

Inverse Trigonometric Function

Q.1. Evaluate the following :

- i. $\sin (\cot^{-1} x)$
- ii. $\sin (2 \sin^{-1} 0.8)$
- iii. $\tan [2 \tan^{-1} (1/5) - \pi / 4]$
- iv. $\sin \cot^{-1} \cos \tan^{-1} x$.
- v. $\cos^{-1} x + \cos^{-1} [x/2 + \{\sqrt{(3 - 3x^2)}/2\}] , 1/2 \leq x \leq 1$.

Solution : 1

i. $\sin (\cot^{-1} x) = \sin [\sin^{-1} 1/\sqrt{(1 + x^2)}] = 1/\sqrt{(1 + x^2)} .$

ii. As $2 \sin^{-1} x = \sin^{-1} 2x \sqrt{(1 - x^2)} ;$

Therefore, $\sin (2 \sin^{-1} 0.8)$
 $= \sin [\sin^{-1} \{2 \times 8/10 \times \sqrt{(1 - 64/100)}\}]$
 $= \sin (\sin^{-1} 0.96) = 0.96 .$

iii. As $2 \tan^{-1} x = \tan^{-1} \{2x / (1 - x^2)\} ;$

Therefore, $\tan [2 \tan^{-1} (1/5) - \pi / 4]$
 $= \tan [\tan^{-1} \{2 \times (1/5) / (1 - 1/25)\} - \tan^{-1} 1]$
 $= \tan [\tan^{-1} (5/12) - \tan^{-1} 1]$
 $= \tan [\tan^{-1} \{(5/12)^{-1} / (1 + 5/12 \times 1)\}]$
 $= \tan \tan^{-1} (-7/17) = -7/17.$

iv. We have to evaluate , $\sin \cot^{-1} \cos \tan^{-1} x .$

Let $\tan^{-1} x = \theta$. Then $\cos \tan^{-1} x = \cos \theta = 1/\sqrt{(1 + x^2)} = z$ (say) .

Let $\cot^{-1} z = \phi$. Then $\cot \phi = z \Rightarrow \sin \phi = 1/\sqrt{(1 + z^2)} = \sqrt{(1 + x^2)}/\sqrt{(2 + x^2)} .$

v. We have to evaluate , $\cos^{-1} x + \cos^{-1} [x/2 + \sqrt{(3 - 3x^2)}/2]$.

Let $\cos^{-1} x = \theta$. Then $\cos \theta = x$.

Now , $1/2 \leq x \leq 1 \Rightarrow \cos^{-1} 1 \leq \cos^{-1} x \leq \cos^{-1} 1/2 \Rightarrow 0 \leq \theta \leq \pi/3$.

Therefore , given expression = $\cos^{-1} x + \cos^{-1} [x \cdot 1/2 + \sqrt{3}/2 \cdot \sqrt{(1 - x^2)}]$

= $\theta + \cos^{-1} [\cos \theta \cdot \cos \pi/3 + \sin \pi/3 \cdot \sin \theta]$

= $\theta + \cos^{-1} \cos (\pi/3 - \theta) = \theta + \pi/3 - \theta = \pi/3$.

Q.2. Show that : $\tan^{-1} 1/2 + \tan^{-1} 1/3 = \pi/4$.

Solution : 2

We know that : $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \{(x + y)/(1 - xy)\}$, if $xy < 1$.

[Here $x = 1/2$, $y = 1/3$, $xy = 1/6 < 1$].

Hence, $\tan^{-1} 1/2 + \tan^{-1} 1/3$

= $\tan^{-1} [(1/2 + 1/3)/(1 - 1/2 \times 1/3)]$

= $\tan^{-1} [(5/6)/(5/6)] = \tan^{-1} 1 = \pi/4$. **[Proved.]**

Q.3. Prove that : $\tan^{-1} x + \cot^{-1} (x + 1) = \tan^{-1} (x^2 + x + 1)$

Solution : 3

L. H. S. = $\tan^{-1} x + \cot^{-1} (x + 1)$

= $\tan^{-1} x + \tan^{-1} \{1/(x + 1)\}$

= $\tan^{-1} [\{x + 1/(x + 1)\}/\{1 - x/(x + 1)\}]$

[As, $x \times \{1/(x + 1)\} = x/(x + 1) < 1$]

$$= \tan^{-1}(x^2 + x + 1) = \text{R. H. S. [Proved.]}$$

Q.4. Prove that : $\sin^{-1}x/\sqrt{1+x^2} + \cos^{-1}(x+1)/\sqrt{x^2+2x+2} = \tan^{-1}(x^2+x+1)$.

Solution : 4

We know that $\sin^{-1}x = \tan^{-1}\{x/\sqrt{1-x^2}\}$

$$\begin{aligned}\text{Therefore, } \sin^{-1}\{x/\sqrt{1+x^2}\} &= \tan^{-1}\left[\frac{x/\sqrt{1+x^2}}{\sqrt{1-x^2/(1+x^2)}}\right] \\ &= \tan^{-1}x.\end{aligned}$$

Also, $\cos^{-1}x = \tan^{-1}\{\sqrt{1-x^2}/x\}$

$$\begin{aligned}\text{Therefore, } \cos^{-1}\{(x+1)/\sqrt{x^2+2x+2}\} &= \tan^{-1}\left[\frac{\sqrt{1-(x+1)^2/(x^2+2x+2)}}{(x+1)/\sqrt{x^2+2x+2}}\right] \\ &= \tan^{-1}\{1/(x+1)\} = \cot^{-1}(x+1).\end{aligned}$$

Now it is the question no. 3 above.

Q.5. Solve for x : $\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1}(2/3)$.

Solution : 5

We have, $\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1}(2/3)$

$$\text{Or, } \tan^{-1}\left[\frac{(2+x) + (2-x)}{1 - (2+x)(2-x)}\right] = \tan^{-1}(2/3) \quad [(2+x)(2-x) < 1]$$

$$\text{Or, } \tan^{-1}\left[\frac{4}{x^2-3}\right] = \tan^{-1}(2/3) \quad [x^2 > 3]$$

$$\text{Or, } 4/(x^2-3) = 2/3, \quad [|x| > \sqrt{3}]$$

$$\text{Or, } x^2-3 = 6 \quad [|x| > \sqrt{3}]$$

$$\text{Or, } x^2 = 9 \quad [|x| > \sqrt{3}]$$

Or, $x = 3$ or -3 , these values satisfy $|x| > \sqrt{3}$.

Q.6. Solve for x : $\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$.

Solution : 6

We have, $\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$

Or, $\tan^{-1} \{(2x + 3x)/(1 - 2x.3x)\} = \pi/4$ $2x.3x < 1$

Or, $5x/(1 - 6x^2) = \tan \pi/4$, $x^2 < 1/6$

Or, $5x/(1 - 6x^2) = 1$, $|x| < 1/\sqrt{6}$

Or, $6x^2 + 5x - 1 = 0$, $-1/\sqrt{6} < x < 1/\sqrt{6}$

Or, $(6x - 1)(x + 1) = 0$, $-1/\sqrt{6} < x < 1/\sqrt{6}$ Or, $x = 1/6$ or -1 , $-1/\sqrt{6} < x < 1/\sqrt{6}$

But only $x = 1/6$ satisfy the condition $-1/\sqrt{6} < x < 1/\sqrt{6}$,

Hence $x = 1/6$ is the solution of the given equation.

Q.7. Prove that : $\tan^{-1} 1/4 + \tan^{-1} 2/9 = \cos^{-1} 2/\sqrt{5}$.

Solution : 7

$\tan^{-1} 1/4 + \tan^{-1} 2/9$

$= \tan^{-1} \{(1/4 + 2/9)/(1 - 1/4 .2/9)\}$

$= \tan^{-1} \{(17/36)/(17/18)\} = \tan^{-1} 1/2$ ----- (1)

we know that, $\cos^{-1} x = \tan^{-1} \{\sqrt{(1 - x^2)}/x\}$

Therefore, $\cos^{-1} 2/\sqrt{5} = \tan^{-1} [\{\sqrt{(1 - 4/5)}\}/(2/\sqrt{5})]$

$= \tan^{-1} 1/2$ ----- (2)

From (1) and (2) we get , $\tan^{-1} 1/4 + \tan^{-1} 2/9 = \cos^{-1} 2/\sqrt{5}$. **[Proved.]**

Q.8. Prove that : $4 \tan^{-1} 1/5 - \tan^{-1} 1/70 + \tan^{-1} 1/99 = \pi/4$.

Solution : 8

$$\begin{aligned}
\text{L. H. S.} &= 2 (\tan^{-1} 1/5) - (\tan^{-1} 1/70 - \tan^{-1} 1/99) \\
&= 2 \tan^{-1} [(2/5)/\{1 - (1/5)^2\}] - \tan^{-1} [(1/70 - 1/99)/(1 + 1/70 \cdot 1/99)] \\
&= 2 \tan^{-1} 5/12 - \tan^{-1} 29/6931 = 2 \tan^{-1} 5/12 - \tan^{-1} 1/239 \\
&= \tan^{-1} [(2 \times 5/12)/\{1 - (5/12)^2\}] - \tan^{-1} 1/239 = \tan^{-1} 120/119 - \tan^{-1} 1/239 \\
&= \tan^{-1} [(120/119 - 1/239)/(1 + 120/119 \times 1/239)] \\
&= \tan^{-1} (28561/28561) = \tan^{-1} 1 = \pi/4 = \text{R. H. S. [Proved.]}
\end{aligned}$$

Q.9. Prove that : $2 \tan^{-1} (1/3) + \cot^{-1} (4) = \tan^{-1} (16/13)$.

Solution : 9

$$\begin{aligned}
\text{L. H. S.} &= 2 \tan^{-1} (1/3) + \cot^{-1} (4) \\
&= 2 \tan^{-1} (1/3) + \tan^{-1} (1/4) \\
&= \tan^{-1} [(2/3)/\{1 - (1/3)^2\}] + \tan^{-1} (1/4) \\
&= \tan^{-1} [(2/3)/(8/9)] + \tan^{-1} (1/4) \\
&= \tan^{-1} (3/4) + \tan^{-1} (1/4) \\
&= \tan^{-1} [(3/4 + 1/4)/(1 - 3/16)] \\
&= \tan^{-1} [1/(13/16)] = \tan^{-1} (16/13) = \text{R. H. S. [Proved.]}
\end{aligned}$$

Q.10. Show that : $\sin^{-1} (1/\sqrt{17}) + \cos^{-1} (9/\sqrt{85}) = \tan^{-1} (1/2)$.

Solution : 10

Let $\sin^{-1} (1/\sqrt{17}) = \theta$, then $\sin \theta = 1/\sqrt{17}$,

Therefore, $\tan \theta = \sin \theta / \cos \theta = \sin \theta / \sqrt{1 - \sin^2 \theta} = (1/\sqrt{17}) / \sqrt{1 - (1/\sqrt{17})^2}$

Or, $\tan \theta = 1/4 \Rightarrow \theta = \tan^{-1} (1/4)$;

Again , let $\cos^{-1} (9/\sqrt{85}) = \phi$, then $\cos \phi = 9/\sqrt{85}$,

Therefore , $\tan \phi = \sin \phi / \cos \phi = \sqrt{(1 - \cos^2 \phi)} / \cos \phi = \sqrt{1 - (9/\sqrt{85})^2} / (9/\sqrt{85})$

Or, $\tan \phi = 2/9 \Rightarrow \phi = \tan^{-1} (2/9)$;

Now, L.H.S. = $\sin^{-1} (1/\sqrt{17}) + \cos^{-1} (9/\sqrt{85})$

= $\tan^{-1} (1/4) + \tan^{-1} (2/9)$

= $\tan^{-1} [(1/4 + 2/9) / \{1 - (1/4)(2/9)\}]$

= $\tan^{-1} [\{(9 + 8)/36\} / \{(36 - 2)/36\}] = \tan^{-1} (17/34) = \tan^{-1} (1/2)$ **[Proved.]**

Q.11. Prove that : $\cot (\pi/4 - 2 \cot^{-1} 3) = 7$.

Solution : 11

L.H.S. = $\cot (\pi/4 - 2 \cot^{-1} 3)$

= $\cot \{ \pi/4 - 2 \tan^{-1} (1/3) \}$ [As, $\cot^{-1} x = \tan^{-1}(1/x)$]

= $\cot [\tan^{-1} (1) - \tan^{-1} \{(2/3)/(1 - 1/9)\}]$

= $\cot [\tan^{-1} (1) - \tan^{-1} \{(2/3)/(8/9)\}]$

= $\cot [\tan^{-1} (1) - \tan^{-1} (3/4)]$

= $\cot [\tan^{-1} \{(1 - 3/4)/(1 + 3/4)\}]$

= $\cot [\tan^{-1} (1/7)] = \cot [\cot^{-1} (7)] = 7$ R.H.S. **[Proved.]**

Q.12. Prove that : $2 (\tan^{-1} 1 + \tan^{-1} 1/2 + \tan^{-1} 1/3) = \pi$.

Solution : 12

L.H.S. = $2 (\tan^{-1} 1 + \tan^{-1} 1/2 + \tan^{-1} 1/3)$

= $2 [\tan^{-1} (1 + 1/2)/(1 - 1/2) + \tan^{-1} 1/3]$

= $2 [\tan^{-1} \{(3/2)/(1/2)\} + \tan^{-1} 1/3]$

$$\begin{aligned}
&= 2 [\tan^{-1} 3 + \tan^{-1} 1/3] \\
&= 2 [\tan^{-1} \{(3 + 1/3)/(1 - 3 \times 1/3)\}] \\
&= 2 [\tan^{-1} \{(10/3)/0\}] \\
&= 2 \tan^{-1} (\infty) = 2 \times \pi/2 = \pi = \text{R.H.S.} \quad \text{[Proved.]}
\end{aligned}$$

Q.13. Show that : $\sin^{-1} 4/5 + \cos^{-1} 2/\sqrt{5} = \cot^{-1} 2/11$.

Solution : 13

We know that : $\sin^{-1} 4/5 = \tan^{-1} 4/3$ and $\cos^{-1} 2/\sqrt{5} = \tan^{-1} 1/2$.

$$\begin{aligned}
\text{L.H.S.} &= \sin^{-1} 4/5 + \cos^{-1} 2/\sqrt{5} \\
&= \tan^{-1} 4/3 + \tan^{-1} 1/2 \\
&= \tan^{-1} [(4/3 + 1/2)/\{1 - (4/3 \times 1/2)\}] \\
&= \tan^{-1} [(11/6)/(6 - 4)/6] = \tan^{-1} [(11/6)/(2/6)] \\
&= \tan^{-1} 11/2 = \cot^{-1} 2/11 = \text{R.H.S.} \quad \text{[Proved.]}
\end{aligned}$$

Q.14. Show that : $\sin^{-1} \sqrt{3}/2 + 2 \tan^{-1} 1/\sqrt{3} = 2\pi/3$.

Solution : 14

$$\begin{aligned}
\text{L.H.S.} &= \sin^{-1} (\sqrt{3}/2) + 2 \tan^{-1} (1/\sqrt{3}) \\
&= \sin^{-1} (\sin 60^\circ) + 2 \tan^{-1} (\tan 30^\circ) \\
&= 60^\circ + 2 \times 30^\circ = 60^\circ + 60^\circ = 120^\circ = 2\pi/3 = \text{R.H.S.} \quad \text{[Proved.]}
\end{aligned}$$

Q.15. Prove that : $\sin^{-1} x/\sqrt{(1+x^2)} + \cos^{-1} (x+1)/\sqrt{(x^2+2x+2)} = \tan^{-1} (x^2+x+1)$.

Solution : 15

$$\text{Let } \sin^{-1} x/\sqrt{(1+x^2)} = \theta \Rightarrow \sin \theta = x/\sqrt{(1+x^2)}$$

$$\text{Therefore, } \tan \theta = \sin \theta / \sqrt{(1 - \sin^2 \theta)}$$

$$= \{x/\sqrt{(1+x^2)}\} / \sqrt{[1 - \{x/\sqrt{(1+x^2)}\}^2]}$$

$$= \{x/\sqrt{(1+x^2)}\} / \{1/\sqrt{(1+x^2)}\}$$

$$= x/1 \Rightarrow \theta = \tan^{-1} x.$$

$$\text{Again let } \cos^{-1} (x+1)/\sqrt{(x^2+2x+2)} = \phi \Rightarrow \cos \phi = (x+1)/\sqrt{(x^2+2x+2)}$$

$$\text{Therefore, } \tan \phi = \sqrt{(1 - \cos^2 \phi)} / \cos \phi = \sqrt{[1 - \{(x+1)/\sqrt{(x^2+2x+2)}\}^2]} / \{(x+1)/\sqrt{(x^2+2x+2)}\}$$

$$= 1/(x+1) \Rightarrow \phi = \tan^{-1} 1/(x+1)$$

$$\text{L.H.S.} = \tan^{-1} x + \tan^{-1} 1/(x+1)$$

$$= \tan^{-1} [\{x + 1/(x+1)\} / \{1 - x/(x+1)\}] = \tan^{-1} (x^2 + x + 1) = \text{R.H.S. [Proved.]}$$

Q.16. Solve for x : $\tan^{-1} (x^{-1}) + \tan^{-1} x + \tan^{-1} (x+1) = \tan^{-1} 3x$.

Solution : 16

$$\tan^{-1} (x - 1) + \tan^{-1} x + \tan^{-1} (x+1) = \tan^{-1} 3x$$

$$\text{Or, } \tan^{-1} (x - 1) + \tan^{-1} (x+1) = \tan^{-1} 3x - \tan^{-1} x$$

$$\text{Or, } \tan^{-1} [(x - 1 + x + 1) / \{1 - (x - 1)(x + 1)\}] = \tan^{-1} [(3x - x) / (1 + 3x.x)]$$

$$\text{Or, } \tan^{-1} 2x/(2 - x^2) = \tan^{-1} 2x/(1 + 3x^2)$$

$$\text{Or, } 2x/(2 - x^2) = 2x/(1 + 3x^2)$$

$$\text{Or, } x[1 + 3x^2 - 2 + x^2] = 0$$

$$\text{Or, } x = 0 \text{ or } 4x^2 - 1 = 0$$

$$\text{Or, } x = \pm 1/2.$$

Q.17. Solve the equation : $\tan^{-1} (2 + x) + \tan^{-1} (2 - x) = \tan^{-1} 2/3$.

Solution : 17

$$\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1} \frac{2}{3}$$

$$\text{Or, } \tan^{-1} \left[\frac{(2+x) + (2-x)}{1 - (2+x)(2-x)} \right] = \tan^{-1} \frac{2}{3}$$

$$\text{Or, } \tan^{-1} \left[\frac{4}{1 - (4 - x^2)} \right] = \tan^{-1} \frac{2}{3} \quad \text{Or, } \tan^{-1} \left[\frac{4}{x^2 - 3} \right] = \tan^{-1} \frac{2}{3}$$

$$\text{Or, } \frac{4}{x^2 - 3} = \frac{2}{3} \quad \text{Or, } 2(x^2 - 3) = 12$$

$$\text{Or, } x^2 - 3 = 6 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3.$$

Q.18. Solve the equation : $\sin^{-1}(6x) + \sin^{-1}(6\sqrt{3}x) = -\pi/2$.

Solution : 18

$$\sin^{-1}(6x) + \sin^{-1}(6\sqrt{3}x) = -\pi/2$$

$$\text{Or, } \sin^{-1} [6x \sqrt{1 - (6\sqrt{3}x)^2} + 6\sqrt{3}x \sqrt{1 - (6x)^2}] = \sin^{-1}(-1)$$

$$\text{Or, } \sin^{-1} [6x\sqrt{1 - 108x^2} + 6\sqrt{3}x \sqrt{1 - 36x^2}] = \sin^{-1}(-1)$$

$$\text{Or, } 6x \sqrt{1 - 108x^2} + 6\sqrt{3}x \sqrt{1 - 36x^2} = -1$$

$$\text{Or, } 6x \sqrt{1 - 108x^2} = -[1 + 6\sqrt{3}x \sqrt{1 - 36x^2}]$$

Squaring both sides we get

$$36x^2(1 - 108x^2) = 1 + 108x^2 - 36 \times 108x^4 + 12\sqrt{3}x \sqrt{1 - 36x^2}$$

$$\text{Or, } 36x^2 - 36 \times 108x^4 = 1 + 108x^2 - 36 \times 108x^4 + 12\sqrt{3}x \sqrt{1 - 36x^2}$$

$$\text{Or, } 72x^2 + 12\sqrt{3}x \sqrt{1 - 36x^2} + 1 = 0$$

$$\text{Or, } 12\sqrt{3}x \sqrt{1 - 36x^2} = -(1 + 72x^2)$$

Squaring both sides again we get

$$432x^2(1 - 36x^2) = 1 + 72 \times 72x^2 + 144x^2$$

$$\text{Or, } 288 \times 72x^4 - 288x^2 + 1 = 0$$

$$\text{Or, } x^2 = \frac{288 \pm \sqrt{(288)^2 - 288 \times 288}}{2 \times 288 \times 72} = \frac{1}{144}$$

Or, $x = \pm 1/12 \Rightarrow x = 1/12$ or $-1/12$.

Q.19. Solve the equation : $\sin^{-1} 2a/(1 + a^2) + \sin^{-1} 2b/(1 + b^2) = 2 \tan^{-1} x$.

Solution : 19

Let $\tan \theta = a$, then $\sin^{-1} 2a/(1 + a^2) = \sin^{-1} [2 \tan \theta / (1 + \tan^2 \theta)]$

$$= \sin^{-1} (\sin 2\theta)$$

$$= 2\theta = 2 \tan^{-1} a.$$

Similarly, $\sin^{-1} 2b/(1 + b^2) = 2 \tan^{-1} b$.

Therefore, $\sin^{-1} 2a/(1 + a^2) + \sin^{-1} 2b/(1 + b^2)$

$$= 2 \tan^{-1} a + 2 \tan^{-1} b = 2 \tan^{-1} (a + b)/(1 - ab), \text{ provided } ab < 1.$$

Therefore, $2 \tan^{-1} (a + b)/(1 - ab) = 2 \tan^{-1} x$

$$\text{Or, } x = (a + b)/(1 - ab).$$

Q.20. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, prove that : $x^2 - y^2 - z^2 + 2yz\sqrt{1 - x^2} = 0$.

Solution : 20

$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$$

$$\text{Or, } \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} z$$

$$\text{Or, } \sin^{-1} [x \sqrt{1 - y^2} + y \sqrt{1 - x^2}] = \pi - \sin^{-1} z$$

$$\text{Or, } x \sqrt{1 - y^2} + y \sqrt{1 - x^2} = \sin(\pi - \sin^{-1} z)$$

$$\text{Or, } x \sqrt{1 - y^2} + y \sqrt{1 - x^2} = z$$

$$\text{Or, } x \sqrt{1 - y^2} = z - y \sqrt{1 - x^2}$$

Squaring we get,

$$x^2(1 - y^2) = z^2 + y^2(1 - x^2) - 2yz\sqrt{1 - x^2}$$

$$\text{Or, } x^2 - y^2 - z^2 + 2yz\sqrt{1 - x^2} = 0 \text{ [Proved.]}$$