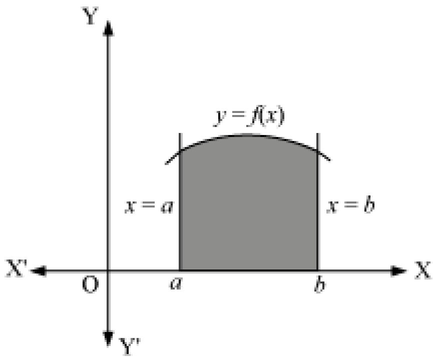
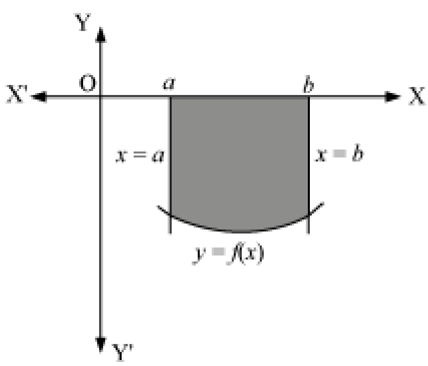
Application of Integrals

Area under Simple Curves

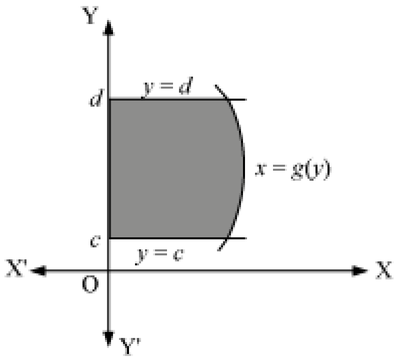
- The area of the region bounded by the curve $y = f(x)$, which is continuous and finite in $[a, b]$ and lies above the x-axis and between the lines $x = a$ and $x = b$ ($b > a$), is given by A = $\int_a^b y\,dx = \int_a^b f(x)\,dx$.



- Area cannot be negative. Hence, the absolute value of the area, $\left|\int_a^b f(x)\,dx\right|$, is taken even if the curve $y = f(x)$ lies below the x-axis (as shown below).



- Let $g(y)$ be continuous and finite in $[c, d]$. The area of the region bounded by the curve $x = g(y)$ when $g(y)$ lies to the right of the y-axis and between the lines $y = c$ and $y = d$ ($d > c$) is given by A = $\int_c^d x\,dy = \int_c^d g(y)\,dy$.

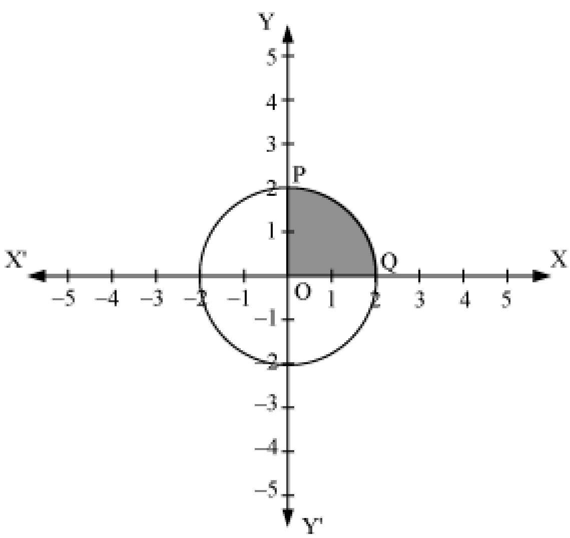


Solved Examples

Example 1

Find the area enclosed by the circle $x^2 + y^2 = 4$.

Solution:



The area enclosed by the given circle is given by

4 ×× Area of the region OPQO

Area of circle $= 4\int_0^2 y\,dx$

$= 4\int_0^2 \sqrt{4-x^2}$ (y lies in the first quadrant ⇒ $y = +\sqrt{4-x^2}$)

$$= 4\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$4[0 + 2\sin^{-1} 1 - 0 - 2\sin^{-1}0]$

$$= 4\left[2 \times \frac{\pi}{2}\right]$$
$$= 4\pi$$

Thus, the required area is 4π.

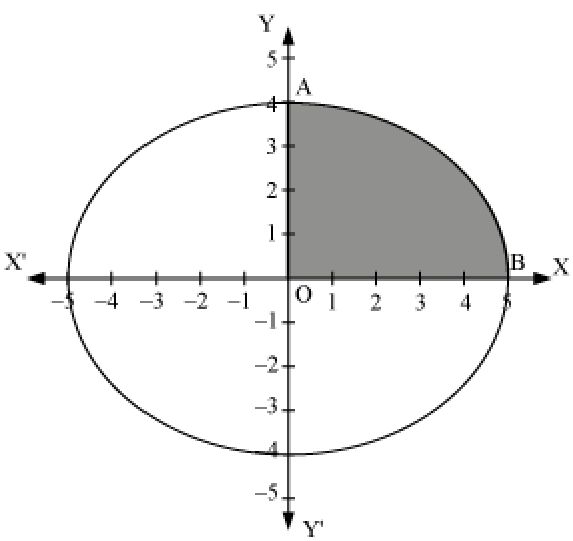
Example 2

Find the area covered by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Solution:

The equation of the given ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

$$\Rightarrow \frac{x^2}{25} = 1 - \frac{y^2}{16} = \frac{1}{16}\left(16 - y^2\right)$$
$$\Rightarrow x = \pm\frac{5}{4}\sqrt{16 - y^2}$$



The area of the given ellipse is given by

4 ×× Area of the region OABO

Area of ellipse $= 4\int_0^4 x\,dy$

$= 4\int_0^4 \frac{5}{4}\sqrt{16-y^2}\,dy$ (*x* lies in the first quadrant. Hence, *x* is positive)

$= 5\int_0^4 \sqrt{16-y^2}\,dy$

$$= 5\left[\frac{y}{2}\sqrt{16-y^2} + \frac{16}{2}\sin^{-1}\frac{y}{4}\right]_0^4$$

$$= 5\left[\frac{4}{2}\sqrt{16-16} + \frac{16}{2}\sin^{-1}1 - 0 - \frac{16}{2}\sin^{-1}0\right]$$

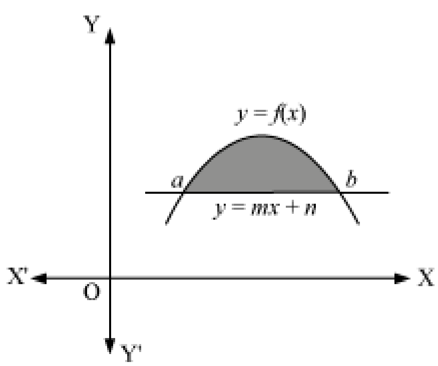
$$= 5\times 8\times\frac{\pi}{2}$$

$$= 20\pi$$

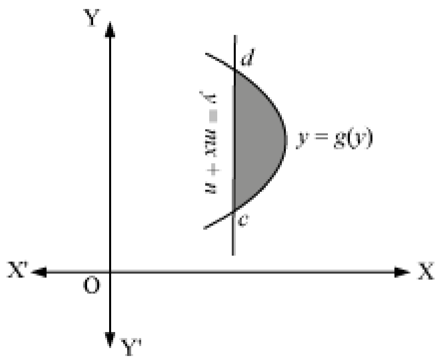
Thus, the required area is 20π.

Area Bounded by a Curve and a Line

- If a line $y = mx + n$ intersects a curve $y = f(x)$ at *a* and *b*, then the area of this curve under the line $y = mx + n$ is given by $A = \int_a^b y\,dx = \int_a^b f(x)\,dx$



- If a line $y = mx + n$ intersects a curve $x = g(y)$ at *c* and *d*, then the area of this curve under the line $y = mx + n$ is given by $A = \int_c^d x\,dy = \int_c^d g(y)\,dy$.



Solved Examples

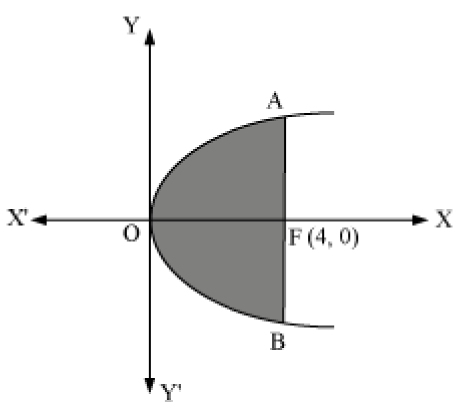
Example 1

Find the area bounded by the parabola $y^2 = 16x$ and its latus rectum.

Solution:

The given equation of the parabola is $y^2 = 16x$. It is symmetrical about the x-axis.

The focus of the given parabola is F (4, 0) and its latus rectum is the line parallel to the y-axis at a distance of 4 units from it.



The required area is given by the area OABO.

Area OABO = 2×(Area OAFO)

$$= 2\int_0^4 y dx$$

$$= 2\int_0^4 4\sqrt{x} dx$$

$$= 2.4.\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4$$

$$= 8\times\frac{2}{3}\left[4^{\frac{3}{2}} - 0\right]$$

$$= \frac{16}{3}\left[(4)^{\frac{3}{2}}\right]$$

$$= \frac{16\times 8}{3}$$

$$= \frac{128}{3}$$

Example 2

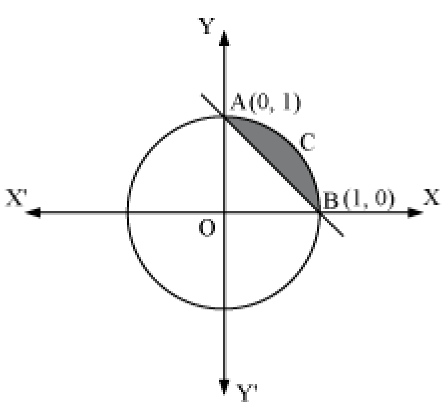
Find the area of the region bounded by $x^2 + y^2 \leq 1$ and $x + y \geq 1$.

Solution:

The area enclosed by the curve $x^2 + y^2 \leq 1$ is the interior of the circle $x^2 + y^2 = 1$.

Also, the area of the region $x + y \geq 1$ is the region lying above the line $x + y = 1$.

The point of intersection of the circle $x^2 + y^2 = 1$ and the line $x + y = 1$ is (0, 1) and (1, 0).



The required area is given by area (ACBA).

Area (ACBA) = Area (AOBCA) − Area (AOB)

$$= \int_0^1 \sqrt{1-x^2}\,dx - \int_0^1 (1-x)\,dx$$

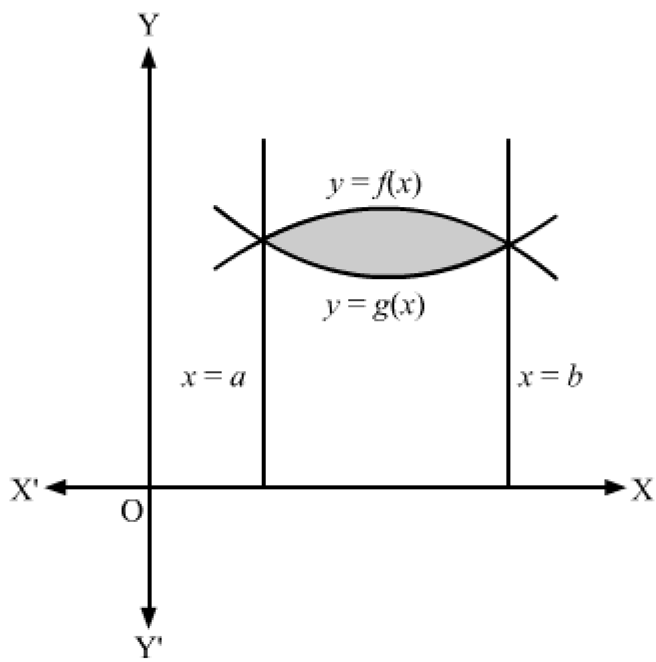
$$= \left[\frac{1}{2}\sin^{-1} x + \frac{x}{2}\sqrt{1-x^2}\right]_0^1 - \left[x - \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\sin^{-1} 1 - 1 + \frac{1}{2}$$

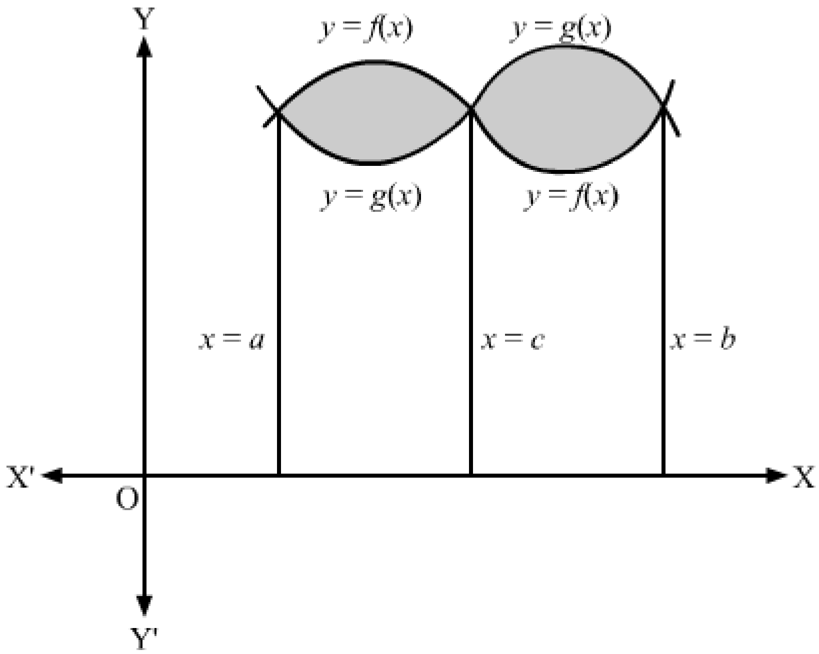
$$= \left(\frac{\pi}{4} - \frac{1}{2}\right)$$

Area between Two Curves

- The area of the region enclosed between two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$, $x = b$ is given by $A = \int_a^b \left[f(x) - g(x)\right]dx$, where $f(x) \geq g(x)$ in $[a, b]$.



- If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then the area enclosed between the curves $f(x)$ and $g(x)$ is given by $A = \int_a^c \left[f(x) - g(x)\right]dx + \int_c^b \left[g(x) - f(x)\right]dx$



Solved Examples

Example 1

Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$.

Solution:

The equations of the two parabolas are

$y^2 = 4ax$ … (1)

$x^2 = 4ay$ … (2)

From (1), we obtain $x = \frac{y^2}{4a}$

Substituting the value of x in (2), we obtain

$$\frac{y^4}{16a^2} = 4ay$$

$\Rightarrow y^4 - 64a^3y = 0$

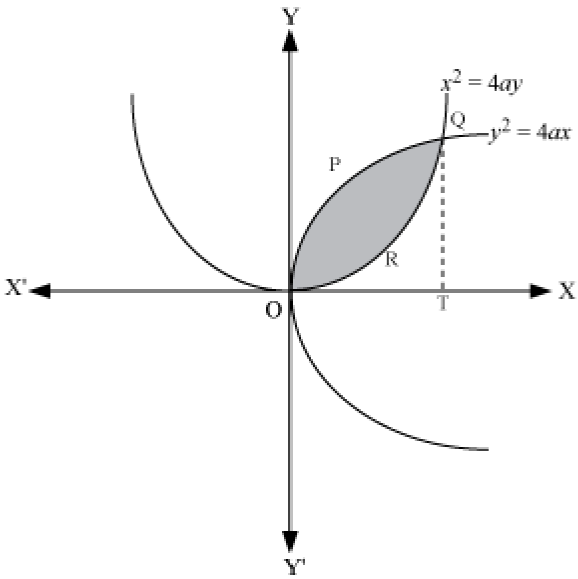
$\Rightarrow y\,(y^3 - 64a^3) = 0$

$\Rightarrow y = 0$ or $y^3 = 64a^3$

$\therefore y = 0$ or $y = 4a$.

From (1), we obtain $y = 0 \Rightarrow x = 0$ and $y = 4a \Rightarrow x = 4a$

Thus, the points of intersection of two parabolas are (0, 0) and ($4a$, $4a$)



Now, draw QT perpendicular to XX′.

∴ Required area = Area OPQRO

Area OPQRO = Area (OPQTO) − Area (ORQTO)

$$= \int_0^{4a} 2\sqrt{ax}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$
$$= 2\sqrt{a} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$
$$= \frac{4}{3}\sqrt{a}\,(4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3$$
$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$
$$= \frac{16}{3}a^2$$

Thus, the required area is $\frac{16}{3}a^2$ square units.

Example 2

Find the area of the region bounded by $x^2 + y^2 \leq 4x, y^2 \geq 2x, x \geq 0, y \geq 0$.

Solution:

We have to find the area of the region lying in the first quadrant as $x \geq 0, y \geq 0$.

The equations of the given curves are

$x^2 + y^2 \leq 4x$ and $y^2 \geq 2x$

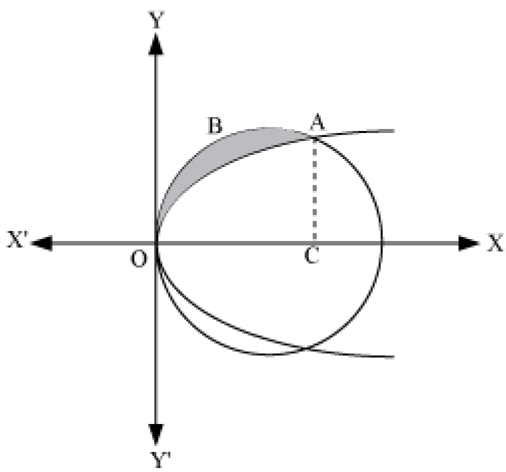
On substituting $y^2 = 2x$ in the first equation, we obtain

$x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0$ or $x = 2$

When $x = 0, y = 0$ and when $x = 2, y = 2$

Thus, the points of intersection of the two curves are O (0, 0) and A (2, 2).

The required area can be diagrammatically represented as



We now draw AC perpendicular to OX.

∴ Required area = Area OBAO

Area OBAO $= \int_0^2 \sqrt{4x - x^2}\,dx - \int_0^2 \sqrt{2x}\,dx = \int_0^2 \sqrt{4 - (2 - x)^2}\,dx - \sqrt{2}\int_0^2 \sqrt{x}\,dx$

Put $2 - x = 2\sin\theta \Rightarrow -dx = 2\cos\theta\, d\theta$

$\therefore$ Area OBAO = $4\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta - \frac{2\sqrt{2}}{3}\left[x^{\frac{3}{2}}\right]_0^2$

$= 4\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta - \frac{2\sqrt{2}}{3}(2)^{\frac{8}{2}}$

$= 4\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} - \frac{2}{3}\times 4$

$= \left(\pi - \frac{8}{3}\right)$