

Successive Differentiation

Q.1. If $y = e^{m \cos^{-1} x}$, prove that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = m^2 y$.

Solution : 1

We have, $y = e^{m \cos^{-1} x}$

Taking log of both sides, we get

$$\log_e y = m \cos^{-1} x \log_e e$$

$$\text{Or, } \log_e y = m \cos^{-1} x [\log_e e = 1]$$

Differentiating both sides with respect to x , we get

$$(1/y) \frac{dy}{dx} = m \cdot (-1)/\sqrt{1 - x^2}$$

$$\text{Or, } \frac{dy}{dx} = -m y / \sqrt{1 - x^2}$$

$$\text{Or, } \frac{dy}{dx} = (-m \times e^{m \cos^{-1} x}) / \sqrt{1 - x^2}$$

Differentiating both sides again w. r. t. x , we get $\frac{d^2y}{dx^2}$

$$= -m [-\sqrt{1 - x^2} \cdot \{m e^{m \cos^{-1} x} / \sqrt{1 - x^2}\} + (2x e^{m \cos^{-1} x}) / \{2\sqrt{1 - x^2}\}] / (1 - x^2)$$

$$\text{Or, } (1 - x^2) \frac{d^2y}{dx^2} = m^2 \cdot e^{m \cos^{-1} x} - m x e^{m \cos^{-1} x} / \sqrt{1 - x^2}$$

$$\text{Or, } (1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = m^2 y \text{ . [Proved.]}$$

Q.2. If $y = e^x (\sin x + \cos x)$, prove that $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$.

Solution : 2

We have $y = e^x (\sin x + \cos x)$

Then differentiating w. r. t. x , we get

$$\frac{dy}{dx} = e^x \cdot (\sin x + \cos x) + e^x (\cos x - \sin x)$$

Or, $dy/dx = 2 e^x \cos x$.

Differentiating again w. r. t. x , we get

$$d^2y/dx^2 = 2 e^x \cos x + 2 e^x (-\sin x)$$

$$\text{Or, } d^2y/dx^2 = 2 e^x (\cos x - \sin x) .$$

$$\text{L. H. S.} = d^2y/dx^2 - 2 dy/dx + 2y$$

$$= 2 e^x (\cos x - \sin x) - 2 \times 2e^x \cos x + 2 e^x (\sin x + \cos x)$$

$$= 2 e^x [\cos x - \sin x - 2 \cos x + \sin x + \cos x]$$

$$= 0 . \text{ [Proved.]}$$

Q.3. If $y = A \cos nx + B \sin nx$, show that $d^2y/dx^2 + n^2 y = 0$.

Solution : 3

We have , $y = A \cos nx + B \sin nx$

Differentiating w. r. t. x , we get

$$dy/dx = -A n \sin nx + B n \cos nx .$$

Differentiating again w. r. t. x , we get

$$d^2y/dx^2 = -(An) n \cos nx - (Bn) n \sin nx$$

$$\text{Or , } d^2y/dx^2 = -n^2 (A \cos nx + B \sin nx) = -n^2 y .$$

$$\text{Hence , } d^2y/dx^2 + n^2 y = 0 . \text{ [Proved.]}$$

Q.4. Find d^2y/dx^2 , when $y = \log (x^2/ex)$.

Solution : 4

$$y = \log (x^2/ex) = \log x^2 - \log ex$$

$$= 2 \log x - x \log e = 2 \log x - x \text{ [As, } \log e = 1]$$

Or, $y = 2 \log x - x$

Differentiating w. r. t. x , we get

$$dy/dx = 2/x - 1.$$

Differentiating again w. r. t. x , we get

$$d^2y/dx^2 = -2/x^2.$$

Q.5. If $y = \operatorname{cosec} x + \cot x$, show that $\sin x \cdot d^2y/dx^2 = y^2$.

Solution : 5

We have, $y = \operatorname{cosec} x + \cot x$ --- (1)

Differentiating w. r. t. x , we get

$$dy/dx = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

$$= -\operatorname{cosec} x (\cot x + \operatorname{cosec} x)$$

$$= -y \operatorname{cosec} x \text{ --- (2) [Using (1)]}$$

differentiating again w. r. t. x , we get

$$d^2y/dx^2 = -[dy/dx \cdot \operatorname{cosec} x + y \cdot (-\operatorname{cosec} x \cot x)]$$

$$= -[(-y \operatorname{cosec} x) \cdot \operatorname{cosec} x - y \operatorname{cosec} x \cot x] \text{ [By (2)]}$$

$$= y \operatorname{cosec}^2 x + y \operatorname{cosec} x \cot x = y \operatorname{cosec} x (\operatorname{cosec} x + \cot x)$$

$$= y \cdot \operatorname{cosec} x \cdot y \text{ [By (1)]} = y^2 \operatorname{cosec} x = y^2/\sin x$$

$$\text{Or, } \sin x \cdot d^2y/dx^2 = y^2. \text{ [Proved.]}$$

Q.6. If $y = \tan^{-1} x$, show that $(1 + x^2) d^2y/dx^2 + 2x dy/dx = 0$.

Solution : 6

We have, $y = \tan^{-1} x$

Differentiating w. r. t. x , we get

$$dy/dx = 1/(1 + x^2)$$

$$\text{Or, } (1 + x^2) dy/dx = 1$$

Differentiating again w. r. t. x , we get

$$(1 + x^2) d^2y/dx^2 + 2x \cdot dy/dx = 0 . \text{ [Proved.]}$$

Q.7. If $y = x + \tan x$, prove that $\cos^2 x \cdot d^2y/dx^2 - 2y + 2x = 0$.

Solution : 7

We have , $y = x + \tan x$

Differentiating w. r. t. x , we get

$$dy/dx = 1 + \sec^2 x$$

Differentiating again w. r. t. x , we get

$$d^2y/dx^2 = 0 + 2 \sec x \cdot \sec x \tan x$$

$$= 2 \sec^2 x \tan x$$

$$= 2 \tan x / \cos^2 x .$$

$$\text{Or, } \cos^2 x \cdot d^2y/dx^2 = 2 \tan x = 2(y - x)$$

$$\text{Or, } \cos^2 x \cdot d^2y/dx^2 - 2(y - x) = 0 . \text{ [Proved.]}$$

Q.8. Find d^2y/dx^2 ; if $x = at^2$, $y = 2 at$.

Solution : 8

We have , $x = at^2 \Rightarrow dx/dt = 2at$,

Also , $y = 2 at \Rightarrow dy/dt = 2 a$.

Therefore , $dy/dx = (dy/dt)/(dx/dt) = (2 a)/(2 at) = 1/t$,

Hence , $d^2y/dx^2 = d/dx (1/t)$

$$= (-1/t^2).dt/dx$$

$$= (-1/t^2)/(dx/dt) = (-1/t^2)/(2 at) = (-1/2) a t^3 .$$

Q.9. If $y = e^{3x}(A + Bx)$, then prove that : $d^2y/dx^2 - 6 dy/dx + 9 y = 0$.

Solution : 9

We have , $y = e^{3x} (A + Bx)$ ----- (1)

Differentiating w. r. t. x , we get

$$dy/dx = e^{3x} . 3 . (A + Bx) + e^{3x} . B$$

$$= 3y + B e^{3x} \text{ ----- (2) [Using (1)]}$$

Differentiating again w. r. t. x , we get

$$d^2y/dx^2 = 3 dy/dx + 3B e^{3x}$$

$$= 3 dy/dx + 3(dy/dx - 3y), \text{ [Using (2)]}$$

$$= 3 dy/dx + 3 dy/dx - 9 y,$$

$$\text{Or, } d^2y/dx^2 - 6dy/dx + 9 y = 0 . \text{ [Proved.]}$$

Q.10. If $y = b e^x + c e^{2x}$, then prove that $d^2y/dx^2 - 3 dy/dx + 2y = 0$.

Solution : 10

We have , $y = b e^x + c e^{2x}$ ----- (1)

Differentiating both sides w. r. t. x , we get

$$dy/dx = b e^x + 2c e^{2x} \text{ ----- (2)}$$

Subtracting (1) from (2) we get

$$dy/dx - y = c e^{2x} \text{ ----- (3)}$$

Differentiating again w. r. t. x , we get

$$d^2y/dx^2 - dy/dx = 2c e^{2x} \text{----- (4)}$$

Multiplying (3) by 2 and then subtracting from (4) we get

$$d^2y/dx^2 - dy/dx - 2 (dy/dx - y) = 0$$

$$\text{Or, } d^2y/dx^2 - 3 dy/dx + 2 y = 0 \text{ [Proved.]}$$

Q.11. If $y = a e^{2x} + b e^{-x}$, then prove that $d^2y/dx^2 - dy/dx - 2y = 0$.

Solution : 11

$$\text{We have, } y = a e^{2x} + b e^{-x} \text{----- (1)}$$

Differentiating both sides w. r. t. x , we get

$$dy/dx = 2a e^{2x} - b e^{-x} \text{----- (2)}$$

Adding (1) and (2) we get

$$dy/dx + y = 3a e^{2x} \text{----- (3)}$$

Differentiating both sides of (3) w. r. t. x , we get

$$d^2y/dx^2 + dy/dx = 3a e^{2x} \cdot 2$$

$$= 2 \cdot (dy/dx + y) \text{ [Using (3)]}$$

$$\text{Or, } d^2y/dx^2 - dy/dx - 2y = 0 \text{ . [Proved.]}$$