

Determinants

Determinants of Matrices upto Order 3

Key Concepts

- To every square matrix $A = [a_{ij}]$ of order n , we can associate a number (real or complex) called the determinant of the square matrix A .
- The determinant of a matrix A is denoted by $|A|$ or $\det A$ or Δ .
- If M is the set of square matrices, K is the set of numbers (real or complex) and $f: M \rightarrow K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of A .

Calculation of Determinants

- Let $A = [a]$ be the matrix of order 1. Accordingly, the determinant of A is defined to be equal to a .

- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2×2 . Accordingly, the determinant of A is defined as $\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{21} a_{12}$

To understand how to find the determinant of a 2×2 matrix with the help of an example,

To understand how to find the determinant of a 3×3 matrix,

- The determinant of a matrix A can be obtained by expanding along any other row or any other column also.
- For easier calculation, we expand the determinant along the row or column that contains the maximum number of zeroes.
- If $A = kB$, where A and B are square matrices of order n , then $|A| = k^n |B|$, where $n = 1, 2, 3$.

Solved Examples

Example 1

Evaluate $\begin{vmatrix} -5 & 4 & 2 \\ 3 & 1 & 6 \\ 2 & 5 & 3 \end{vmatrix}$.

Solution:

$$\begin{aligned}\Delta &= \begin{vmatrix} -5 & 4 & 2 \\ 3 & 1 & 6 \\ 2 & 5 & 3 \end{vmatrix} \\ &= -5(1 \times 3 - 5 \times 6) - 4(3 \times 3 - 2 \times 6) + 2(3 \times 5 - 2 \times 1) \\ &= -5(3 - 30) - 4(9 - 12) + 2(15 - 2) \\ &= -5(-27) - 4(-3) + 2(13) \\ &= 135 + 12 + 26 \\ &= 173\end{aligned}$$

Example 2

Find the value of a if $\begin{vmatrix} a & -3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -a \\ 2 & 5 \end{vmatrix}$.

Solution:

We have

$$\begin{aligned}\begin{vmatrix} a & -3 \\ 2 & 4 \end{vmatrix} &= \begin{vmatrix} 1 & -a \\ 2 & 5 \end{vmatrix} \\ \Rightarrow a \times 4 - 2 \times (-3) &= 1 \times 5 - 2 \times (-a) \\ \Rightarrow 4a + 6 &= 5 + 2a \\ \Rightarrow 4a - 2a &= 5 - 6 \\ \Rightarrow 2a &= -1 \\ \Rightarrow a &= -\frac{1}{2}\end{aligned}$$

Example 3

Find the equation which x satisfies if $\begin{vmatrix} x & x+1 \\ 2 & -3x \end{vmatrix} = 4$.

Solution:

We have

$$\begin{vmatrix} x & x+1 \\ 2 & -3x \end{vmatrix} = 4$$

$$\Rightarrow x \times (-3x) - 2 \times (x + 1) = 4$$

$$\Rightarrow -3x^2 - 2x - 2 = 4$$

$$\Rightarrow -3x^2 - 2x - 6 = 0$$

$$\Rightarrow 3x^2 + 2x + 6 = 0$$

This is the required equation that x satisfies.

Properties of Determinants

- The value of the determinant remains unchanged if its rows and columns are interchanged.
- The interchange of two rows or columns can be symbolically written as $C_i \leftrightarrow R_i$, where R_i is the i^{th} row and C_i is the i^{th} column.

• For example,
$$\begin{vmatrix} -9 & 8 & 1 \\ -5 & 5 & 3 \\ -1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} -9 & -5 & -1 \\ 8 & 5 & 2 \\ 1 & 3 & 4 \end{vmatrix}$$

- If A is a square matrix, then $\det(A) = \det(A')$ i.e., $|A| = |A'|$
- If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.
- The interchange of two rows is written as $R_i \leftrightarrow R_j$ and the interchange of two columns is written as $C_i \leftrightarrow C_j$.

- For example,

$$\begin{vmatrix} 1 & 1 & -2 \\ 8 & 6 & -3 \\ 5 & 4 & -9 \end{vmatrix} = - \begin{vmatrix} 5 & 4 & -9 \\ 8 & 6 & -3 \\ 1 & 1 & -2 \end{vmatrix}, \text{ by applying } R_1 \leftrightarrow R_3$$

- If any two rows (or columns) of a determinant are identical or proportional, then the value of the determinant is zero.

For example,
$$\begin{vmatrix} 4 & -9 & 11 \\ 12 & 3 & 13 \\ 8 & -18 & 22 \end{vmatrix} = 2 \begin{vmatrix} 4 & -9 & 11 \\ 12 & 3 & 13 \\ 4 & -9 & 11 \end{vmatrix} = 0$$

- If each element of a row (or a column) of determinant is multiplied by a constant α , then its determinant value gets multiplied by α . For example:

Consider
$$A = \begin{vmatrix} -6 & 2 & 8 \\ 3 & 1 & 2 \\ -1 & 4 & 7 \end{vmatrix}$$

Now, $|A| = 64$

$$|3A| = \begin{vmatrix} -6 & 2 & 8 \\ 9 & 3 & 6 \\ -1 & 4 & 7 \end{vmatrix} = 192 = 3 \times 64 = 3|A|$$

- If some or all elements of a row or a column in a determinant can be expressed as the sum of two (or more) elements, then the determinant can be expressed as the sum of two (or more) determinants.
For example,
- If to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row or column are added, then the value of the determinant remains unchanged.
- In other words, the value of the determinant remains same, if we apply the operation $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$. That is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \lambda a_3 & b_1 + \lambda b_3 & c_1 + \lambda c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Solved Examples

Example 1: Evaluate:
$$\begin{vmatrix} 270 & 255 & 234 \\ 255 & 230 & 213 \\ 234 & 213 & 186 \end{vmatrix}$$

Solution:

$$\Delta = \begin{vmatrix} 270 & 255 & 234 \\ 255 & 230 & 213 \\ 234 & 213 & 186 \end{vmatrix}$$

By applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$, we obtain

$$\Delta = \begin{vmatrix} 36 & 21 & 234 \\ 42 & 17 & 213 \\ 48 & 27 & 186 \end{vmatrix}$$

By applying $C_1 \rightarrow C_1 - 2C_2$ and $C_3 \rightarrow C_3 - 10C_2$, we obtain

$$\Delta = \begin{vmatrix} -6 & 21 & 24 \\ 8 & 17 & 43 \\ -6 & 27 & -84 \end{vmatrix}$$

By applying $R_3 \rightarrow R_3 - R_1$ and $R_2 \rightarrow R_2 + \frac{4}{3}R_1$, we obtain

$$\Delta = \begin{vmatrix} -6 & 21 & 24 \\ 0 & 45 & 75 \\ 0 & 6 & -108 \end{vmatrix} = -54 \begin{vmatrix} 1 & 7 & 8 \\ 0 & 15 & 25 \\ 0 & 2 & -36 \end{vmatrix} = -54(-540 - 50) = 31860$$

Example 2: Prove that:
$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2 \\ 10x+8y & 8x & 3 \end{vmatrix} = x^2$$

Solution:

$$\begin{aligned}
\Delta &= \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2 \\ 10x+8y & 8x & 3 \end{vmatrix} \\
&= \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 10x & 8x & 3 \end{vmatrix} + \begin{vmatrix} y & x & x \\ 4y & 4x & 2 \\ 8y & 8x & 3 \end{vmatrix} \\
&= \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 10x & 8x & 3 \end{vmatrix} + xy \begin{vmatrix} 1 & 1 & x \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix} \\
&= \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 10x & 8x & 3 \end{vmatrix} + 0 \\
&= \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 10x & 8x & 3 \end{vmatrix}
\end{aligned}$$

By applying $R_3 \rightarrow R_3 - 2R_2$, we obtain

$$\Delta = \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 0 & 0 & -1 \end{vmatrix}$$

By applying $R_2 \rightarrow R_2 - 4R_1$, we obtain

$$\Delta = \begin{vmatrix} x & x & x \\ x & 0 & 2-4x \\ 0 & 0 & -1 \end{vmatrix}$$

By expanding along R_3 , we obtain

$$\Delta = (-1) \times (0 - x^2) = x^2$$

Hence proved.

Area of Triangle

Area of Triangle

- The area of a triangle whose vertices are (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) is given by the determinant:

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- The area of a triangle is always positive, therefore, the absolute value of the above mentioned determinant is always taken.
- If the area of a triangle is given, then both positive and negative values of the determinant are used for calculation.
- The area of the triangle formed by three collinear points is zero.

Solved Examples

Example 1: Using determinants, find the equation of the line joining the points $\left(\frac{1}{2}, \frac{1}{3}\right)$ and $(-4, 5)$.

Solution:

Let the given points be A $\left(\frac{1}{2}, \frac{1}{3}\right)$ and B $(-4, 5)$.

Let $P(x, y)$ be the point lying on the line AB. Then, the points A, P, and B will be collinear and hence, the area of the triangle formed by joining these three points is zero.

$$\therefore \frac{1}{2} \begin{vmatrix} \frac{1}{2} & \frac{1}{3} & 1 \\ -4 & 5 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2} \begin{vmatrix} 5 & 1 \\ y & 1 \end{vmatrix} - \frac{1}{3} \begin{vmatrix} -4 & 1 \\ x & 1 \end{vmatrix} + 1 \begin{vmatrix} -4 & 5 \\ x & y \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}(5-y) - \frac{1}{3}(-4-x) + 1(-4y-5x) = 0$$

$$\begin{aligned}
&\Rightarrow \frac{5}{2} - \frac{1}{2}y + \frac{1}{3}x + \frac{4}{3} - 4y - 5x = 0 \\
&\Rightarrow x\left(\frac{1}{3} - 5\right) + y\left(-\frac{1}{2} - 4\right) + \left(\frac{5}{2} + \frac{4}{3}\right) = 0 \\
&\Rightarrow -\frac{14x}{3} - \frac{9y}{2} + \frac{23}{6} = 0 \\
&\Rightarrow \frac{14}{3}x + \frac{9y}{2} - \frac{23}{6} = 0
\end{aligned}$$

$$28x + 27y - 23 = 0$$

This is the required equation of the line joining the two given points.

Example 2: Find the value of k , if the area of the triangle formed by joining the vertices $(5, 5)$, $(k, 1)$, and $(3, -1)$ is 16 square units.

Solution:

The area of the triangle formed by joining the vertices $(5, 5)$, $(k, 1)$, and $(3, -1)$ is given by,

$$\begin{aligned}
&\frac{1}{2} \begin{vmatrix} 5 & 5 & 1 \\ k & 1 & 1 \\ 3 & -1 & 1 \end{vmatrix} \\
&= \frac{1}{2} [5(1+1) - 5(k-3) + 1(-k-3)] \\
&= \frac{1}{2} [5(2) - 5(k-3) + 1(-k-3)] \\
&= \frac{1}{2} [10 - 5k + 15 - k - 3] \\
&= \frac{1}{2} [-6k + 22] = -3k + 11
\end{aligned}$$

It is given that the area of the triangle is 16 square units. Therefore, we have

$$-3k + 11 = \pm 16$$

$$\Rightarrow -3k = 16 - 11 \text{ or } -3k = -16 - 11$$

$$\Rightarrow -3k = 5 \text{ or } -3k = -27$$

$$\Rightarrow k = -\frac{5}{3} \text{ or } k = 9$$

Thus, the value of k is $-\frac{5}{3}$ or 9 .

Minors and Cofactors

Minors

- Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which the element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij} .
- Minor of an element of a determinant of order n ($n \geq 2$) is a determinant of order $(n - 1)$.
- For example, the minor of the element a_{22} in the determinant

$$\begin{vmatrix} 10 & 2 & -8 \\ 11 & 21 & 6 \\ 3 & 9 & 5 \end{vmatrix} \text{ is given by, } M_{22} = \begin{vmatrix} 10 & -8 \\ 3 & 5 \end{vmatrix} = 10 \times 5 - 3 \times (-8) = 50 + 24 = 74$$

Cofactors

- Cofactor of an element a_{ij} , denoted by A_{ij} , is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij} .

For example, the cofactor of the element a_{23} in the determinant

$$\begin{vmatrix} -1 & 2 & -1 \\ 3 & 5 & 4 \\ 5 & -2 & -3 \end{vmatrix} \text{ is given by,}$$

$$A_{23} = (-1)^{2+3} M_{23} = -1[2 - 10] = -1 \times -8 = 8$$

- The value of a determinant is equal to the sum of the product of elements of any row (or column) with their corresponding cofactors. That is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

- If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. That is, $a_{12}A_{13} + a_{22}A_{23} + a_{32}A_{33} = 0$
- To understand how to find the minor and cofactor of an element in a better way.

Solved Examples

Example 1: Find the minors and cofactors of the elements of the determinant: $\begin{vmatrix} x & x+y \\ 2x+y & -x \end{vmatrix}$

Solution:

The given determinant is $\begin{vmatrix} x & x+y \\ 2x+y & -x \end{vmatrix}$.

$$M_{11} = -x A_{11} = (-1)^{1+1} (-x) = -x$$

$$M_{12} = 2x + y A_{12} = (-1)^{1+2} (2x + y) = -(2x + y)$$

$$M_{21} = x + y A_{21} = (-1)^{2+1} (x + y) = -(x + y)$$

$$M_{22} = x A_{22} = (-1)^{2+2} (x) = x$$

Example 2: Using cofactors of elements of second column, evaluate: $\begin{vmatrix} 2 & 1 & 6 \\ -3 & 4 & 9 \\ 5 & 8 & -2 \end{vmatrix}$

Solution:

The given determinant is $\Delta = \begin{vmatrix} 2 & 1 & 6 \\ -3 & 4 & 9 \\ 5 & 8 & -2 \end{vmatrix}$

$$M_{12} = 6 - 45 = -39 \quad A_{12} = (-1)^{1+2} M_{12} = -1 \times -39 = 39$$

$$M_{22} = -4 - 30 = -34 \quad A_{22} = (-1)^{2+2} M_{22} = -34$$

$$M_{32} = 18 + 18 = 36 \quad A_{32} = (-1)^{3+2} M_{32} = (-1) \times 36 = -36$$

We know that the value of a determinant is equal to the sum of the product of elements of any column (or row) with their corresponding cofactors.

$$\begin{aligned}
\therefore \Delta &= a_{12} A_{12} + a_{22} A_{22} + a_{32} A_{32} \\
&= 1 \times 39 + 4 \times (-34) + 8 \times (-36) \\
&= 39 - 136 - 288 \\
&= -385
\end{aligned}$$

Thus, the value of the given determinant is -385 .

Adjoint and Inverse of a Matrix

Adjoint of a Matrix

- The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij} . Adjoint of the matrix A is denoted by $\text{adj } A$.

- For the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the adjoint is given by:

$$\text{adj } A = \text{Transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

- For a square matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ of order 2, the $\text{adj } A$ can be obtained by interchanging a_{11} and a_{22} and by changing the signs of a_{12} and a_{21} .

$$\text{adj } A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- If A is a square matrix of order n , then

$$A (\text{adj } A) = (\text{adj } A) A = |A| I, \text{ where } I \text{ is the identity matrix of order } n.$$

- A square matrix A is said to be singular if $|A| = 0$, otherwise A is said to be non-singular (i.e., if $|A| \neq 0$).
- If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

- The determinant of the product of matrices is equal to the product of their respective determinants. That is, $|AB| = |A| \cdot |B|$, where A and B are square matrices of the same order.
- If A is a non-singular matrix of order n , then $|adj A| = |A|^{n-1}$

Inverse of a Matrix

- A square matrix A is invertible if and only if A is a non-singular matrix. If A is an invertible matrix of order n , then there exists a square matrix B of order n , such that $AB = BA = I$, where I is the identity matrix of order n . Here, matrix B is called the inverse of matrix A and vice-versa (written as $A^{-1} = B$ and $B^{-1} = A$).

It can be noted that $(A^{-1})^{-1} = A$

The matrix $\begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}$ is not invertible, since $\begin{vmatrix} 4 & 8 \\ 3 & 6 \end{vmatrix} = 24 - 24 = 0$.

- The inverse of an invertible square matrix A is given by $A^{-1} = \frac{1}{|A|} \cdot (adj A)$
- If A and B are two square matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$

Solved Examples

Example 1:

If $A = \begin{bmatrix} 12 & 4 \\ 10 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ 6 & -7 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1} A^{-1}$

Solution:

$$\text{We have, } AB = \begin{bmatrix} 12 & 4 \\ 10 & -2 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 6 & -7 \end{bmatrix} = \begin{bmatrix} 36+24 & -60-28 \\ 30-12 & -50+14 \end{bmatrix} = \begin{bmatrix} 60 & -88 \\ 18 & -36 \end{bmatrix}$$

$$\text{Now, } |AB| = 60 \times (-36) - 18 \times (-88) = -2160 + 1584 = -576 \neq 0$$

Therefore, inverse of AB exists.

$$adj(AB) = \begin{bmatrix} -36 & 88 \\ -18 & 60 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB) = \frac{-1}{576} \begin{bmatrix} -36 & 88 \\ -18 & 60 \end{bmatrix}$$

$$\text{Now, } |A| = 12 \times (-2) - 10 \times 4 = -24 - 40 = -64 \neq 0$$

$$|B| = 3 \times (-7) - 6 \times (-5) = -21 + 30 = 9 \neq 0$$

$\therefore A^{-1}$ and B^{-1} exists.

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-64} \begin{bmatrix} -2 & -4 \\ -10 & 12 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B = \frac{1}{9} \begin{bmatrix} -7 & 5 \\ -6 & 3 \end{bmatrix}$$

$$\therefore B^{-1}A^{-1} = \frac{-1}{64 \times 9} \begin{bmatrix} -7 & 5 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ -10 & 12 \end{bmatrix} = \frac{-1}{576} \begin{bmatrix} 14-50 & 28+60 \\ 12-30 & 24+36 \end{bmatrix} = \frac{-1}{576} \begin{bmatrix} -36 & 88 \\ -18 & 60 \end{bmatrix}$$

$$\text{Hence, } (AB)^{-1} = B^{-1}A^{-1}$$

Example 2:

$$\text{If } A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \\ -2 & 2 & -3 \end{bmatrix}, \text{ then find } A^{-1}.$$

Solution:

$$\text{For the given matrix A, we have } |A| = 1(3-2) - 2(-9+2) + 1(6-2) = 1 + 14 + 4 = 19$$

$$\text{Now, } A_{11} = 1, A_{12} = 7, A_{13} = 4,$$

$$A_{21} = 8, A_{22} = -1, A_{23} = -6,$$

$$A_{31} = 3, A_{32} = 2, A_{33} = -7$$

$$\therefore \text{adj } A = \begin{bmatrix} 1 & 8 & 3 \\ 7 & -1 & 2 \\ 4 & -6 & -7 \end{bmatrix}$$

$$= \frac{1}{|A|} \text{adj } A = \frac{1}{19} \begin{bmatrix} 1 & 8 & 3 \\ 7 & -1 & 2 \\ 4 & -6 & -7 \end{bmatrix}$$

Thus, A^{-1}

Consistency of System of Linear Equations

- A system of equations is said to be consistent, if its solution (one or more) exists.
- A system of equations is said to be inconsistent, if its solution does not exist.
- A system of linear equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{can be written as } AX = B, \text{ where}$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

For the system of equations $AX = B$, we have the following possibilities:

- If $|A| \neq 0$, i.e. if A is non-singular, then there exists a unique solution.
- If $|A| = 0$, we calculate $(\text{adj } A) B$.

If $(\text{adj } A) B \neq O$, (O being zero matrix), then solution does not exist and the system of equations is inconsistent.

If $(\text{adj } A) B = O$, then system may be either consistent or inconsistent depending on whether the system has either infinitely many solutions or no solution.

- To understand how to check whether the given system of equations is consistent or inconsistent.

Solved Examples

Example 1:

Determine whether the solution of the following system of equations exists or not.

$$12x - 9y - 7 = 0$$

$$-4x + 3y + 11 = 0$$

Solution:

The given system of linear equations is

$$12x - 9y - 7 = 0 \text{ or } 12x - 9y = 7$$

$$-4x + 3y + 11 = 0 \text{ or } -4x + 3y = -11$$

This system can be written as $AX = B$, where

$$A = \begin{bmatrix} 12 & -9 \\ -4 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } B = \begin{bmatrix} 7 \\ -11 \end{bmatrix}$$

$$\text{Now, } |A| = 12 \times 3 - (-9) \times (-4) = 36 - 36 = 0$$

$\therefore A$ is a singular matrix.

Now, we calculate $(adj A) B$.

$$adj A = \begin{bmatrix} 3 & 9 \\ 4 & 12 \end{bmatrix}$$

$$\therefore (adj A) B = \begin{bmatrix} 3 & 9 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} 7 \\ -11 \end{bmatrix} = \begin{bmatrix} -78 \\ -104 \end{bmatrix} \neq O$$

Hence, the given system of equations is inconsistent.

Example 2:

Examine the consistency of the system of following equations:

$$2x - y + 3z = 0$$

$$5x + 2y + z = 7$$

$$x - 6y - 2z = 9$$

Solution:

The given system of linear equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 2 & 1 \\ 1 & -6 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix}$$

We have,

$$|A| = 2[-4+6] + 1[-10-1] + 3[-30-2] = 2 \times 2 + 1 \times (-11) + 3 \times (-32) = 4 - 11 - 96 \\ = 4 - 107 = -103 \neq 0$$

$\therefore |A| \neq 0$ i.e., A is a non-singular matrix.

Hence, the given system of equations is consistent.

Solution of System of Linear Equations

- $a_1x + b_1y + c_1z = d_1$
 $a_2x + b_2y + c_2z = d_2$
 $a_3x + b_3y + c_3z = d_3$

A system of above linear equations, which can also be written in the form of $AX = B$, can be solved by using matrix method.

- If A is non-singular, i.e. if $|A| \neq 0$, then its inverse exists and the unique solution of the system of equations $AX = B$ is given by $X = A^{-1}B$.
- If A is singular, i.e. $|A| = 0$, then in this case $(adj A) \cdot B$ is calculated and the following cases arise:
- If $(adj A) \cdot B \neq 0$ (0, being null matrix), then the solution of the system of equations $AX = B$ does not exist.
- If $(adj A) \cdot B = 0$, then the system of equations may either have infinitely many solutions or no solution.

Solved Examples

Example 1: Solve the following system of equations by matrix method.

$$4x - 3y = 17$$

$$-7x - 8y = 10$$

Solution:

The given system of linear equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 4 & -3 \\ -7 & -8 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } B = \begin{bmatrix} 17 \\ 10 \end{bmatrix}$$

$$\text{Now, } |A| = 4 \times (-8) - (-3) \times (-7) = -32 - 21 = -53 \neq 0$$

Hence, A is a non-singular matrix. Therefore, the given system of linear equations has a unique solution which is given by,

$$X = A^{-1}B$$

$$A^{-1} = \frac{1}{|A|} \cdot (\text{adj}A) = \frac{1}{-53} \begin{bmatrix} -8 & 3 \\ 7 & 4 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{53} \begin{bmatrix} -8 & 3 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 17 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{53} \begin{bmatrix} -8 \times 17 + 3 \times 10 \\ 7 \times 17 + 4 \times 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{53} \begin{bmatrix} -136 + 30 \\ 119 + 40 \end{bmatrix} = -\frac{1}{53} \begin{bmatrix} -106 \\ 159 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Thus, the solution of the given system of linear equations is $x = 2$ and $y = -3$.

Example 2: The cost of 2 burgers, 1 packet of French fries, and 1 cold drink is Rs 90. The cost of 1 burger, 2 packets of French fries, and 1 cold drink is Rs 105. The cost of 3 burgers, 1 packet of French fries, and 2 cold drinks is Rs 125. Find the cost of one burger, one packet of French fries, and 1 cold drink by matrix method.

Solution:

Let the respective cost of one burger, one packet of French fries, and one cold drink be x, y , and z .

According to the given information, we obtain

$$2x + y + z = 90$$

$$x + 2y + z = 105$$

$$3x + y + 2z = 125$$

This system can be written as $AX=B$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} 90 \\ 105 \\ 125 \end{bmatrix}$$

$$|A| = 2(4-1) - 1(2-3) + 1(1-6) = 6 + 1 - 5 = 2 \neq 0$$

Now,

$$A_{11} = 4-1=3, A_{12} = -(2-3)=1, A_{13} = 1-6=-5$$

$$A_{21} = -(2-1)=-1, A_{22} = 4-3=1, A_{23} = -(2-3)=1$$

$$A_{31} = 1-2=-1, A_{32} = -(2-1)=-1, A_{33} = 4-1=3$$

$$\text{adj}A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{2} \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 3 \end{bmatrix}$$

Now, $X = A^{-1}B$

Therefore, we obtain

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 3 \end{bmatrix} \begin{bmatrix} 90 \\ 105 \\ 125 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 270 - 105 - 125 \\ 90 + 105 - 125 \\ -450 + 105 + 375 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 40 \\ 70 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 35 \\ 15 \end{bmatrix}$$

$$\therefore x = 20, y = 35, z = 15$$

Thus, the respective costs of one burger, one packet of French fries, and one cold drink are Rs 20, Rs 35, and Rs 15.