

Derivatives of Implicit Function

Q.1. If $y = x \log y$, show that $x \frac{dy}{dx} = \frac{y^2}{(y - x)}$.

Solution : 1

We have $y = x \log y$,

Differentiating both sides with respect to x we get , $\frac{dy}{dx} = x \cdot (1/y)$.

$$\frac{dy}{dx} + \log y \cdot 1$$

$$\text{Or, } (1 - x/y) \frac{dy}{dx} = \log y$$

$$\text{Or, } [(y - x)/y] \frac{dy}{dx} = \log y$$

$$\text{Or, } \frac{dy}{dx} = \frac{y \log y}{(y - x)}$$

$$\text{Or, } x \frac{dy}{dx} = \frac{x y \log y}{(y - x)}$$

$$= \frac{y(x \log y)}{(y - x)} \text{ [As, } x \log y = y]$$

$$= \frac{y^2}{(y - x)}.$$

Q.2. If $y \log x = x - y$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

Solution : 2

We have , $y \log x = x - y$,

Differentiating with respect to x we get ,

$$y \times \frac{1}{x} + \log x \times \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\text{Or, } (1 + \log x) \frac{dy}{dx} = 1 - y/x \text{ --- --- --- (i)}$$

We are given, $y \log x = x - y$

$$\text{Or, } y \log x + y = x$$

$$\text{Or, } y (1 + \log x) = x$$

$$\text{Or, } y/x = 1/(1 + \log x) \text{ --- --- --- (ii)}$$

Hence, putting y/x from (ii) to (i) we get,

$$(1 + \log x) \frac{dy}{dx} = 1 - \frac{1}{(1 + \log x)}$$
$$= (1 + \log x - 1) / (1 + \log x) = \log x / (1 + \log x)$$

Therefore, $\frac{dy}{dx} = \log x / (1 + \log x)^2$. **[Proved.]**

Q.3. If $\log (x^2 + y^2) = 2 \tan^{-1} (y/x)$, show that $\frac{dy}{dx} = (x + y)/(x - y)$.

Solution : 3

We have, $\log (x^2 + y^2) = 2 \tan^{-1} (y/x)$,

Differentiating with respect to x we get,

$$\frac{1}{(x^2 + y^2)} \cdot \frac{d}{dx} (x^2 + y^2) = 2 \cdot \frac{1}{\{1 + (y/x)^2\}} \cdot \frac{d}{dx} (y/x)$$

$$\text{Or, } \frac{1}{(x^2 + y^2)} \{ \frac{d}{dx} (x^2) + \frac{d}{dx} (y^2) \} = \frac{2x^2}{(x^2 + y^2)} \cdot [\{x \cdot \frac{dy}{dx} - y \cdot 1\} / x^2]$$

$$\text{Or, } \frac{1}{(x^2 + y^2)} (2x + 2y \frac{dy}{dx}) = \frac{2}{(x^2 + y^2)} \cdot \{x \frac{dy}{dx} - y\}$$

$$\text{Or, } 2 (x + y \frac{dy}{dx}) = 2 (x \frac{dy}{dx} - y)$$

$$\text{Or, } x + y \frac{dy}{dx} = x \frac{dy}{dx} - y$$

$$\text{Or, } (y - x) \frac{dy}{dx} = - (x + y)$$

$$\text{Or, } \frac{dy}{dx} = (x + y)/(x - y). \text{ **[Proved.]}**$$

Q.4. If $x \sqrt{1 + y} + y \sqrt{1 + x} = 0$, prove that $\frac{dy}{dx} = - 1/(x + 1)^2$.

Solution : 4

We have, $x \sqrt{1 + y} + y \sqrt{1 + x} = 0$

$$x \sqrt{1 + y} = - y \sqrt{1 + x}$$

Squaring both sides we get,

$$x^2 (1 + y) = y^2 (1 + x)$$

$$\text{Or, } x^2 - y^2 = y^2 x - x^2 y$$

$$\text{Or, } (x + y)(x - y) = -xy(x - y)$$

$$\text{Or, } x + y = -xy \text{ [As } x = y \text{ does not satisfy the given equation, } x - y \neq 0 \text{]}$$

$$\text{Or, } x = -y - xy = > y(1 + x) = -x \text{ Or, } y = -x/(1 + x)$$

$$\text{Therefore, } dy/dx = - [\{(1 + x) \cdot 1 - x(0 + 1)\} / \{(1 + x)^2\}] = -1/(1 + x)^2. \text{ [Proved.]}$$

Q.5. If $y\sqrt{1 - x^2} + x\sqrt{1 - y^2} = 1$, prove that $dy/dx = -\sqrt{(1 - y^2)/(1 - x^2)}$.

Solution : 5

Let $x = \sin \theta$ and $y = \cos \phi$, then

$$y\sqrt{1 - x^2} + x\sqrt{1 - y^2} = 1 \text{ reduces to}$$

$$\cos \phi \sqrt{1 - \sin^2 \theta} + \sin \theta \sqrt{1 - \cos^2 \phi} = 1$$

$$\text{Or, } \cos \phi \cos \theta + \sin \theta \sin \phi = 1$$

$$\text{Or, } \cos(\theta - \phi) = 1$$

$$\text{Or, } \theta - \phi = \cos^{-1} 1$$

$$\text{Or, } \sin^{-1} x - \cos^{-1} y = \cos^{-1} 1$$

$$\text{Or, } \sin^{-1} x - \cos^{-1} y = 0$$

$$[\text{As, } \cos^{-1} 1 = 0]$$

$$\text{Differentiating we get, } 1/\sqrt{1 - x^2} - (-1)/\sqrt{1 - y^2} dy/dx = 0$$

$$\text{Or, } 1/\sqrt{1 - y^2} dy/dx = -1/\sqrt{1 - x^2}$$

$$\text{Or, } dy/dx = -\sqrt{(1 - y^2)/(1 - x^2)}. \text{ [Proved.]}$$

Q.6. If $\sin y = x \sin(a + y)$, prove that $dy/dx = \sin^2(a + y)/\sin a$.

Solution : 6

$$\text{We have } \sin y = x \sin(a + y)$$

Differentiating both sides with respect to y we get ,

$$dx/dy = [\sin (a + y) \cdot \cos y - \sin y \cdot \cos (a + y)] / \sin^2 (a + y) .$$

$$= \sin (a + y - y) / \sin^2 (a + y)$$

$$= \sin a / \sin^2 (a + y)$$

$$\text{Hence , } dy/dx = \sin^2 (a + y) / \sin a . \text{ [Proved.]}$$

Q.7. If $y = x \sin y$, prove that $x \cdot dy/dx = y/(1 - x \cos y)$.

Solution : 7

We have $y = x \sin y$

Differentiating both sides with respect to x we get

$$dy/dx = x \cdot \cos y \, dy/dx + 1 \cdot \sin y$$

$$\text{Or, } dy/dx (1 - x \cos y) = \sin y$$

$$\text{Or, } dy/dx = \sin y / (1 - x \cos y)$$

$$\text{Or, } x \cdot dy/dx = (x \cdot \sin y) / (1 - x \cos y) \text{ [As, } y = x \sin y]$$

$$= y / (1 - x \cos y) \text{ [Proved.]}$$