

PRINCIPLE OF MATHEMATICAL INDUCTION

4.1 Overview

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of n , where n is a positive integer.

4.1.1 The principle of mathematical induction

Let $P(n)$ be a given statement involving the natural number n such that

- (i) The statement is true for $n = 1$, i.e., $P(1)$ is true (or true for any fixed natural number) and
- (ii) If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of $P(k)$ implies the truth of $P(k + 1)$. Then $P(n)$ is true for all natural numbers n .

4.2 Solved Examples

Short Answer Type

Prove statements in Examples 1 to 5, by using the Principle of Mathematical Induction for all $n \in \mathbf{N}$, that :

Example 1 $1 + 3 + 5 + \dots + (2n - 1) = n^2$

Solution Let the given statement $P(n)$ be defined as $P(n) : 1 + 3 + 5 + \dots + (2n - 1) = n^2$, for $n \in \mathbf{N}$. Note that $P(1)$ is true, since

$$P(1) : 1 = 1^2$$

Assume that $P(k)$ is true for some $k \in \mathbf{N}$, i.e.,

$$P(k) : 1 + 3 + 5 + \dots + (2k - 1) = k^2$$

Now, to prove that $P(k + 1)$ is true, we have

$$\begin{aligned} 1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) && \text{(Why?)} \\ &= k^2 + 2k + 1 = (k + 1)^2 \end{aligned}$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbf{N}$.

Example 2 $\sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}$, for all natural numbers $n \geq 2$.

Solution Let the given statement $P(n)$, be given as

$$P(n) : \sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}, \text{ for all natural numbers } n \geq 2.$$

We observe that

$$\begin{aligned} P(2) : \sum_{t=1}^{2-1} t(t+1) &= \sum_{t=1}^1 t(t+1) = 1.2 = \frac{1.2.3}{3} \\ &= \frac{2.(2-1).(2+1)}{3} \end{aligned}$$

Thus, $P(n)$ is true for $n = 2$.

Assume that $P(n)$ is true for $n = k \in \mathbf{N}$.

$$\text{i.e.,} \quad P(k) : \sum_{t=1}^{k-1} t(t+1) = \frac{k(k-1)(k+1)}{3}$$

To prove that $P(k + 1)$ is true, we have

$$\begin{aligned} \sum_{t=1}^{(k+1)-1} t(t+1) &= \sum_{t=1}^k t(t+1) \\ &= \sum_{t=1}^{k-1} t(t+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1) \\ &= k(k+1) \left[\frac{k-1+3}{3} \right] = \frac{k(k+1)(k+2)}{3} \\ &= \frac{(k+1)((k+1)-1)((k+1)+1)}{3} \end{aligned}$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers $n \geq 2$.

Example 3 $\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$, for all natural numbers, $n \geq 2$.

Solution Let the given statement be $P(n)$, i.e.,

$$P(n) : \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for all natural numbers, } n \geq 2$$

We, observe that $P(2)$ is true, since

$$\left(1 - \frac{1}{2^2}\right) = 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4} = \frac{2+1}{2 \times 2}$$

Assume that $P(n)$ is true for some $k \in \mathbf{N}$, i.e.,

$$P(k) : \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$$

Now, to prove that $P(k+1)$ is true, we have

$$\begin{aligned} & \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{k^2}\right) \cdot \left(1 - \frac{1}{(k+1)^2}\right) \\ &= \frac{k+1}{2k} \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k^2 + 2k}{2k(k+1)} = \frac{(k+1)+1}{2(k+1)} \end{aligned}$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers, $n \geq 2$.

Example 4 $2^{2n} - 1$ is divisible by 3.

Solution Let the statement $P(n)$ given as

$P(n) : 2^{2n} - 1$ is divisible by 3, for every natural number n .

We observe that $P(1)$ is true, since

$$2^2 - 1 = 4 - 1 = 3.1 \text{ is divisible by 3.}$$

Assume that $P(n)$ is true for some natural number k , i.e.,

$P(k) : 2^{2k} - 1$ is divisible by 3, i.e., $2^{2k} - 1 = 3q$, where $q \in \mathbf{N}$

Now, to prove that $P(k+1)$ is true, we have

$$\begin{aligned} P(k+1) : 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 = 2^{2k} \cdot 2^2 - 1 \\ &= 2^{2k} \cdot 4 - 1 = 3 \cdot 2^{2k} + (2^{2k} - 1) \end{aligned}$$

$$\begin{aligned}
 &= 3 \cdot 2^{2k} + 3q \\
 &= 3(2^{2k} + q) = 3m, \text{ where } m \in \mathbb{N}
 \end{aligned}$$

Thus $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction $P(n)$ is true for all natural numbers n .

Example 5 $2n + 1 < 2^n$, for all natural numbers $n \geq 3$.

Solution Let $P(n)$ be the given statement, i.e., $P(n) : (2n + 1) < 2^n$ for all natural numbers, $n \geq 3$. We observe that $P(3)$ is true, since

$$2 \cdot 3 + 1 = 7 < 8 = 2^3$$

Assume that $P(n)$ is true for some natural number k , i.e., $2k + 1 < 2^k$

To prove $P(k + 1)$ is true, we have to show that $2(k + 1) + 1 < 2^{k+1}$. Now, we have

$$\begin{aligned}
 2(k + 1) + 1 &= 2k + 3 \\
 &= 2k + 1 + 2 < 2^k + 2 < 2^k \cdot 2 = 2^{k+1}.
 \end{aligned}$$

Thus $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction $P(n)$ is true for all natural numbers, $n \geq 3$.

Long Answer Type

Example 6 Define the sequence $a_1, a_2, a_3, \dots$ as follows :

$a_1 = 2, a_n = 5a_{n-1}$, for all natural numbers $n \geq 2$.

- Write the first four terms of the sequence.
- Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula $a_n = 2 \cdot 5^{n-1}$ for all natural numbers.

Solution

- We have $a_1 = 2$

$$a_2 = 5a_{2-1} = 5a_1 = 5 \cdot 2 = 10$$

$$a_3 = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$$

$$a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = 250$$

- Let $P(n)$ be the statement, i.e.,

$P(n) : a_n = 2 \cdot 5^{n-1}$ for all natural numbers. We observe that $P(1)$ is true

Assume that $P(n)$ is true for some natural number k , i.e., $P(k) : a_k = 2 \cdot 5^{k-1}$.

Now to prove that $P(k + 1)$ is true, we have

$$\begin{aligned} P(k+1) : a_{k+1} &= 5.a_k = 5 \cdot (2.5^{k-1}) \\ &= 2.5^k = 2.5^{(k+1)-1} \end{aligned}$$

Thus $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers.

Example 7 The distributive law from algebra says that for all real numbers c , a_1 and a_2 , we have $c(a_1 + a_2) = ca_1 + ca_2$.

Use this law and mathematical induction to prove that, for all natural numbers, $n \geq 2$, if $c, a_1, a_2, \dots, a_n$ are any real numbers, then

$$c(a_1 + a_2 + \dots + a_n) = ca_1 + ca_2 + \dots + ca_n$$

Solution Let $P(n)$ be the given statement, i.e.,

$P(n) : c(a_1 + a_2 + \dots + a_n) = ca_1 + ca_2 + \dots + ca_n$ for all natural numbers $n \geq 2$, for $c, a_1, a_2, \dots, a_n \in \mathbf{R}$.

We observe that $P(2)$ is true since

$$c(a_1 + a_2) = ca_1 + ca_2 \quad (\text{by distributive law})$$

Assume that $P(n)$ is true for some natural number k , where $k > 2$, i.e.,

$$P(k) : c(a_1 + a_2 + \dots + a_k) = ca_1 + ca_2 + \dots + ca_k$$

Now to prove $P(k+1)$ is true, we have

$$\begin{aligned} P(k+1) : c(a_1 + a_2 + \dots + a_k + a_{k+1}) \\ &= c((a_1 + a_2 + \dots + a_k) + a_{k+1}) \\ &= c(a_1 + a_2 + \dots + a_k) + ca_{k+1} \quad (\text{by distributive law}) \\ &= ca_1 + ca_2 + \dots + ca_k + ca_{k+1} \end{aligned}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of Mathematical Induction, $P(n)$ is true for all natural numbers $n \geq 2$.

Example 8 Prove by induction that for all natural number n

$\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n-1)\beta)$

$$\begin{aligned} &= \frac{\sin(\alpha + \frac{n-1}{2}\beta) \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \end{aligned}$$

Solution Consider $P(n) : \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n-1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}, \text{ for all natural number } n.$$

We observe that

P (1) is true, since

$$P(1) : \sin \alpha = \frac{\sin(\alpha+0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

Assume that P(n) is true for some natural numbers k, i.e.,

P (k) : $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin (\alpha + (k-1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Now, to prove that P (k + 1) is true, we have

P (k + 1) : $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin (\alpha + (k-1)\beta) + \sin (\alpha + k\beta)$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \sin(\alpha + k\beta)$$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \sin(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) + \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$\begin{aligned}
 &= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2 \sin \frac{\beta}{2}} \\
 &= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right) \sin\left(\frac{k\beta + \beta}{2}\right)}{\sin \frac{\beta}{2}} \\
 &= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right) \sin(k+1)\left(\frac{\beta}{2}\right)}{\sin \frac{\beta}{2}}
 \end{aligned}$$

Thus $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction $P(n)$ is true for all natural number n .

Example 9 Prove by the Principle of Mathematical Induction that

$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1$ for all natural numbers n .

Solution Let $P(n)$ be the given statement, that is,

$P(n) : 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1$ for all natural numbers n .

Note that $P(1)$ is true, since

$$P(1) : 1 \times 1! = 1 = 2 - 1 = 2! - 1.$$

Assume that $P(n)$ is true for some natural number k , i.e.,

$$P(k) : 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! = (k+1)! - 1$$

To prove $P(k+1)$ is true, we have

$$\begin{aligned}
 P(k+1) : & 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! + (k+1) \times (k+1)! \\
 &= (k+1)! - 1 + (k+1)! \times (k+1) \\
 &= (k+1+1) (k+1)! - 1 \\
 &= (k+2) (k+1)! - 1 = ((k+2)! - 1
 \end{aligned}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all natural number n .

Example 10 Show by the Principle of Mathematical Induction that the sum S_n of the n term of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 \dots$ is given by

$$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Solution Here $P(n) : S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{when } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{when } n \text{ is odd} \end{cases}$

Also, note that any term T_n of the series is given by

$$T_n = \begin{cases} n^2 & \text{if } n \text{ is odd} \\ 2n^2 & \text{if } n \text{ is even} \end{cases}$$

We observe that $P(1)$ is true since

$$P(1) : S_1 = 1^2 = 1 = \frac{1 \cdot 2}{2} = \frac{1^2 \cdot (1+1)}{2}$$

Assume that $P(k)$ is true for some natural number k , i.e.

Case 1 When k is odd, then $k + 1$ is even. We have

$$\begin{aligned} P(k+1) : S_{k+1} &= 1^2 + 2 \times 2^2 + \dots + k^2 + 2 \times (k+1)^2 \\ &= \frac{k^2(k+1)}{2} + 2 \times (k+1)^2 \\ &= \frac{(k+1)}{2} [k^2 + 4(k+1)] \quad (\text{as } k \text{ is odd, } 1^2 + 2 \times 2^2 + \dots + k^2 = k^2 \frac{(k+1)}{2}) \\ &= \frac{k+1}{2} [k^2 + 4k + 4] \\ &= \frac{k+1}{2} (k+2)^2 = (k+1) \frac{[(k+1)+1]^2}{2} \end{aligned}$$

So $P(k+1)$ is true, whenever $P(k)$ is true in the case when k is odd.

Case 2 When k is even, then $k + 1$ is odd.

$$\begin{aligned}
 \text{Now, } P(k+1) : 1^2 + 2 \times 2^2 + \dots + 2 \cdot k^2 + (k+1)^2 \\
 = \frac{k(k+1)^2}{2} + (k+1)^2 \quad (\text{as } k \text{ is even, } 1^2 + 2 \times 2^2 + \dots + 2k^2 = k \frac{(k+1)^2}{2}) \\
 = \frac{(k+1)^2(k+2)}{2} = \frac{(k+1)^2((k+1)+1)}{2}
 \end{aligned}$$

Therefore, $P(k+1)$ is true, whenever $P(k)$ is true for the case when k is even. Thus $P(k+1)$ is true whenever $P(k)$ is true for any natural numbers k . Hence, $P(n)$ true for all natural numbers.

Objective Type Questions

Choose the correct answer in Examples 11 and 12 (M.C.Q.)

Example 11 Let $P(n) : "2^n < (1 \times 2 \times 3 \times \dots \times n)"$. Then the smallest positive integer for which $P(n)$ is true is

- (A) 1 (B) 2 (C) 3 (D) 4

Solution Answer is D, since

$$P(1) : 2 < 1 \text{ is false}$$

$$P(2) : 2^2 < 1 \times 2 \text{ is false}$$

$$P(3) : 2^3 < 1 \times 2 \times 3 \text{ is false}$$

But $P(4) : 2^4 < 1 \times 2 \times 3 \times 4$ is true

Example 12 A student was asked to prove a statement $P(n)$ by induction. He proved that $P(k+1)$ is true whenever $P(k)$ is true for all $k > 5 \in \mathbf{N}$ and also that $P(5)$ is true. On the basis of this he could conclude that $P(n)$ is true

- (A) for all $n \in \mathbf{N}$ (B) for all $n > 5$
 (C) for all $n \geq 5$ (D) for all $n < 5$

Solution Answer is (C), since $P(5)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true. Fill in the blanks in Example 13 and 14.

Example 13 If $P(n) : "2 \cdot 4^{2n+1} + 3^{3n+1}$ is divisible by λ for all $n \in \mathbf{N}"$ is true, then the value of λ is _____

Solution Now, for $n = 1$,

$$2 \cdot 4^{2+1} + 3^{3+1} = 2 \cdot 4^3 + 3^4 = 2 \cdot 64 + 81 = 128 + 81 = 209,$$

$$\text{for } n = 2, 2 \cdot 4^5 + 3^7 = 8 \cdot 256 + 2187 = 2048 + 2187 = 4235$$

Note that the H.C.F. of 209 and 4235 is 11. So $2 \cdot 4^{2n+1} + 3^{3n+1}$ is divisible by 11. Hence, λ is 11

Example 14 If $P(n)$: “ $49^n + 16^n + k$ is divisible by 64 for $n \in \mathbf{N}$ ” is true, then the least negative integral value of k is _____.

Solution For $n = 1$, $P(1)$: $65 + k$ is divisible by 64.

Thus k , should be -1 since, $65 - 1 = 64$ is divisible by 64.

Example 15 State whether the following proof (by mathematical induction) is true or false for the statement.

$$P(n): 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof By the Principle of Mathematical induction, $P(n)$ is true for $n = 1$,

$1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$. Again for some $k \geq 1$, $k^2 = \frac{k(k+1)(2k+1)}{6}$. Now we prove that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Solution False

Since in the inductive step both the inductive hypothesis and what is to be proved are wrong.

4.3 EXERCISE

Short Answer Type

1. Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.
2. Give an example of a statement $P(n)$ which is true for all n . Justify your answer.
Prove each of the statements in Exercises 3 - 16 by the Principle of Mathematical Induction :
3. $4^n - 1$ is divisible by 3, for each natural number n .
4. $2^{3n} - 1$ is divisible by 7, for all natural numbers n .
5. $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n .
6. $3^{2n} - 1$ is divisible by 8, for all natural numbers n .

7. For any natural number n , $7^n - 2^n$ is divisible by 5.
8. For any natural number n , $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.
9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.
10. $n(n^2 + 5)$ is divisible by 6, for each natural number n .
11. $n^2 < 2^n$ for all natural numbers $n \geq 5$.
12. $2n < (n + 2)!$ for all natural number n .
13. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.
14. $2 + 4 + 6 + \dots + 2n = n^2 + n$ for all natural numbers n .
15. $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$ for all natural numbers n .
16. $1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1)$ for all natural numbers n .

Long Answer Type

Use the Principle of Mathematical Induction in the following Exercises.

17. A sequence $a_1, a_2, a_3 \dots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all natural numbers.
18. A sequence $b_0, b_1, b_2 \dots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k . Show that $b_n = 5 + 4n$ for all natural number n using mathematical induction.
19. A sequence $d_1, d_2, d_3 \dots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all $n \in \mathbf{N}$.
20. Prove that for all $n \in \mathbf{N}$

$$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots + \cos (\alpha + (n - 1) \beta)$$

$$= \frac{\cos \left(\alpha + \left(\frac{n-1}{2} \right) \beta \right) \sin \left(\frac{n\beta}{2} \right)}{\sin \frac{\beta}{2}}$$

21. Prove that, $\cos \theta \cos 2\theta \cos 2^2\theta \dots \cos 2^{n-1}\theta = \frac{\sin 2^n \theta}{2^n \sin \theta}$, for all $n \in \mathbf{N}$.

$$\frac{\sin n\theta \sin \frac{(n+1)\theta}{2}}{2}$$

22. Prove that, $\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\sin \frac{n\theta}{2} \sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}}$, for all $n \in \mathbf{N}$.

23. Show that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number for all $n \in \mathbf{N}$.
24. Prove that $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24}$, for all natural numbers $n > 1$.
25. Prove that number of subsets of a set containing n distinct elements is 2^n , for all $n \in \mathbf{N}$.

Objective Type Questions

Choose the correct answers in Exercises 26 to 30 (M.C.Q.).

26. If $10^n + 3 \cdot 4^{n+2} + k$ is divisible by 9 for all $n \in \mathbf{N}$, then the least positive integral value of k is
 (A) 5 (B) 3 (C) 7 (D) 1
27. For all $n \in \mathbf{N}$, $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by
 (A) 19 (B) 17 (C) 23 (D) 25
28. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is
 (A) 1 (B) 2 (C) 3 (D) 4

Fill in the blanks in the following :

29. If $P(n) : 2n < n!, n \in \mathbf{N}$, then $P(n)$ is true for all $n \geq$ _____.

State whether the following statement is true or false. Justify.

30. Let $P(n)$ be a statement and let $P(k) \Rightarrow P(k + 1)$, for some natural number k , then $P(n)$ is true for all $n \in \mathbf{N}$.

