

Integration of some Functions

Q.1. Evaluate $\int x^2/(x^2 - 4) dx$.

Solution : 1

$$\begin{aligned}\int x^2/(x^2 - 4) dx &= \int (x^2 - 4 + 4)/(x^2 - 4) dx = \int \{1 + 4/(x^2 - 4)\} dx \\&= x + 4 \times \{1/(2 \times 2)\} \log [(x - 2)/(x + 2)] + c \\&= x + \log [(x - 2)/(x + 2)] + c.\end{aligned}$$

Q.2. Integrate the following : $(2 \sin x + 3 \cos x)/(5 \sin x + 12 \cos x)$.

Solution : 2

$$\begin{aligned}\text{Let } 2 \sin x + 3 \cos x &\equiv A(5 \sin x + 12 \cos x) + B \cdot d/dx(5 \sin x + 12 \cos x) \\&\equiv A(5 \sin x + 12 \cos x) + B(5 \cos x - 12 \sin x) \\&\equiv \sin x (5A - 12B) + \cos x (12A + 5B)\end{aligned}$$

Equating the coefficients of $\sin x$ and $\cos x$, we get

$$5A - 12B = 2, \quad 12A + 5B = 3.$$

Solving, we get $A = 46/169$ and $B = -9/169$.

Therefore, $2 \sin x + 3 \cos x = 46/169(5 \sin x + 12 \cos x) - 9/169(5 \cos x - 12 \sin x)$.

Therefore, $\int [(2 \sin x + 3 \cos x)/(5 \sin x + 12 \cos x)] \cdot dx$

$$\begin{aligned}&= (46/169) \int [(5 \sin x + 12 \cos x)/(5 \sin x + 12 \cos x)] \cdot dx - (9/169) \int [(5 \cos x - 12 \sin x)/(5 \sin x + 12 \cos x)] \cdot dx \\&= (46/169) \int dx - (9/169) \int [(5 \cos x - 12 \sin x)/(5 \sin x + 12 \cos x)] \cdot dx \\&= (46/169)x - (9/169) \int dt/t, \text{ [where } 5 \sin x + 12 \cos x = t] = (46/169)x - \log |t| + c \\&= (46/169)x - (9/169) \log |5 \sin x + 12 \cos x| + c.\end{aligned}$$

Q.3. Evaluate : $\int [1/(1 + \tan x)].dx$.

Solution : 3

$$\text{Let } I = \int [1/(1 + \tan x)].dx$$

$$1/(1 + \tan x) = 1/(1 + \sin x/\cos x) = \cos x/(\cos x + \sin x)$$

$$= 1/2 [(\cos x + \sin x + \cos x - \sin x)/(\cos x + \sin x)]$$

$$= 1/2 [1 + (\cos x - \sin x)/(\cos x + \sin x)]$$

$$\text{Therefore, } I = 1/2 [\int 1.dx + \int [(\cos x - \sin x)/(\cos x + \sin x)].dx$$

$$= 1/2 [x + \log |\sin x + \cos x|] + c.$$

Q.4. Evaluate : $\int dx/(a \sin x + b \cos x)$.

Solution : 4

$$\text{Let } I = \int dx/(a \sin x + b \cos x).$$

$$\text{Put } a = r \cos \alpha \text{ and } b = r \sin \alpha, \text{ so that } r^2 = a^2 + b^2 \text{ and } \alpha = \tan^{-1} (b/a)$$

$$\text{Therefore, } I = \int dx/(a \sin x + b \cos x) = \int dx/(r \cos \alpha \sin x + r \sin \alpha \cos x)$$

$$= (1/r) \int dx/\sin (x + \alpha) = (1/r) \int \operatorname{cosec} (x + \alpha).dx$$

$$= (1/r) \log |\tan \{(x + \alpha)/2\}| + c$$

$$= \{1/\sqrt{(a^2 + b^2)}\} \log |\tan \{x/2 + 1/2 \tan^{-1} (b/a)\}| + c.$$

Q.5. Evaluate : $\int d\theta/(\sin^4 \theta + \cos^4 \theta)$.

Solution : 5

$$\text{Let } I = \int d\theta/(\sin^4 \theta + \cos^4 \theta).$$

Dividing both num. and denom. by $\cos^4 \theta$, we get

$$I = \int \sec^4 \theta d\theta/(1 + \tan^4 \theta); \text{ Put } \tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt$$

Therefore, $I = \int [(1 + t^2)/(1 + t^4)]dt$, Dividing Num. and Denom. By t^2 , we get

$$I = \int [\{1 + (1/t^2)\}/\{t^2 + (1/t^2)\}]dt \text{ Put } t - 1/t = z \Rightarrow (1 + 1/t^2)dt = dz$$

$$\text{And } t^2 + 1/t^2 = (t - 1/t)^2 + 2 = 2 + z^2,$$

$$\text{Therefore, } I = \int dz/(2 + z^2) = 1/\sqrt{2} \tan^{-1} (z/\sqrt{2}) + c$$

$$= 1/\sqrt{2} \tan^{-1} \{(t^2 - 1)t/\sqrt{2}\} + c = 1/\sqrt{2} \tan^{-1} \{(\tan^2 \theta - 1)/\tan \theta \sqrt{2}\} + c.$$

Q.6. Evaluate : $\int dx/(a^2 \sin^2 x + b^2 \cos^2 x)$.

Solution : 6

$$\text{Let } I = \int dx/(a^2 \sin^2 x + b^2 \cos^2 x) \text{ [Dividing Nr and Dr by } \cos^2 x]$$

$$= \int (\sec^2 x \, dx)/(a^2 \tan^2 x + b^2)$$

$$= \int dt/(a^2 t^2 + b^2) \text{ [Putting } \tan x$$

$$= t \text{ so that } \sec^2 x \, dx = dt] = 1/a^2 \int dt/\{t^2 + (b/a)^2\}$$

$$= (1/a^2) \{1/(b/a)\} \tan^{-1} \{t/(b/a)\} + c = (1/ab) \tan^{-1} (at/b) + c$$

$$= (1/ab) \tan^{-1} \{(a \tan x)/b\} + c.$$

Q.7. Evaluate : $\int dx/(2 + \cos x)$.

Solution : 7

$$\int dx/(2 + \cos x) = \int dx/\{1 + (1 + \cos x)\}$$

$$= \int dx/\{1 + 2 \cos^2 (x/2)\}$$

$$= \int [\{\sec^2 (x/2)\}/\{\sec^2 (x/2) + 2\}].dx$$

$$\text{[Dividing Nr \& Dr by } \cos^2 (x/2)]$$

$$= \int [\{\sec^2 (x/2)\}/\{3 + \tan^2 (x/2)\}].dx$$

$$\text{[Put } \tan (x/2) = t \text{ so that } [\{\sec^2 (x/2)\}/(1/2)]dx = dt = 2 \int dt/(3 + t^2)]$$

$$= 2 \int dt / \{(\sqrt{3})^2 + t^2\} = 2.(1/\sqrt{3}) \tan^{-1} (t/\sqrt{3}) + c$$

$$= (2/\sqrt{3}) \tan^{-1} [\{\tan (x/2)\}/\sqrt{3}] + c.$$

Q.8. Evaluate : $\int dx/(5 + 4 \cos x)$.

Solution : 8

$$\int dx/(5 + 4 \cos x) = \int dx/[5 + 4 \{1 - \tan^2 (x/2)\}/\{1 + \tan^2 (x/2)\}]$$

$$= \int [\{1 + \tan^2 (x/2)\}/\{9 + \tan^2 (x/2)\}].dx$$

[Put $\tan (x/2) = t$, then $\sec^2 (x/2) dx = dt$]

$$= 2 \int dt/(32 + t^2)$$

$$= (2/3) \tan^{-1} (t/3) + c$$

$$= (2/3) \tan^{-1} [\{\tan (x/2)\}/3] + c.$$

Q.9. Evaluate : $\int [x/\sqrt{4 - x^4}].dx$.

Solution : 9

Let $I = \int [x/\sqrt{4 - x^4}].dx$, Put $x^2 = t$ then $2x dx = dt$ or, $x dx = 1/2 dt$

Therefore, $I = \int [(1/2 dt)/\sqrt{4 - t^2}]$

$$= (1/2) \int dt/\sqrt{2^2 - t^2}$$

$$= (1/2) \sin^{-1} (t/2) + c$$

$$= (1/2) \sin^{-1} (x^2/2) + c.$$

Q.10. Evaluate : $\int [\sqrt{a - x}/\sqrt{a + x}].dx$.

Solution : 10

$$\begin{aligned}
\text{Let } I &= \int [\sqrt{(a-x)}/\sqrt{(a+x)}].dx \\
&= \int [\sqrt{(a-x)}/\sqrt{(a+x)}].[\sqrt{(a-x)}/\sqrt{(a-x)}].dx \\
&= \int [(a-x)/\sqrt{(a^2-x^2)}].dx = a \int dx/\sqrt{(a^2-x^2)} - \int [x/\sqrt{(a^2-x^2)}].dx \\
&= a \sin^{-1}(x/a) + 1/2 \int dt/\sqrt{(t)} [\\
&\text{Where } a^2 - x^2 = t \text{ so that } -2x dx = dt] \\
&\text{Therefore, } I = a \sin^{-1}(x/a) + (1/2) \cdot 2\sqrt{(t)} + c \\
&= a \sin^{-1}(x/a) + \sqrt{(a^2-x^2)} + c.
\end{aligned}$$

Q.11. Evaluate : $\int [(6x+5)/\sqrt{(6+x-2x^2)}].dx.$

Solution : 11

$$\begin{aligned}
\text{Let } I &= \int [(6x+5)/\sqrt{(6+x-2x^2)}].dx \\
\text{Let } 6x+5 &= A \cdot \frac{d}{dx}(6+x-2x^2) + B = A(1-4x) + B. \\
&\text{Equating coefficients of } x \text{ and constants, we get} \\
&-4A = 6 \text{ and } A+B = 5 \Rightarrow A = -3/2, B = 13/2. \\
&\text{Therefore, } I = \int [-3/2(1-4x) + 13/2]/\sqrt{(6+x-2x^2)}.dx \\
&= -3/2 \int [(1-4x)/\sqrt{(6+x-2x^2)}].dx + 13/2 \int dx/\sqrt{(6+x-2x^2)} \\
&= -3/2 [\{\sqrt{(6+x-2x^2)}\}/(1/2)] + 13/2 \int [1/\sqrt{\{2(3+x/2-x^2)\}}].dx \\
&= -3\sqrt{(6+x-2x^2)} + \{13/(2\sqrt{2})\} \int [1/\sqrt{\{(7/4)^2 - (x-1/4)^2\}}].dx \\
&= -3\sqrt{(6+x-2x^2)} + \{13/(2\sqrt{2})\} \sin^{-1} \{(x-1/4)/(7/4)\} + c \\
&= -3\sqrt{(6+x-2x^2)} + \{13/(2\sqrt{2})\} \sin^{-1} \{(4x-1)/7\} + c.
\end{aligned}$$

Q.12. Evaluate : $\int [\sqrt{\{(1+x)/x\}}].dx.$

Solution : 12

$$\text{Let } I = \int [\sqrt{\{(1+x)/x\}}].dx$$

$$= \int [\sqrt{\{(1+x)/x\}}].[\sqrt{\{(1+x)/(1+x)\}}].dx$$

$$= \int [(1+x)/\sqrt{x(1+x)}].dx = \int [(1+x)/\sqrt{(x^2+x)}].dx$$

$$\text{Let } 1+x = A. \frac{d}{dx}(x^2+x) + B = A(2x+1) + B$$

Equating coefficients of like terms, we get $A = 1/2$, $B = 1/2$.

$$\text{Therefore, } I = \int [\{1/2(2x+1) + 1/2\}/\sqrt{(x^2+x)}].dx$$

$$= 1/2 \int [(2x+1)/\sqrt{(x^2+x)}].dx + 1/2 \int [1/\sqrt{(x^2+x)}].dx$$

$$= I_1 + I_2 \quad I_1 = 1/2 \int [(2x+1)/\sqrt{(x^2+x)}].dx \quad [\text{Put } x^2+x = t \text{ then } (2x+1)dx = dt]$$

$$\text{Therefore, } I_1 = 1/2 \int dt/\sqrt{t} = \sqrt{t};$$

$$\text{And } I_2 = 1/2 \int [1/\sqrt{(x^2+x)}].dx$$

$$= 1/2 \int [1/\sqrt{\{(x+1/2)^2 - (1/2)^2\}}].dx$$

$$= 1/2 \log |(x+1/2) + \sqrt{(x^2+x)}|$$

$$\text{Therefore, } I = I_1 + I_2 = \sqrt{(x^2+x)} + 1/2 \log |(2x+1)/2 + \sqrt{(x^2+x)}| + c.$$

Q.13. Evaluate : $\int [\sqrt{(2ax-x^2)}].dx.$

Solution : 13

$$\text{Let } I = \int [\sqrt{(2ax-x^2)}].dx$$

$$= \int [\sqrt{\{a^2 - (x-a)^2\}}].dx \quad [\text{Put } x-a = t \text{ then } dx = dt]$$

$$\text{Therefore, } I = \int \sqrt{(a^2-t^2)}.dt$$

$$= (1/2)t \sqrt{(a^2-t^2)} + (1/2) a^2 \sin^{-1}(t/a) + c$$

$$= 1/2 (x-a) \sqrt{\{a^2 - (x-a)^2\}} + (1/2) a^2 \sin^{-1} \{(x-a)/a\} + c$$

$$= 1/2 (x-a) \sqrt{(2ax-x^2)} + (1/2) a^2 \sin^{-1} \{(x-a)/a\} + c.$$

Q.14. Evaluate the following integral : $\int e^{2x}/(2+e^x) dx.$

Solution : 14

Put $2 + e^x = t$, then $e^x dx = dt$

Therefore, $\int e^{2x}/(2 + e^x) dx = \int [(t - 2)^2/t] \cdot dt/(t - 2)$

$$= \int [(t - 2)/t] dt$$

$$= t - 2 \log | t | + c$$

$$= e^x - 2 \log | e^x + 2 | + c.$$