Relations and Functions

Types of Relations

Equivalence Relation

- A relation R in a set A is called reflexive if $(a, a) \in R$ for every $a \in A$.
- For example: A relation R in set $A = \left[0, \frac{\pi}{2}\right]$ defined by $R = \{\sin a = \sin b;\ a, b \in A\}$ is a reflexive relation since $\sin a = \sin a \forall a \in A$.
- A relation R in a set A is called symmetric if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.
- For example: A relation in the set $A = \left[0, \frac{\pi}{2}\right]$ defined by $R = \{\sin a = \sin b;\ a, b \in A\}$ is a symmetric relation. Since for $a, b \in A$, $\sin a = \sin b$ implies $\sin b = \sin a$. So, $(a, b) \in R \Rightarrow (b, a) \in R$.
- A relation R in a set A is called transitive if $(a_1, a_2) \in R$, and $(a_2, a_3) \in R$ together imply that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.
- For example: A relation in the set $A = \left[0, \frac{\pi}{2}\right]$ defined by $R = \{\sin a = \sin b, a, b \in A\}$ is a transitive relation. Since for $a, b, c \in A$, let $(a, b), (b, c) \in R$.

  $\Rightarrow \sin a = \sin b$ and $\sin b = \sin c$

  $\Rightarrow \sin a = \sin c$

  $\Rightarrow (a, c) \in R$

- A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.
- For example: Relation R in the set $A = \left[0, \frac{\pi}{2}\right]$ defined by $R = \{\sin a = \sin b;\ a, b \in A\}$ is an equivalence relation.

Equivalence Classes

- Every arbitrary equivalence relation R in a set X divides X into mutually disjoint subsets (A_i) called partitions or subdivisions of X satisfying the following conditions:

- All elements of A_i are related to0020each other for all i.
- No element of A_i is related to any element of A_j whenever $i \neq j$.
- $A_i \cup A_j = X$ and $A_i \cap A_j = \Phi$, $i \neq j$

These subsets (A_i) are called equivalence classes.

- For an equivalence relation in a set X, the equivalence class containing $a \in X$, denoted by $[a]$, is the subset of X containing all elements b related to a.

Trivial Relations

- Trivial relations are of two types:
- Empty relation
- Universal relation
- A relation in a set A is called an empty relation if no element of A is related to any element of A, i.e., $R = \Phi \subset A \times A$.
- For example: Consider a relation R in set $A = \{2, 4, 6\}$ defined by $R = \{(a, b): a + b \text{ is odd}, \text{ where } a, b \in A\}$. The relation R is an empty relation since for any pair $(a, b) \in A \times A$, $a + b$ is always even.
- A relation R in a set A is called a universal relation if each element of A is related to every element of A, i.e., $R = A \times A$.
- For example: Let A be the set of all students of class XI. Let R be a relation in set A defined by $R = \{(a, b)$: the sum of the ages of a and b is greater than 10 years$\}$. The relation R is a universal relation because it is obvious that the sum of the ages of two students of class XI is always greater than 10 years.

Solved Examples

Example 1

Check whether the relation R in the set of all vowels defined by $R = \{(u, u), (u, a), (a, u)\}$ is reflexive, symmetric or transitive?

Solution:

The relation R is defined in the set $\{a, e, i, o, u\}$ as $R = \{(u, u), (u, a), (a, u)\}$.

The relation R is not reflexive as $(a, a), (e, e)$ $(i, i), (o, o) \notin R$.

Now, (u, a) and $(a, u) \in R$

Hence, R is symmetric.

Now, (u, u) $(u, a) \in R$ implies $(u, a) \in R$

Also, $(u, a), (a, u) \in R$ implies $(u, u) \in R$

Hence, R is transitive.

Thus, the relation R is symmetric and transitive but not reflexive.

Example 2

Show that the relation R defined in the set of real numbers as $R = \{(a, b)$ $a = b$ or $a = -b$ for $a, b \in \mathbf{R}\}$ is an equivalence relation. Also, find its equivalence classes.

Solution:

A relation R in $\mathbf{R}$ is defined as $R = \{(a, b): a = b$ or $a = -b$, for $a, b \in \mathbf{R}\}$

Clearly, $(a, a) \in R$ for every $a \in \mathbf{R}$, since $a = a$.

$\therefore R$ is reflexive.

Now, let $(a, b) \in R$ for $a, b \in \mathbf{R}$

$\Rightarrow a = b$ or $a = -b$

$\Rightarrow b = a$ or $b = -a$

$\Rightarrow (b, a) \in R$

$\therefore R$ is symmetric.

Now, let $(a, b), (b, c) \in R$, for $a, b, c \in \mathbf{R}$

$\therefore a = b$ or $a = -b$ and $b = c$ or $b = -c$

Case I

$a = b, b = c$

$\Rightarrow a = c$

$\Rightarrow (a, c) \in R$

Case II

$a = b, b = -c$

$\Rightarrow a = -c$

$\Rightarrow (a, c) \in R$

Case III

$a = -b, b = c$

$\Rightarrow a = -c$

$\Rightarrow (a, c) \in R$

Case IV

$a = -b, b = -c$

$\Rightarrow a = c$

$\Rightarrow (a, c) \in R$

Thus, $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

Equivalence class of 0 = $[0] = \{0\}$

Equivalence class of 1 = $[1] = \{1, -1\}$

Equivalence class of 2 = $[2] = \{2, -2\}$ and so on. . .

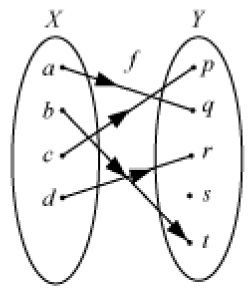
There are an infinite number of equivalence classes.

For every $a \in \mathbf{R}$, $[a] = \{a, -a\}$.

Types of Functions

- A function $f: X \to Y$ is said to be **one-one** (or **injective**) if the images of distinct elements of X under f are distinct. In other words, a function f is one-one if for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

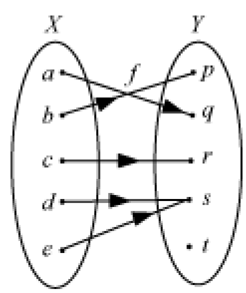
  An example of a one-one function from X to Y is shown in the following diagram.



One–One

A function $f: X \to Y$ is said to be **many-one** if the image of distinct elements of X under f are not distinct i.e., a function that is not one-one is called a many-one function.

An example of a many-one function from X to Y is shown in the following diagram.



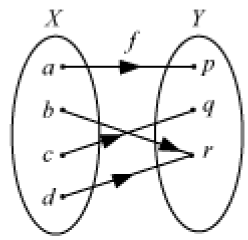
Many–One

In this case, two elements $f(d) = f(e) = s$.

- A function $f: X \to Y$ is defined as **onto** (or **surjective**) if every element of Y is the image of some element of x in X under f. In other words, f is onto if and only if, $y \in Y$, there exist $x \in X$ such that $f(x) = y$.

- $f: X \to Y$ is onto if and only if the range of $f = Y$.

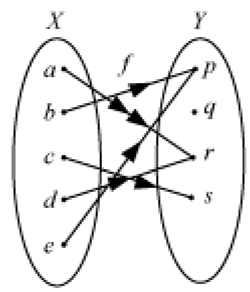
- An example of an onto function from X to Y is shown in the following diagram.



Onto

-

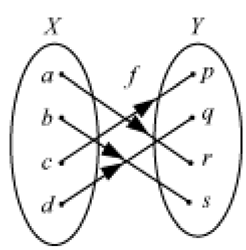
- A function from X to Y that is not onto is shown in the following diagram.



Not onto

- A function $f: X \to Y$ is said to be **bijective** if it is both one-one and onto.

  A bijective function from X to Y is shown in the following diagram.



Bijective

Solved Examples

Example 1

Check whether the function $h: \mathbf{R} \to \mathbf{R}$ defined by $h(x) = |2x - 5|$ is an injective function.

Solution:

The given function i.e., h: **R** → **R** is defined by

$h(x) = |2x-5|$

It can be observed that 5, 0 ∈ **R** (considering domain). Hence, we have

$h(5) = |2\times 5-5| = |10-5| = 5$

$h(0) = |2\times 0-5| = |0-5| = 5$

∴ $h(5) = h(0)$.

Hence, the given function i.e., $h(x)$ is not an injective function.

Example 2

Check whether the function f: **R** → **R** defined by $f(x) = x^5 + 4$ is a bijective function.

Solution:

We know that a function is bijective if it is both one-one and onto.

Now, let $x_1, x_2 \in$ **R** such that $f(x_1) = f(x_2)$. Accordingly,

$f(x_1) = f(x_2)$

$$\Rightarrow x_1^5 + 4 = x_2^5 + 4$$
$$\Rightarrow x_1^5 = x_2^5$$
$$\Rightarrow x_1 = x_2$$

Therefore, the function f is a one-one function.

It is clear that for every $y \in$ **R**, there exists $(y-4)^{\frac{1}{5}} \in$ **R** such that $f\left[(y-4)^{\frac{1}{5}}\right] = \left[(y-4)^{\frac{1}{5}}\right]^5 + 4 = (y-4)+4 = y$

Therefore, the function f is an onto function.

Hence, the given function f is a bijective function.

Example 3

Check whether the function f: **N** → **N** defined by $f(x) = 4^x$ is an onto function.

Solution:

The given function f: **N** → **N** is defined by

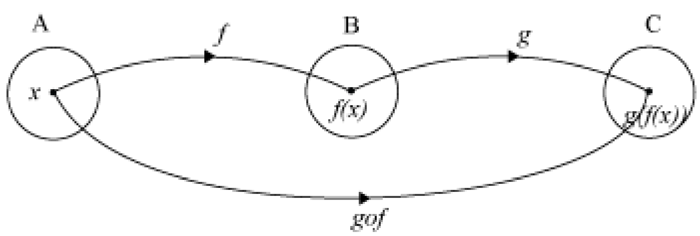
$f(x) = 4^x$

We can clearly observe that 2 ∈ **N** (co-domain). However, there does not exist any element y ∈ **N** (domain) whose image is 2.

Hence, the given function f is not an onto function.

Composition of Two functions

Let f: $A \to B$ and g: $B \to C$ be two functions. Accordingly, the composition of f and g is denoted by gof and is defined as the function gof: A → C given by $gof(x) = g(f(x))$, for all x∈A.



- For example: If f: **N** → **N** is defined by $f(x) = x + 1$ for all x∈**N** and g: **N** → **N** is defined by $g(x) = x^2$ for all x∈**N**, then gof : **N** → **N** is given by

  $gof(x) = g(f(x)) = g(x + 1) = (x + 1)^2$, where x∈**N**.

  Also, $fog(x) = f(g(x)) = f(x^2) = x^2 + 1$ for all x∈**N**

- If f: $A \to B$ and g: $B \to C$ are one-one, then gof: $A \to C$ is also one-one.
- If f: $A \to B$ and g: $B \to C$ are onto, then gof: $A \to C$ is also onto.
- If the composite function gof is one-one, then the function f is also one-one. However, the function g may or may not be one-one.
- If the composite function gof is onto, then the function g is also onto. However, the function f may or may not be onto.

Solved Examples

Example 1

Let f: **R** → **R** be given by $f(x) = 12x^2 - x - 11$ and g: **R** → **R** be given by $g(x) = x^2$. Find $fo(gog)$.

Solution:

It is given that

f: **R** → **R** is defined by $f(x) = 12x^2 - x - 11$

g: **R** → **R** is defined by $g(x) = x^2$

Now, $(gog)(x) = g(g(x))$

$= g(x^2)$

$= (x^2)^2$
$= x^4$

$(fo(gog))(x) = f((gog)(x))$

$= f(x^4)$

$= 12(x^4)^2 - x^4 - 11$

$= 12x^8 - x^4 - 11$

Example 2

Let f: **R** − $\left\{\frac{11}{3}\right\}$ → **R** − $\left\{-\frac{4}{3}\right\}$ be defined by $f(x) = \frac{4x-5}{11-3x}$ and g: **R** − $\left\{\frac{-4}{3}\right\}$ → **R** − $\left\{\frac{11}{3}\right\}$ be defined by $g(x) = \frac{11x+5}{4+3x}$. Show that $fog = I_A$ and $gof = I_B$, where I_A and I_B are identity functions on A and B respectively and A = **R** − $\left\{\frac{-4}{3}\right\}$ and B = **R** − $\left\{\frac{11}{3}\right\}$.

Solution:

$(fog)(x) = f(g(x))$

$$= f\left(\frac{11x+5}{4+3x}\right)$$

$$= \frac{4\left(\frac{11x+5}{4+3x}\right)-5}{11-3\left(\frac{11x+5}{4+3x}\right)}$$

$$= \frac{44x+20-20-15x}{44+33x-33x-15}$$
$$= \frac{29x}{29}$$
$$= x$$

$(gof)(x) = g(f(x))$

$$= g\left(\frac{4x-5}{11-3x}\right)$$
$$= \frac{11\left(\frac{4x-5}{11-3x}\right)+5}{4+3\left(\frac{4x-5}{11-3x}\right)}$$
$$= \frac{44x-55+55-15x}{44-12x+12x-15}$$
$$= \frac{29x}{29}$$
$$= x$$

Thus, $(fog)(x) = x$ for all $x \in A \Rightarrow fog = I_A$ and $(gof)(x) = x$ for all $x \in B \Rightarrow gof = I_B$.

Hence proved.

Example 3

Let f: **R** → **R** be defined as $f(x) = 2x^{\frac{1}{3}}$; g: **R** → **R** be defined as $g(x) = x + 2$ and h: **R** → **R** be defined as $h(x) = 4x + 9$. Find $fo(g + h)$ and $(fog) + (foh)$.

Solution:

$(g + h)$: **R** → **R** is given by:

$(g + h)(x) = g(x) + h(x)$

$= (x + 2) + (4x + 9)$
$= 5x + 11$

$\therefore fo(g + h)(x) = f((g + h)(x))$

$= f(5x + 11)$

$= 2(5x+11)^{\frac{1}{3}}$

Now, $(fog)(x) = f(g(x))$

$= f(x + 2)$

$= 2(x+2)^{\frac{1}{3}}$

$(foh)(x) = f(h(x))$

$= f(4x + 9)$

$= 2(4x+9)^{\frac{1}{3}}$

$\therefore (fog + foh)(x) = (fog)(x) + (foh)(x)$

$$= 2(x+2)^{\frac{1}{3}} + 2(4x+9)^{\frac{1}{3}}$$
$$= 2\left[(x+2)^{\frac{1}{3}} + (4x+9)^{\frac{1}{3}}\right]$$

Invertible Functions

Key Concepts

- A function $f: X \to Y$ is said to be invertible if there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$.
- The function g is called the inverse of f and it is denoted by f^{-1}.
- A function f is invertible if and only if f is one-one and onto.
- If $f: X \to Y$ and $g: Y \to Z$ are invertible functions, then gof is also invertible and $(gof)^{-1} = f^{-1}\, og^{-1}$

Solved Examples

Example 1

Determine whether the following functions have inverse or not. Find the inverse, if it exists.

(i) f: {10, 12, 15} → {3, 7, 9, 10, 14}is defined as f = {(12, 9), (15, 7), (10, 10)}.

(ii) g : {2, 4, 6, 8} → {1, 3, 5} is defined as g : {(4, 3), (8, 3), (2, 1), (6, 5)}.

(iii) h : {11, 16}→ {7, 14} is defined as h :{(11, 7), (16, 14)}.

Solution:

(i) The given function f is one-one. However, f is not onto since the elements 3, 14 ∈ {3, 7, 9, 10, 14} are not the image of any element in {10, 12, 15} under f.

Hence, function f is not invertible.

(ii) The given function g is onto. However, g is not one-one since, $g(4) = g(8) = 3$.

Hence, the function g is not invertible.

(iii) Clearly, the given function h is both one-one and onto. Hence, h is invertible.

The inverse of h is given by h^{-1} = {(7, 11), (14, 16)}.

Example 2

Determine whether the functions f and g, defined below, are inverses of each other or not.

f: **R** − {4} → **R** − {−3} is given as $f(x)=\frac{-3x}{x-4}$, and

g: **R** − {−3} → **R** − {4} is given as $g(x)=\frac{4x}{x+3}$

Solution:

We have

$$(fog)(x)=f(g(x))=f\left(\frac{4x}{x+3}\right)=\frac{-3\left(\frac{4x}{x+3}\right)}{\frac{4x}{x+3}-4}=\frac{-12x}{4x-4x-12}=\frac{-12x}{-12}=x$$

$$(gof)(x)=g(f(x))=g\left(\frac{-3x}{x-4}\right)=\frac{4\left(-\frac{3x}{x-4}\right)}{\left(\frac{-3x}{x-4}\right)+3}=\frac{-12x}{-3x+3x-12}=\frac{-12x}{-12}=x$$

Thus, $(fog)(x) = x \forall x \in B,$ where $B = \mathbf{R} - \{-3\}$ and $(gof)(x) = x \forall x \in A,$ $A = \mathbf{R} - \{4\}$.

$\therefore gof = I_A$ and $fog = I_B$.

Thus, functions *f* and *g* are the inverses of each other.

Example 3

Let f: $\mathbf{R}_+ \rightarrow [-3, \infty)$ be defined as $f(x) = 4x^2 - 5x - 3$ where $\mathbf{R}_+$ is the set of all positive real numbers. Show that *f* is invertible and find the inverse of *f*.

Solution:

f: $\mathbf{R}_+ \rightarrow [-3, \infty)$ is defined as $f(x) = 4x^2 - 5x - 3$.

Let *y* be an arbitrary element of $[-3, \infty)$.

Let $y = 4x^2 - 5x - 3$

$$\Rightarrow y = \left(2x - \frac{5}{4}\right)^2 - 3 - \frac{25}{16} = \left(2x - \frac{5}{4}\right)^2 - \frac{73}{16}$$

$$\Rightarrow \left(2x - \frac{5}{4}\right)^2 = y + \frac{73}{16}$$

$$\Rightarrow 2x - \frac{5}{4} = \sqrt{y + \frac{73}{16}} \qquad \left[y \geq -3 \Rightarrow y + \frac{73}{16} \geq 0\right]$$

$$x = \frac{1}{2}\sqrt{y + \frac{73}{16}} + \frac{5}{8}$$

$\therefore f$ is onto.

Hence, Range $f = [-3, \infty)$.

Let us define g: $[-3, \infty) \rightarrow \mathbf{R}_+$ as

$$g(y) = \frac{1}{2}\sqrt{y + \frac{73}{16}} + \frac{5}{8}$$

Now, we have

$$(gof)(x) = g(f(x)) = g(4x^2 - 5x - 3) = g\left(\left(2x - \frac{5}{4}\right)^2 - \frac{73}{16}\right) = \frac{1}{2}\sqrt{\left(2x - \frac{5}{4}\right)^2} + \frac{5}{8} = x - \frac{5}{8} + \frac{5}{8} = x$$

$$(fog)(y) = f(g(y)) = f\left(\left(\frac{1}{2}\sqrt{y + \frac{73}{16}}\right) + \frac{5}{8}\right) = \left[2 \times \left(\left(\frac{1}{2}\sqrt{y + \frac{73}{16}}\right) + \frac{5}{8}\right) - \frac{5}{4}\right]^2 - \frac{73}{16} = y + \frac{73}{16} - \frac{73}{16} = y$$

$$\therefore gof = I_{R_+} \text{ and } fog = I_{[-3,\infty)}$$

Thus, *f* is invertible and its inverse is given by

$$f^{-1}(y) = g(y) = \frac{1}{2}\left(\sqrt{y + \frac{73}{16}}\right) + \frac{5}{8}$$

Binary Operations

Definition of Binary Operation Properties

- A binary operation * on a set *A* is a function * from *A* × *A* → *A*. We denote *(*a, b*) by *a* * *b*.

  For example, the operation * defined on **N** as $a * b = a^2b$ is a binary operation since * carries each pair (*a, b*) to a unique element a^2b in **N.**

Properties of Binary Operation

- A binary operation * on a set *A* is called commutative, if *a* * *b* = *b* * *a*, for every *a, b* ∈ *A*.

  For example, *: **R** × **R** → **R** defined by *a* * *b* = 11 (*a* + *b* + *ab*) is commutative since *a* * *b* = 11(*a* + *b* + *ab*) and *b* * *a* = 11(*b* + *a* + *ba*). Therefore, *a* * *b* = *b* * *a.*

- A binary operation * on a set *A* is called associative, if (*a* * *b*) * *c* = *a* * (*b* * *c*), for every *a*, *b*, *c* ∈ *A*.

  For example, *: **N** × **N** → **N** defined by *a* * *b* = 5 + *a* + *b* is associative.

  a * (*b* * *c*) = 10 + *a* + *b* + *c* = 5 + (5+ *a* + *b*) + *c* = ((*a* * *b*) * *c*

- For a binary operation *: *A* × *A* → *A*, an element *e* ∈ *A*, if it exists, is called its identity element, if *a* * *e* = *a* = *e* * *a*, for every *a* ∈ *A*.

  For example: 1 is the identity for multiplication on **R**.

- Given a binary operation $*: A \times A \rightarrow A$ with the identity element e in A, an element $a \in A$ is said to be invertible with respect to the operation *, if there exists an element $b \in A$, such that $a * b = e = b * a$, and b is called the inverse of a and is denoted by a^{-1}.

  For example: $-a$ is the inverse of a for the addition operation on **R**, where 0 is the identity element.

Binary Operation Table

- When the number of elements in set A is small, we can express a binary operation * on A through a table called operation table.
- For an operation $*: A \times A \rightarrow A$, if $A = \{a_1, a_2 \ldots a_n\}$, then the operation table will have n rows and n columns with $(i, j)^{\text{th}}$ entry being $a_i * a_j$.
- Given any operation with n rows and n columns with each entry being an element of $A = \{a_1, a_2 \ldots a_n\}$, we can define a binary operation * on A given by $a_i * a_j$ = entry in i^{th} row and j^{th} column of the operation table

  Example: We can define a binary operation * on $A = \{a, b, c\}$ as follows:

*	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

Here, $a * b = b = b * a$

$a * c = c = c * a$

$b * c = c = c * b$

∴ The operation * is commutative.

Solved Examples

Example 1:

A binary operation $A \times A \to A$, where $A = \{a, b, c\}$, is defined as follows:

*	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*a*	*c*	*b*

Determine whether the operation * is commutative and associative. Also, find the identity for the operation *, if it exists.

Solution:

From the table, it can be observed that

$a * b = a = b * a$

$a * c = a = c * a$

$b * c = c = c * b$

The given binary operation * is commutative since for all $x, y, \in A = \{a, b, c\}$.

$x * y = y * x$

Now, consider $a * (b * c) = a * c = a$

$(a * b) * c = a * c = a$

Thus, $a * (b * c) = (a * b) * c$

Similarly, we can prove that $(x * y) * z = x * (y * z)$ for all $x, y, z \in A$.

Thus, the given binary operation * is associative.

Also, we can observe that for any element $x \in A$, we have $x * a = x = a * x$.

Thus, a is the identity element for the given binary operation *.

Example 2:

Determine whether the binary operation on the set **R**, defined by $a * b = \frac{2a+5b}{4}$, $a, b \in$ **R,** is commutative or not.

Solution:

We have *: **R** × **R** → **R** defined by $a * b = \frac{2a+5b}{4}$, $a, b \in$ **R.**

We know that a binary operation * defined on set A is commutative, if $a * b = b * a$ &mnForE $a, b \in A$.

Now, $a * b = \frac{2a+5b}{4}$ and $b * a = \frac{2b+5a}{4}$

$\therefore a * b \neq b * a$

Hence, the given binary operation * is not commutative.

Example 3:

A binary operation * on the set {5, 6, 9} is defined by the following table:

*	5	6	9
5	5	6	9
6	6	9	5
9	9	5	6

Compute (5 * 9) * 6 and 5 * (9 * 6). Are they equal?

Solution:

From the given binary operation table, we have (5 * 9) = 9

∴(5 * 9) * 6 = 9 * 6 = 5

Then, (9 * 6) = 5

∴5 * (9 * 6) = 5 * 5 = 5

Thus, (5 * 9) * 6 = 5 * (9 * 6)