

Lagrange's M. V. Theorem

Q.1. Use Lagrange's mean value theorem to determine a point P on the curve $y = \sqrt{x - 2}$ defined in the interval $[2, 3]$, where the tangent is parallel to the chord joining the end points on the curve.

Solution : 1

$y = \sqrt{x - 2}$ defined in $[2, 3]$

$dy/dx = 1/\{2\sqrt{x - 2}\}$ which exists for all $x \in (2, 3)$.

Therefore, (1) y is continuous in $[2, 3]$ and (2) y is differentiable in $(2, 3)$.

Therefore, $1/\{2\sqrt{x - 2}\} = [f(3) - f(2)]/(3 - 2) = (1 - 0)/(3 - 2) = 1$

Or, $1/2 = \sqrt{x - 2}$

Or, $1/4 = x - 2$

Or, $x = 2 + 1/4 = 9/4 \in (2, 3)$

Therefore, $y = \sqrt{9/4 - 2} = 1/2$.

Therefore, P is $(9/4, 1/2)$.

Q.2. Verify Lagrange's mean value theorem for the function $f(x) = 3x^2 - 5x + 1$ defined in interval $[2, 5]$.

Solution : 2

We have, $f(x) = 3x^2 - 5x + 1$ where, $x \in [2, 5]$.

(1) $f(x)$ is a polynomial function, hence continuous in the interval $[2, 5]$.

(2) $f(x)$ is a polynomial function, hence differentiable in the interval $(2, 5)$.

(3) $f(5) = 3(5)^2 - 5 \times 5 + 1 = 51$, $f(2) = 3(2)^2 - 5 \times 2 + 1 = 3$.

Also, $f'(x) = 6x - 5 \Rightarrow f'(c) = 6c - 5$.

Now, $f'(c) = [f(b) - f(a)]/(b - a)$

$$\text{Or , } 6c - 5 = [f(5) - f(2)]/(5 - 2) = (51 - 3)/(5 - 2) = 48/3 = 16$$

$$\text{Or , } 6c = 16 + 5 = 21 \Rightarrow c = 21/6 \in (2, 5) .$$

Hence , **Lagrange's mean value theorem is verified.**

Q.3. Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1).

Solution : 3

We have $y = (x - 3)^2$ where $x \in [3, 4]$.

(1) $f(x)$ is a polynomial function and is continuous in $[3, 4]$.

(2) $f'(x) = 2(x - 3)$, which exist for all $x \in (3, 4)$.

Thus both the conditions of Lagrange's mean value theorem is satisfied.

Hence , there must exist a point $c \in (3, 4)$ such that $f'(c) = [f(4) - f(3)]/(4 - 3) = 1$.

$$\text{Now , } f'(c) = 1 \Rightarrow 2(c - 3) = 1 \Rightarrow c = 7/2 \in (3, 4) .$$

$$\text{Thus when } x = 7/2 , y = (7/2 - 3)^2 = 1/4 .$$

Hence , the required point is $(7/2, 1/4)$ on the parabola , the tangent is parallel to the chord joining (3, 0) and (4, 1) .

Q.4. Examine the validity and conclusion of Lagrange's mean value theorem for the function $f(x) = x(x - 1)(x - 2)$, for every $x \in [0, 1/2]$.

Solution : 4

$$\text{We have } f(x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x .$$

(1) $f(x)$ is polynomial function and is continuous in $[0, 1/2]$.

(2) $f'(x) = 3x^2 - 6x + 2$, which exists and hence is differentiable in $(0, 1/2)$.

$$\text{Now } f'(c) = 3c^2 - 6c + 2 , f(0) = 0 , f(1/2) = 1/2(1/2 - 1)(1/2 - 2) = 3/8 .$$

$$[f(b) - f(a)]/(b - a) = f'(c)$$

$$\text{Or, } (3/8 - 0)/(1/2 - 0) = 3c^2 - 6c + 2$$

$$\text{Or, } 3/4 = 3c^2 - 6c + 2$$

$$\text{Or, } 12c^2 - 24c + 5 = 0$$

$$\text{Or, } c = [24 \pm \sqrt{(576 - 240)}] / 24 = [24 \pm \sqrt{336}] / 24 = 1 \pm [(\sqrt{21})/6] .$$

$$\text{Or, } c = 1 - \sqrt{21}/6 = 0.236 \text{ (approx).}$$

And $c = 1 + \sqrt{21}/6 > 1/2$, is not acceptable.

Q.5. Is Lagrange's mean value theorem applicable to : $f(x) = 4 - (6 - x)^{2/3}$ in $[5, 7]$.

Solution : 5

We have $f(x) = 4 - (6 - x)^{2/3}$.

Therefore , $f'(x) = (-2/3) \times (6 - x)^{-1/3} \times (-1) = 2/[3(6 - x)^{1/3}]$, which does not exist at $x = 6$, and $6 \in (5, 7)$.

Therefore $f(x)$ is not differentiable in the interval $(5, 7)$.

Hence, Lagrange's mean value theorem is not applicable.

Q.6. Show that the function $f(x) = x^2 - 6x + 1$ satisfies the Lagrange's Mean Value Theorem. Also find the co-ordinate of a point at which the tangent to the curve represented by the above function is parallel to the chord joining $A(1, -4)$ and $B(3, -8)$.

Solution : 6

We have $f(x) = x^2 - 6x + 1$,

(1) $f(x)$ being a polynomial function is continuous.

(2) $f'(x) = 2x - 6$, i.e. $f(x)$ is differentiable.

$$\text{Slope of line AB} = f'(x) = \{-8 - (-4)\}/(3 - 1) = -2.$$

$$\text{Or, } 2x - 6 = -2$$

$$\text{Or, } x = 2 \text{ and } y = f(x) = (2)^2 - 6 \times 2 + 1 = -7.$$

Hence , point is $(2, -7)$.