

Derivative of Inverse Trigonometric Function

Q.1. Find dy/dx if $y = \tan^{-1} \{\sqrt{(1+x^2)} - 1\}/x$.

Solution : 1

We have $y = \tan^{-1} \{\sqrt{(1+x^2)} - 1\}/x$,

Let $x = \tan \theta$ then, $\theta = \tan^{-1} x$,

$$\begin{aligned}\text{Therefore, } y &= \tan^{-1} \{\sqrt{(1+\tan^2\theta)} - 1\}/\tan \theta = \tan^{-1} [(\sec \theta - 1)/\tan \theta] \\ &= \tan^{-1} [(1 - \cos \theta) / \sin \theta] = \tan^{-1} [2 \sin^2(\theta/2) / \{2 \sin(\theta/2) \cos(\theta/2)\}] \\ &= \tan^{-1} [\tan(\theta/2)] = \theta/2 = 1/2 \tan^{-1} x,\end{aligned}$$

Therefore, $dy/dx = 1/\{2(1+x^2)\}$.

Q.2. If $y = \sin^{-1} \sqrt{(1-x^2)}$, prove that $(1-x^2) dy/dx - xy = 1$.

Solution : 2

$$y = \sin^{-1} \sqrt{(1-x^2)}$$

Differentiating both sides with respect to x we get,

$$\begin{aligned}dy/dx &= \sqrt{(1-x^2)} \frac{d}{dx}(\sin^{-1}x) - \sin^{-1}x \frac{d}{dx}\{\sqrt{(1-x^2)}\} / \{\sqrt{(1-x^2)}\}^2 \\ &= \sqrt{(1-x^2)} \times 1/\sqrt{(1-x^2)} = \sin^{-1}x \times 1/[2\sqrt{(1-x^2)} (-2x) / (1-x^2)] \\ (1-x^2) dy/dx &= 1 + (x \sin^{-1}x) / (\sqrt{(1-x^2)}) \\ (1-x^2) dy/dx &= 1 + xy \quad [y = \sin^{-1}x / \sqrt{(1-x^2)}] \\ (1-x^2) dy/dx - xy &= 1. \quad \text{[Proved.]}\end{aligned}$$

Q.3. If $y = \tan^{-1} [2x/(1-x^2)]$, prove that $dy/dx = 2/(1+x^2)$.

Solution : 3

$y = \tan^{-1} [2x/(1 - x^2)]$, put $x = \tan \theta$, then $\theta = \tan^{-1} x$.

Then , $y = \tan^{-1} [2 \tan \theta / (1 - \tan^2 \theta)] = \tan^{-1} [\tan^2 \theta]$

$$= 2 \theta = 2 \tan^{-1} x$$

Therefore, $dy/dx = 2 \cdot [1/(1 + x^2)] = 2/(1 + x^2)$. **[Proved.]**

Q.4. If $y = \sec^{-1} [(\sqrt{x} + 1)/\sqrt{x - 1}] + \sin^{-1} [(\sqrt{x - 1})/(\sqrt{x} + 1)]$, find dy/dx .

Solution : 4

$$\sec^{-1} [(\sqrt{x} + 1)/(\sqrt{x - 1})] = \cos^{-1} [(\sqrt{x - 1})/(\sqrt{x} + 1)]$$

Therefore , $y = \cos^{-1} [(\sqrt{x} - 1)/(\sqrt{x} + 1)] + \sin^{-1} [(\sqrt{x - 1})/(\sqrt{x} + 1)] = \pi/2$

Hence , $dy / dx = 0$.

Q.5. If $y = \sin^{-1} \sqrt{1 - x} + \cos^{-1} (\sqrt{x})$.

Solution : 5

Put $x = \cos^2 \theta$, then

$$y = \sin^{-1} \sqrt{1 - \cos^2 \theta} + \cos^{-1} (\sqrt{\cos^2 \theta})$$

$$= \sin^{-1} (\sin \theta) + \cos^{-1} (\cos \theta)$$

$$= \theta + \theta = 2 \theta = 2 \cos^{-1} (\sqrt{x}) ,$$

Therefore , $dy/dx = 2 \cdot (-1)/\sqrt{1 - (\sqrt{x})^2} \cdot d/dx (\sqrt{x})$

$$= 2 \cdot (-1)/\sqrt{1 - x} \cdot (-1)/2\sqrt{x}$$

$$= (-1)/\sqrt{x(1 - x)}$$

$$= (-1)/\sqrt{x - x^2} .$$

Q.6. If $y = \tan^{-1} \left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right]$, find dy/dx .

Solution : 6

We have $y = \tan^{-1} \left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right]$,

Putting $x = \cos^2 \theta$, we get

$$y = \tan^{-1} \left[\frac{\sqrt{1+\cos^2 \theta} - \sqrt{1-\cos^2 \theta}}{\sqrt{1+\cos^2 \theta} + \sqrt{1-\cos^2 \theta}} \right]$$

$$= \tan^{-1} \left[\frac{\sqrt{2\cos^2 \theta} - \sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 \theta} + \sqrt{2\sin^2 \theta}} \right]$$

$$= \tan^{-1} \left[\frac{(\cos \theta - \sin \theta)}{(\cos \theta + \sin \theta)} \right]$$

$$= \tan^{-1} \left[\frac{(1 - \tan \theta)}{(1 + \tan \theta)} \right]$$

$$= \tan^{-1} [\tan(\pi/4 - \theta)] = \pi/4 - \theta = \pi/4 - (1/2)\cos^{-1} x,$$

$$\text{Therefore, } dy/dx = -1/2 \cdot (-1)/\sqrt{1-x^2} = 1/[2\sqrt{1-x^2}].$$

Q.7. If $y = \tan^{-1} \left[\frac{(\cos x + \sin x)}{(\cos x - \sin x)} \right]$, find dy/dx .

Solution : 7

$$y = \tan^{-1} \left[\frac{(\cos x + \sin x)}{(\cos x - \sin x)} \right]$$

$$= \tan^{-1} \left[\frac{(1 + \tan x)}{(1 - \tan x)} \right]$$

$$= \tan^{-1} [\tan(\pi/4 + x)] = \pi/4 + x,$$

$$\text{Therefore, } dy/dx = 0 + 1 = 1.$$

Q.8. Differentiate : $\tan^{-1} [\cos x/(1 + \sin x)]$ w. r. t. x .

Solution : 8

$$\text{We have, } y = \tan^{-1} [\cos x/(1 + \sin x)]$$

$$= \tan^{-1} [\sin(\pi/2 - x)/\{1 + \cos(\pi/2 - x)\}]$$

$$= \tan^{-1} \left[\frac{\{2 \sin (\pi/4 - x/2) \cos (\pi/4 - x/2)\}}{\{2 \cos^2 (\pi/4 - x/2)\}} \right]$$

$$= \tan^{-1} [\tan (\pi/4 - x/2)] = \pi/4 - x/2 .$$

Therefore , $dy/dx = -1/2$.

Q.9. Differentiate : $\tan^{-1} [(1 - \cos x)/\sin x]$ w. r. t. x .

Solution : 9

We have , $y = \tan^{-1} [(1 - \cos x)/\sin x]$

$$= \tan^{-1} [(2 \sin^2 x/2)/(2 \sin x/2 \cos x/2)]$$

$$= \tan^{-1} [\tan x/2] = x/2 .$$

Therefore , $dy/dx = 1/2$.

Q.10. Using a suitable substitution find the derivative of $\tan^{-1} [4\sqrt{x}/(1 - 4x)]$.

Solution : 10

Let $y = \tan^{-1} [4\sqrt{x}/(1 - 4x)]$ and let $2\sqrt{x} = \tan \theta$

Therefore, $y = \tan^{-1} [2 \tan \theta / (1 - \tan^2 \theta)]$

$$= \tan^{-1} \tan 2\theta$$

$$= 2\theta$$

$$= 2 \tan^{-1} (2\sqrt{x})$$

Then, dy/dx

$$= \{2/(1 + 4x)\} (2/2\sqrt{x}) = 2/[\sqrt{x}(1 + 4x)].$$