Inverse Trigonometric Functions

Inverse Trigonometric Functions

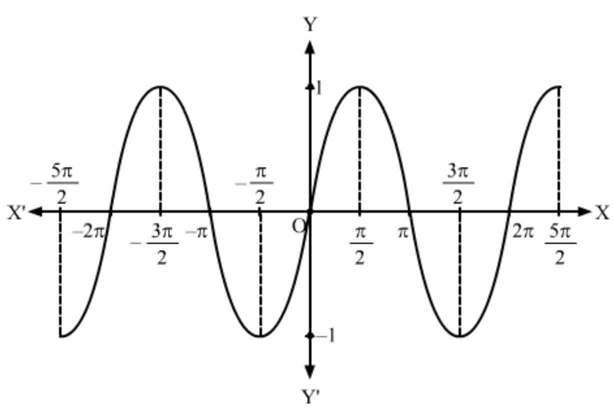
- Trigonometric functions are not one-one and onto in their usual natural domain. Hence, they are not invertible.

    They can be made one-one and onto (i.e., invertible) by restricting their domain. In this case, the

- Range of the inverse of trigonometric function is the proper subset of the domain of that trigonometric function.

- The branch of the inverse trigonometric function with the restricted range is called the Principal Value Branch.

- There are 6 inverse trigonometric functions. They can be described as



$y = \sin x$

Trigonometric function	$y = \sin x$
Inverse trigonometric function	$y = \sin^{-1} x$
Domain	$[-1, 1]$
Principal Value Branch (Range)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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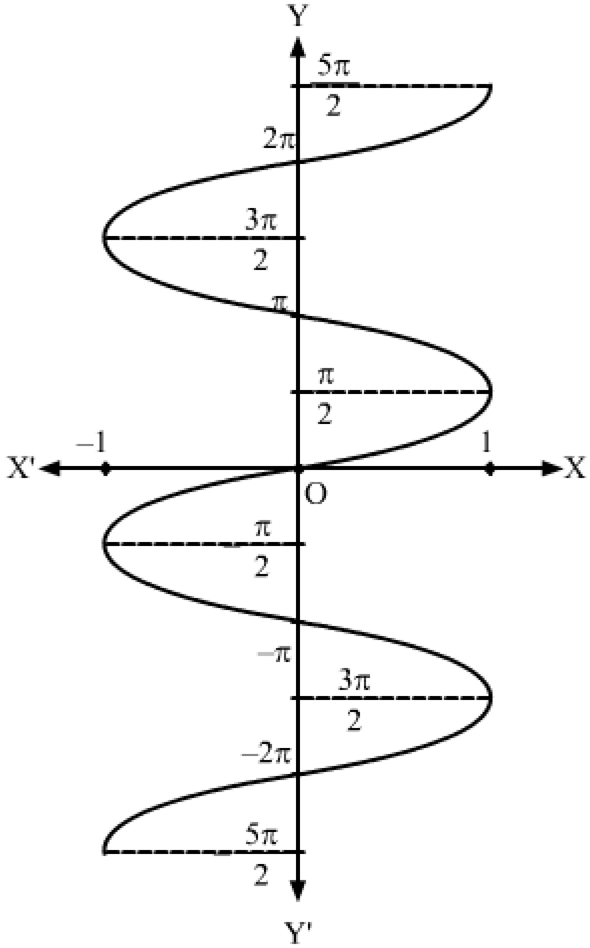
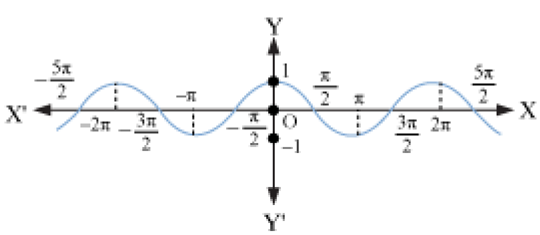
<tr><td>

$y = \sin^{-1}x$</td><td></td><td></td></tr>
<tr><td rowspan="4">

$y = \cos x$</td><td>Trigonometric function</td><td>$y = \cos x$</td></tr>
<tr><td>Inverse trigonometric function</td><td>$y = \cos^{-1}x$</td></tr>
<tr><td>Domain</td><td>$[-1, 1]$</td></tr>
<tr><td>Principal Value Branch (Range)</td><td>$[0, \pi]$</td></tr>
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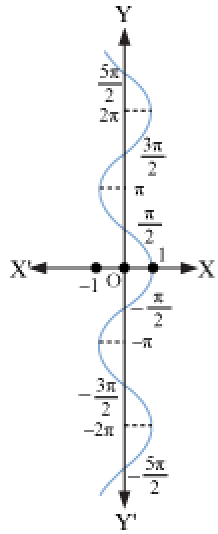
<td>

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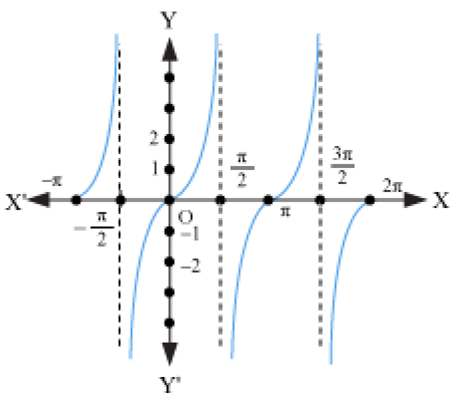
<td rowspan="4">

$y = \tan x$</td>
<td>Trigonometric function</td>
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<tr>
<td>Inverse trigonometric function</td>
<td>$y = \tan^{-1}x$</td>
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<tr>
<td>Domain</td>
<td>R</td>
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<tr>
<td>Principal Value Branch (Range)</td>
<td>$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$</td>
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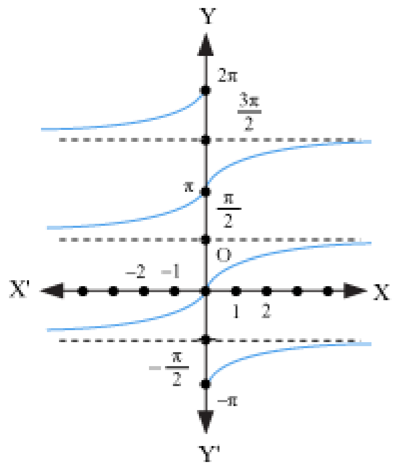
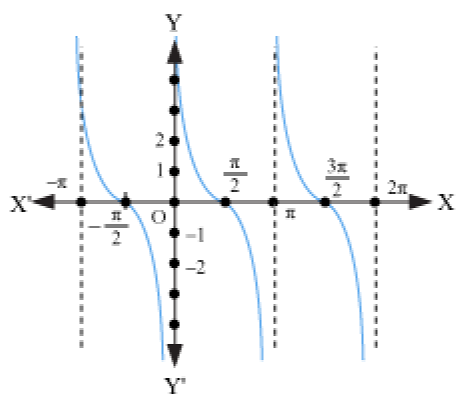
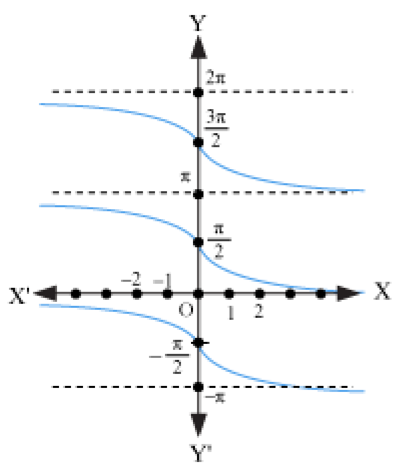
<tr><td>
$y = \tan^{-1}x$</td><td></td><td></td></tr>
<tr><td rowspan="5">
$y = \cot x$

$y = \cot^{-1}x$</td><td>Trigonometric function</td><td>$y = \cot x$</td></tr>
<tr><td>Inverse trigonometric function</td><td>$y = \cot^{-1}x$</td></tr>
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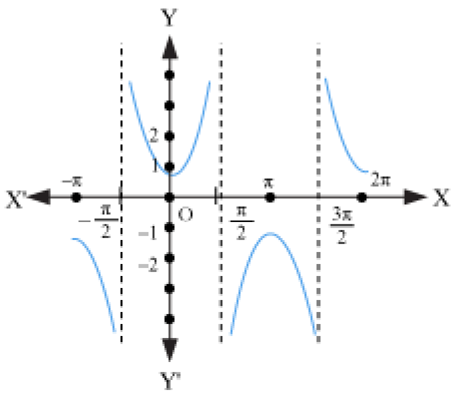
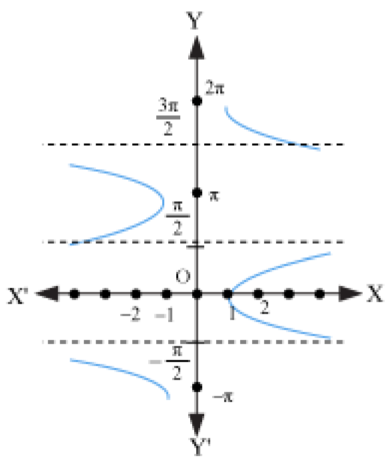
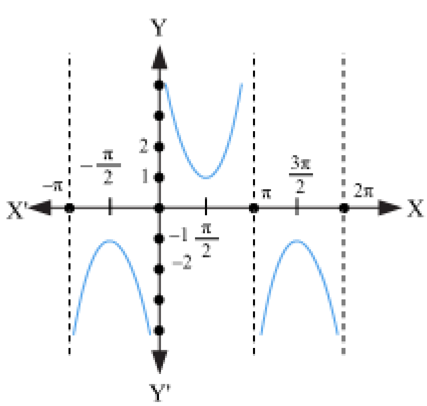
<tr><td rowspan="4">

$y = \sec x$


$y = \sec^{-1} x$</td><td>Trigonometric function</td><td>$y = \sec x$</td></tr>
<tr><td>Inverse trigonometric function</td><td>$y = \sec^{-1}x$</td></tr>
<tr><td>Domain</td><td>$\mathbf{R} - (-1, 1)$</td></tr>
<tr><td>Principal Value Branch (Range)</td><td>$[0, \pi] - \left\{\frac{\pi}{2}\right\}$</td></tr>
<tr><td rowspan="4">

$y = \operatorname{cosec} x$</td><td>Trigonometric function</td><td>$y = \operatorname{cosec} x$</td></tr>
<tr><td>Inverse trigonometric function</td><td>$y = \operatorname{cosec}^{-1}x$</td></tr>
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<tr><td>Principal Value Branch (Range)</td><td>$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$</td></tr>
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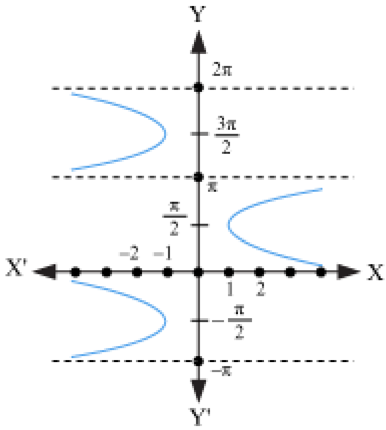 $y = \text{cosec}^{-1}x$		

Solved Examples

Example 1

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Solution:

$$\text{Let } \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$$

Accordingly, $\cos y = \frac{\sqrt{3}}{2}$

We know that the range of principal value branch of $\cos^{-1}$ is $[0, \pi]$.

$$\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right), \text{ where } \frac{\pi}{6} \in [0, \pi]$$

Thus, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

Example 2

Find the value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+3\cos^{-1}\left(\frac{1}{2}\right)+\tan^{-1}(1)$.

Solution:

$$\text{Let } \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=y$$

$$\text{Accordingly, } \sin y=\frac{\sqrt{3}}{2}.$$

The range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

$$\sin y=\frac{\sqrt{3}}{2}=\sin\left(\frac{\pi}{3}\right), \text{ where } \frac{\pi}{3}\in\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3} \qquad ...(1)$$

$$\text{Let } \cos^{-1}\left(\frac{1}{2}\right)=u. \text{ Accordingly, } \cos u=\frac{1}{2}$$

The range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$.

$$\cos u=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right), \text{ where } \frac{\pi}{3}\in[0,\ \pi]$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3} \qquad ...(2)$$

$$\text{Let } \tan^{-1}(1)=v. \text{ Accordingly, } \tan v=1.$$

The range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$.

$$\tan v=1=\tan\left(\frac{\pi}{4}\right), \text{ where } \frac{\pi}{4}\in\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}(1)=\frac{\pi}{4} \qquad ...(3)$$

From (1), (2) and (3), we obtain

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+3\cos^{-1}\left(\frac{1}{2}\right)+\tan^{-1}(1)$$
$$=\frac{\pi}{3}+3.\frac{\pi}{3}+\frac{\pi}{4}$$
$$=\frac{4\pi+12\pi+3\pi}{12}$$
$$=\frac{19\pi}{12}$$

Properties of Inverse Trigonometric Functions

- **Properties of $\sin^{-1}x$**
- $y = \sin^{-1} x \Rightarrow x = \sin y$
- $x = \sin y \Rightarrow y = \sin^{-1} x$
- $\sin (\sin^{-1} x) = x$
- $\sin^{-1} (\sin x) = x$
- $\sin^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1}x$
- $\sin^{-1} (- x) = - \sin^{-1}x$
- **Properties of $\cos^{-1}x$.**
- $y = \cos^{-1} x \Rightarrow x = \cos y$
- $x = \cos y \Rightarrow y = \cos^{-1} x$
- $\cos (\cos^{-1} x) = x$
- $\cos^{-1} (\cos x) = x$
- $\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1}x$
- $\cos^{-1} (- x) = \pi - \cos^{-1}x$
- **Properties of $\tan^{-1}x$**

- $x = \tan y \Rightarrow y = \tan^{-1} x$
- $\tan^{-1} (\tan x) = x$
- $\tan (\tan^{-1} x) = x$
- $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$
- $\tan^{-1} (-x) = -\tan^{-1} x$
- **Properties of $\cot^{-1}x$.**
- $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$
- $\cot^{-1} (-x) = \pi - \cot^{-1} x$
- **Properties of $\operatorname{cosec}^{-1}x$.**
- $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$
- $\operatorname{cosec}^{-1} (-x) = -\operatorname{cosec}^{-1} x$
- **Properties of $\sec^{-1}x$.**
- $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$
- $\sec^{-1} (-x) = \pi - \sec^{-1} x$
- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$$\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

- To understand the proof of the formula $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$,
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

  $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy > -1$

- To understand the proof of the formula of $\tan^{-1} x + \tan^{-1} y$,
- $$2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & |x| \le 1 \\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & x \ge 1 \\ \tan^{-1}\left(\frac{2x}{1-x^2}\right), & -1 < x < 1 \end{cases}$$
- To understand the proof of the formula $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$
- $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

  $2\cos^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \frac{1}{\sqrt{2}} \le x \le 1$

Solved Examples

Example 1

Solve the equation $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$.

Solution:

$$\left(\tan^{-1} x\right)^2 + \left(\cot^{-1} x\right)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \left(\tan^{-1} x + \cot^{-1} x\right)^2 - 2\tan^{-1} x \cot^{-1} x = \frac{5\pi^2}{8} \qquad \left[a^2 + b^2 = (a+b)^2 - 2ab\right]$$

$$\Rightarrow \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1} x . \cot^{-1} x = \frac{5\pi^2}{8} \qquad \left[\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

$$\Rightarrow 2\tan^{-1} x \cot^{-1} x = \frac{\pi^2}{4} - \frac{5\pi^2}{8}$$

$$\Rightarrow 2\tan^{-1} x \cot^{-1} x = -\frac{3\pi^2}{8}$$

$$\Rightarrow 2\tan^{-1} x\left(\frac{\pi}{2} - \tan^{-1} x\right) = -\frac{3\pi^2}{8}$$

$$\Rightarrow \pi \tan^{-1} x - 2\left(\tan^{-1} x\right)^2 = -\frac{3\pi^2}{8}$$

Let $\tan^{-1} x = y$

$$\Rightarrow 8\pi y - 16y^2 = -3\pi^2$$

$$\Rightarrow 16y^2 - 8\pi y - 3\pi^2 = 0$$

$$\Rightarrow y = \frac{8\pi \pm \sqrt{64\pi^2 + 192\pi^2}}{32} = \frac{8\pi \pm 16\pi}{32}$$

$$\Rightarrow y = \frac{\pi \pm 2\pi}{4}$$

$$\Rightarrow y = -\frac{\pi}{4} \text{ and } \frac{3\pi}{4}$$

$$\Rightarrow \tan^{-1} x = -\frac{\pi}{4} \text{ and } \tan^{-1} x = \frac{3\pi}{4}$$

The principal value branch of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \tan^{-1} x = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow x = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow x = -1$$

Example 2

If $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$, $x > 0$, then prove that $x=\pm\frac{1}{\sqrt{3}}$.

Solution:

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow 2\tan^{-1}\left(\frac{1-x}{1+x}\right)=\tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left\{\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right\}=\tan^{-1}x$$

$$\Rightarrow \frac{2\left(1-x^2\right)}{4x}=x$$

$$\Rightarrow 1-x^2=2x^2$$

$$\Rightarrow x^2=\frac{1}{3}$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{3}}$$