$

$$\begin{aligned}
&= (2/3)(x^{3/2})(\log x)^2 - 4/3[\log x \{(x^{3/2})/(3/2)\} - \int (1/x)\{(x^{3/2})/(3/2)\}] \\
&= (2/3)(x^{3/2})(\log x)^2 - 4/3[(2/3)(x^{3/2})\log x - (2/3)\int (x^{1/2})] + c \\
&= (x^{3/2})[(2/3)(\log x)^2 - (8/9)(\log x) + 16/27] + c.
\end{aligned}$$

Q.10. Evaluate : $\int x \tan^{-1} x . dx$

Solution : 10

$$\begin{aligned}
\text{Let } I &= \int x \tan^{-1} x . dx = (\tan^{-1} x) \int x . dx - \int [d/dx(\tan^{-1} x) \int x . dx] . dx \\
&= (\tan^{-1} x)(x^2/2) - \int [\{1/(1 + x^2)\}(x^2/2)] . dx \\
&= (x^2/2)(\tan^{-1} x) - (1/2) \int \{x^2/(1 + x^2)\} dx \\
&= (x^2/2)(\tan^{-1} x) - (1/2) \int [1 - 1/(1 + x^2)] . dx \\
&= (x^2/2)(\tan^{-1} x) - (1/2)[x - \tan^{-1} x] + c \\
&= (1/2)[(x^2 + 1) \tan^{-1} x - x] + c.
\end{aligned}$$

Q.11. Evaluate : $\int \log \{x + \sqrt{(a^2 + x^2)}\} . dx$.

Solution : 11

$$\begin{aligned}
\text{Let } I &= \int \log \{x + \sqrt{(a^2 + x^2)}\} . dx = \int [\log \{x + \sqrt{(a^2 + x^2)}\}] . 1 . dx \\
&= \log \{x + \sqrt{(a^2 + x^2)}\} \int 1 . dx - \int d/dx[\log \{x + \sqrt{(a^2 + x^2)}\} \int 1 . dx \\
&= x \log \{x + \sqrt{(a^2 + x^2)}\} - \int \{1/\sqrt{(a^2 + x^2)}\} . x . dx \\
&= x \log \{x + \sqrt{(a^2 + x^2)}\} - (1/2) \int \{(a^2 + x^2)^{-1/2}\} (2x) . dx \\
&= x \log \{x + \sqrt{(a^2 + x^2)}\} - (1/2) \{(a^2 + x^2)^{-1/2}\} / (1/2) + c \\
&= x \log \{x + \sqrt{(a^2 + x^2)}\} - \sqrt{(a^2 + x^2)} + c.
\end{aligned}$$

Q.12. Evaluate : $\int \cos 2x \log \sin x . dx$.

Solution : 12

$$\begin{aligned}
\text{Let } I &= \int \cos 2x \log \sin x . dx = \int (\log \sin x) . \cos 2x . dx \\
&= (\log \sin x) \int \cos 2x . dx - \int [d/dx(\log \sin x) \int \cos 2x . dx] . dx \\
&= (\log \sin x) (\sin 2x/2) - \int (\cos x / \sin x) (\sin 2x/2) dx \\
&= (\log \sin x) \{ (2 \sin x . \cos x) / 2 \} - \int (\cos x / \sin x) \{ (2 \sin x . \cos x) / 2 \} . dx \\
&= \sin x . \cos x . \log \sin x - \int \cos^2 x . dx = \sin x . \cos x . \log \sin x - \int \{ (1 + \cos 2x) / 2 \} . dx \\
&= \sin x \cos x \log \sin x - 1/2 (x + \sin 2x/2) + c \\
&= \sin x \cos x \log \sin x - 1/2 (x + \sin x \cos x) + c.
\end{aligned}$$

Q.13. Evaluate : $\int [(x \sin^{-1} x) / \sqrt{(1 - x^2)}] . dx.$

Solution : 13

$$\begin{aligned}
\text{Let } I &= \int [(x \sin^{-1} x) / \sqrt{(1 - x^2)}] . dx \text{ [put } x = \sin \theta, \text{ then } dx = \cos \theta d\theta] \\
I &= \int \{ \theta . \sin \theta \} / \sqrt{(1 - \sin^2 \theta)} \} . \cos \theta d\theta \\
&= \int \theta \sin \theta d\theta = \theta (-\cos \theta) - \int 1 . (-\cos \theta) d\theta \\
&= -\theta \cos \theta + \sin \theta + c = -\theta \sqrt{(1 - \sin^2 \theta)} + \sin \theta + c \\
&= -\sqrt{(1 - x^2)} \sin^{-1} x + x + c.
\end{aligned}$$

Q.14. Evaluate : $\int [(x + \sin x) / (1 + \cos x)] . dx.$

Solution : 14

$$\begin{aligned}
\text{Let } I &= \int [(x + \sin x) / (1 + \cos x)] . dx \\
&= \int [(x + 2 \sin 1/2 x . \cos 1/2 x) / (1 + 2 \cos^2 1/2 x - 1)] . dx \\
&= \int [(x + 2 \sin 1/2 x \cos 1/2 x) / 2 \cos^2 1/2 x] . dx \\
&= 1/2 \int x . \sec^2 1/2 x . dx + \int \tan 1/2 x . dx
\end{aligned}$$

$$\begin{aligned}
&= 1/2 [x.(2 \tan 1/2 x) - \int 1.(2 \tan 1/2 x).dx + \int \tan 1/2 x.dx \\
&= x. \tan 1/2 x - \int \tan 1/2 x.dx + \int \tan 1/2 x.dx \\
&= x. \tan 1/2 x + c.
\end{aligned}$$

Q.15. Evaluate : $\int [xe^x/(x + 1)^2].dx$.

Solution : 15

$$\begin{aligned}
\text{Let } I &= \int [xe^x/(x + 1)^2].dx = \int (x.e^x)\{(x + 1)^{-2}\}.dx \\
&= (x.e^x)\int [(x + 1)^{-2}].dx - \int [d/dx(x.e^x)\int \{(x + 1)^{-2}\}d^x].d^x \\
&= -xe^x/(x + 1) - \int (xe^x + e^x.1)(-1/(x + 1)).dx \\
&= -xe^x/(x + 1) + \int e^x(x + 1)/(x + 1).dx \\
&= -xe^x/(x + 1) + \int e^x.dx \\
&= -xe^x/(x + 1) + ex + c \\
&= e^x/(x + 1) + c.
\end{aligned}$$