

Continuity and Differentiability

Continuity of a Function

- Suppose a function f is a real valued function defined on a subset of real numbers. Let c be a point in the domain of f . Then, f is continuous at c if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

- A function is continuous at $x = c$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$

i.e., $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} f(x) = f(c)$

- If f is not continuous at point c , then f is said to be discontinuous at c and ' c ' is called the point of discontinuity of f .

- Consider the function
$$f(x) = \begin{cases} x^2 + 2, & x = 0 \\ 1, & x \neq 0 \end{cases}$$

At point $x = 0$,

$$\text{Left hand limit} = \lim_{x \rightarrow 0^-} f(x) = 1; \text{ Right hand limit} = \lim_{x \rightarrow 0^+} f(x) = 1$$

$$f(0) = 0^2 + 2 = 2$$

$$\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} f(x) \neq f(0)$$

Therefore, f is not continuous at $x = 0$.

At point $x = 1$,

$$\text{Left hand limit} = \lim_{x \rightarrow 1^-} f(x) = 1; \text{ Right hand limit} = \lim_{x \rightarrow 1^+} f(x) = 1$$

$$f(1) = 1$$

$$\therefore \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} f(x) = f(1)$$

Therefore, f is continuous at $x = 1$.

- A real function f is said to be continuous if it is continuous at every point in the domain of f .

- The continuity of function can be written as:
- Suppose f is a function defined on close interval $[a, b]$. Then, for f to be continuous, it needs to be continuous at every point of $[a, b]$ including a and b .

Thus, the continuity of f at a means $\lim_{x \rightarrow a^+} f(x) = f(a)$.

The continuity of f at b means $\lim_{x \rightarrow b^-} f(x) = f(b)$.

- Suppose a function f is defined at a point c . Then, f is continuous in the domain $\{c\}$.
- To understand how to check the continuity of a function at a point,

Concept of Infinity

- Analyse the function $f(x) = \frac{1}{x}$ near $x = 0$. Here, x can approach 0 either from the left of 0 or from the right of 0.
- When x approaches 0 from the right, then we have the following values:

•

x	1	0.5	0.25	$0.01 = 10^{-2}$	$0.0001 = 10^{-4}$	10^{-n}
$f(x)$	1	2	4	10^2	10^4	10^n

•

As x gets closer to 0 from the right, the value of $f(x)$ keeps on increasing. When $f(x)$ keeps on increasing, it is said to approach infinity(∞).

Mathematically, it is written as $\lim_{x \rightarrow 0^+} f(x) = \infty$, where $f(x) = \frac{1}{x}, x \neq 0$.

- When x approaches 0 from the left, then we have the following values:

x	-1	-0.5	-0.25	$-0.01 = -10^{-2}$	$-0.0001 = 10^{-4}$	-10^{-n}
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$f(x)$	-1	-2	-4	-100	-10^4	-10^n
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- As x gets closer to 0 from the left, the value of $f(x)$ keeps on decreasing. When $f(x)$ keeps on decreasing, it is said to approach infinity ($-\infty$). Mathematically, it is

written as $\lim_{x \rightarrow 0^-} f(x) = -\infty$, where $f(x) = \frac{1}{x}, x \neq 0$.

Solved Examples

Example 1

Check whether the given function is continuous or not.

$$f(x) = \begin{cases} \frac{1}{4}x + 1, & x \leq 4 \\ \frac{8}{x}, & x > 4 \end{cases}$$

Solution:

Let us first check the continuity of f at $x = 4$.

$$\lim_{x \rightarrow 4^-} f(x) = \frac{4}{4} + 1 = 2, \quad \lim_{x \rightarrow 4^+} f(x) = \frac{8}{4} = 2, \quad f(4) = \frac{4}{4} + 1 = 2$$

$$\therefore \lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x) = f(4)$$

Hence, f is continuous at $x = 4$.

Let c be a real number such that $c < 4$.

Accordingly, $\lim_{x \rightarrow c} f(x) = \frac{c}{4} + 1 = f(c)$.

Hence, f is continuous at all real numbers less than 4.

Let c be a real number such that $c > 4$.

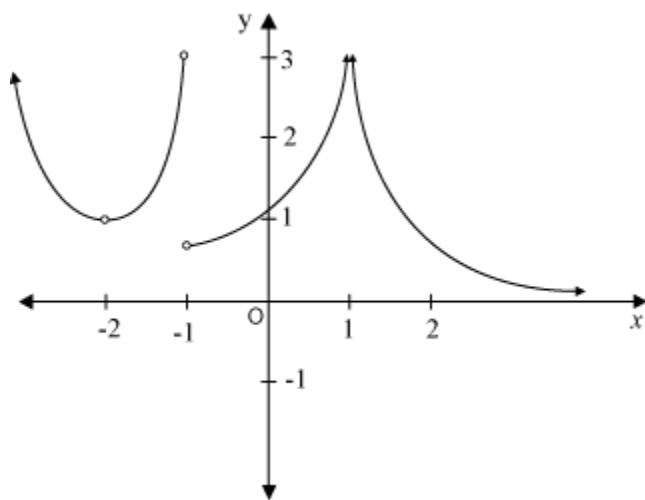
Accordingly, $\lim_{x \rightarrow c} f(x) = \frac{8}{c} = f(c)$

Hence, f is continuous at all real numbers greater than 4.

Thus, f is continuous at all points. Hence, f is a continuous function.

Example 2

Consider the graph of $f(x)$ given below.



State the conditions of continuity that are not met at each discontinuous point within the interval $[-2, 2]$.

Solution:

Consider the given function at point $x = -2$.

f is not defined at point $x = -2$.

Therefore, f is discontinuous at $x = -2$.

Consider the given function at point $x = -1$.

Function f is not defined at $x = -1$. Hence, it is discontinuous at $x = -1$.

Consider the point $x = 1$.

The function f is not defined at $x = 1$. Hence, it is discontinuous at $x = 1$.

Algebra of Continuous Functions

- Suppose f and g are two real functions continuous at a real number c , then
- $f + g$ is continuous at $x = c$

- $f - g$ is continuous at $x = c$
- $f \cdot g$ is continuous at $x = c$
- $\left(\frac{f}{g}\right)$ is continuous at $x = c$ (provided $g(c) \neq 0$)
- If f is a constant function, i.e. $f(x) = k$ for some real number k , then
- The function $f \cdot g = (k \cdot g)$ defined by $(k \cdot g)(x) = k \cdot g(x)$ is also continuous, provided g is continuous.
- The function $\left(\frac{k}{g}\right)$ defined by $\frac{k}{g}(x) = \frac{k}{g(x)}$ is also continuous, provided g is continuous and $g(x) \neq 0$.

All polynomial functions, sine function, and cosine function are continuous.

- Suppose f and g are real-valued functions such that $(f \circ g)$ is defined at c . If g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c .

Solved Examples

Example 1:

For what values of x is the function $f(x) = \frac{|\sin x|}{4 - \sqrt{x^2 - 9}}$ continuous?

Solution:

Let $g(x) = \sin x$, $h(x) = |x|$

Then, numerator of $f(x) = |\sin x| = h(g(x))$

Since g and h are continuous functions, the numerator of $f(x)$ is also continuous for all real x .

[Functional composition of 2 continuous functions is also continuous]

Now, consider the denominator of $f(x)$, which is $4 - \sqrt{x^2 - 9}$.

Let $g(x) = 4$, $h(x) = x^2 - 9$, and $k(x) = \sqrt{x}$

Functions g and h are continuous for all values of x since both are polynomials.

Function k is continuous for all $x \geq 0$

$$\text{Now, } h(x) = x^2 - 9 = (x + 3)(x - 3)$$

$$\Rightarrow h(x) = 0, \text{ when } x = 3 \text{ or } x = -3$$

$$\therefore h(x) \geq 0 \text{ for } x \geq 3 \text{ and } x \leq -3$$

$$\Rightarrow k(h(x)) = \sqrt{x^2 - 9} \text{ is continuous for } x \geq 3 \text{ and } x \leq -3$$

Thus, the denominator of $f(x) = \frac{|\sin x|}{4 - \sqrt{x^2 - 9}}$ is continuous for $x \geq 3$ and $x \leq -3$.

However, for the function f to be defined, denominator should never be 0.

$$\text{If } 4 - \sqrt{x^2 - 9} = 0,$$

$$\text{then } x^2 - 9 = 16$$

$$\Rightarrow x = \pm 5$$

Thus, denominator is zero, if $x = 5$ or $x = -5$.

$$\therefore f(x) = \frac{|\sin x|}{4 - \sqrt{x^2 - 9}} \quad (x \neq 5, x \neq -5) \text{ is continuous for } x \geq 3 \text{ and } x \leq -3.$$

Example 2:

For what values of x is the function $f(x) = \frac{x^2}{2} + \frac{\sin x}{\cos x}$ continuous?

Solution:

$$\text{Let } h(x) = \frac{x^2}{2}, g(x) = \sin x, k(x) = \cos x$$

Now, h , being a polynomial function, is a continuous function.

g and k are continuous functions.

$$\Rightarrow \left(\frac{g}{k} \right)(x) = \frac{g(x)}{k(x)} = \frac{\sin x}{\cos x} \text{ is continuous, provided } \cos x \neq 0.$$

Now, $\cos x = 0$ for $x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}$

Thus, $\left(\frac{g}{k}\right)$ is not continuous for $x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}$

$\Rightarrow h + \left(\frac{g}{k}\right)$ is not continuous for $x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}$

Thus, $f(x)$ is continuous everywhere except at the points $x = (2n + 1) \frac{\pi}{2}, n \in \mathbf{Z}$

Differentiability of a Function

- The derivative of a real function f at a point c of its domain is defined by

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}.$$

- A real function f is said to be differentiable in an interval $[a, b]$ if it is differentiable at every point of $[a, b]$. Similarly, f is differentiable in interval (a, b) if it is differentiable at every point of (a, b) .
- If a function f is differentiable at a point c , then it is also continuous at that point.
- Every differentiable function is continuous. However, every continuous function may or may not be differentiable.

Solved Examples

Example 1

Check the continuity and differentiability of the function $f(x) = |x|$ at $x = 0$.

Solution:

The given function is $f(x) = |x|$.

At $x = 0$, it can be observed that

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0, \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Therefore, f is continuous at point $x = 0$.

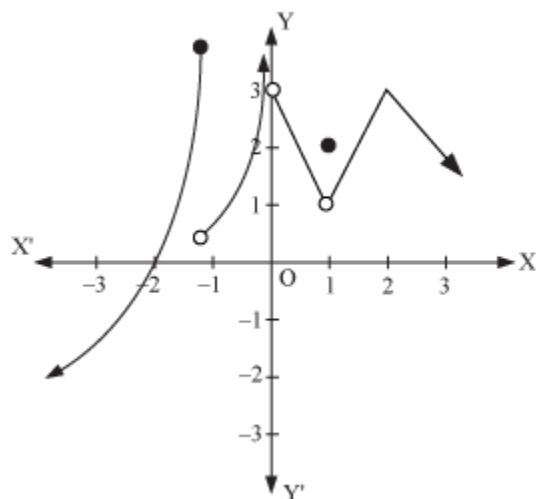
$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$$

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1$$

Therefore, $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ does not exist. Hence, f is not differentiable at $x = 0$.

Example 2

The graph of a function f given as



At what points is the function not necessarily differentiable?

Solution:

Consider the function at $x = -1$. It is seen that $\lim_{x \rightarrow -1^-} f(x) = f(-1) \neq \lim_{x \rightarrow -1^+} f(x)$.

$\therefore f$ is not continuous at $x = -1$. Hence, it is not differentiable at $x = -1$.

Consider the function at $x = 0$. It is seen that f is not defined at $x = 0$.

$\therefore f$ is not continuous. Hence, it is not differentiable at $x = 0$.

Consider the function at $x = 1$. It is seen that $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) \neq f(1)$
 because $\lim_{x \rightarrow 1^-} f(x) = 1$, $\lim_{x \rightarrow 1^+} f(x) = 1$, $f(1) = 2$.

$\therefore f$ is not continuous. Hence, it is not differentiable at $x = 1$.

Thus, the function is not necessarily differentiable at $x = -1$, $x = 0$ and $x = 1$.

Chain Rule to Find the Derivatives

Chain Rule

- Let f be a real-valued function, which is a composite of 2 functions u and v . i.e., $f = v \circ u$.

Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, then

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

The above rule is called chain rule.

Suppose f is a real-valued function, which is a composite of 3 functions u , v , and w i.e., $f = (w \circ v \circ u)$ or $f = w(v(u(x)))$.
 If $t = v(x)$ and $s = u(t)$, then

$$\frac{df}{dx} = \frac{d(w \circ v \circ u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Derivatives of Implicit Functions

- When a relationship between x and y can be expressed in such a way that $y = f(x)$, then y is said to be an explicit function of x .

Example:

$$x^2 + 1 - y = 0$$

$\Rightarrow y = x^2 + 1$, which is an explicit function of x .

- When a function between two variables x and y is represented by an equation such that x and y are neither the subject of the equation, the function is said to be an implicit function of x and y .

For example:

$x^5 + 4x^2y^3 - 3y^4 = -6$ is an implicit function of x and y .

- Implicit function can be differentiated using chain rule.

Consider the equation $x^5 + 4x^2y^3 - 3y^4 = -6$

$$\begin{aligned} & \frac{dy}{dx} \\ \text{We can find } & \frac{dy}{dx} \text{ by differentiating both sides as} \\ \frac{d}{dx}(x^5) + 4 \frac{d}{dx}(x^2y^3) - 3 \frac{d}{dx}(y^4) &= \frac{d}{dx}(-6) \\ \Rightarrow 5x^4 + 4x^2 \frac{d}{dx}(y^3) + 4y^3 \frac{d}{dx}(x^2) - 3(4y^3) \cdot \frac{dy}{dx} &= 0 \quad \text{[Using product rule and chain rule]} \\ \Rightarrow 5x^4 + 4x^2 \cdot 3y^2 \cdot \frac{dy}{dx} + 8xy^3 - 12y^3 \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx}(12x^2y^2 - 12y^3) &= -8xy^3 - 5x^4 \\ \Rightarrow \frac{dy}{dx} &= -\frac{8xy^3 + 5x^4}{12x^2y^2 - 12y^3} \end{aligned}$$

Derivatives of Inverse Trigonometric Functions

- Inverse trigonometric functions are continuous functions and hence, differentiable.
- Inverse trigonometric functions can be differentiated using chain rule.
- Differentiation of various inverse trigonometric functions is as follows:

$f(x)$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	$\sec^{-1} x$	$\operatorname{cosec}^{-1} x$
$f'(x)$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$\frac{-1}{1+x^2}$	$\frac{1}{x\sqrt{x^2-1}}$	$\frac{-1}{x\sqrt{x^2-1}}$
Domain of f'	$(-1, 1)$	$(-1, 1)$	R	R	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1) \cup (1, \infty)$

Solved Examples

Example 1:

Find $\frac{dy}{dx}$ from the equation $y \cos x + x \sin y = 0$

Solution:

Differentiating both sides of the given equation,

$$y \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx} \cdot y + x \frac{d}{dx} \cdot \sin y + \sin y \cdot \frac{d}{dx} \cdot x = 0$$

$$\Rightarrow y(-\sin x) + \cos x \cdot \frac{dy}{dx} + x \cos y \frac{dy}{dx} + \sin y = 0$$

$$\Rightarrow -y \sin x + (\cos x + x \cos y) \frac{dy}{dx} + \sin y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \sin x - \sin y}{\cos x + x \cos y}$$

Example 2:

Differentiate: $y = \left(\sin^{-1}(x^3) \right)^4$

Solution:

Applying chain rule, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\left(\sin^{-1}(x^3) \right)^4 \right] \\ &= 4 \cdot \left(\sin^{-1}(x^3) \right)^3 \cdot \frac{d}{dx} \left(\sin^{-1}(x^3) \right) \\ &= 4 \left(\sin^{-1}(x^3) \right)^3 \cdot \frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx} (x^3) \\ &= 4 \left(\sin^{-1}(x^3) \right)^3 \cdot \frac{3x^2}{\sqrt{1-x^6}} \\ &= \frac{12x^2}{\sqrt{1-x^6}} \cdot \left(\sin^{-1}(x^3) \right)^3 \end{aligned}$$

Example 3:

Find the derivative of $y = \sec(\tan(x^3))$

Solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\sec(\tan(x^3))] \\ &= \sec(\tan(x^3)) \cdot \tan(\tan(x^3)) \cdot \frac{d}{dx} (\tan(x^3)) \\ &= \sec(\tan(x^3)) \cdot \tan(\tan(x^3)) \cdot \sec^2(x^3) \frac{d}{dx} (x^3) \\ &= 3x^2 \sec(\tan(x^3)) \cdot \tan(\tan(x^3)) \cdot \sec^2(x^3)\end{aligned}$$

Exponential Function

- The exponential function with positive base $b > 1$ is the function $y = f(x) = b^x$
- Some properties of exponential functions are:
- The domain of an exponential function is $\mathbb{R}$.
- The range of an exponential function is $\mathbb{R}^+$.
- Point $(0, 1)$ always lies on the graph of exponential function.
- Exponential function is always increasing.
- For very large negative values of x , exponential function tends to 0.
- The exponential function with base 10 is known as common exponential function.
- The number that lies between 2 and 3 and is equal to the sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \dots$ is denoted by e . The exponential function with base e is known as a natural exponential function.
- The derivative of e^x with respect to x is $\frac{d}{dx}(e^x) = e^x$

Solved Examples

Example 1

Find the derivative of $y = \cos 2x \left(e^{x^2-1} \right)$.

Solution:

On applying chain rule, we obtain

$$\frac{dy}{dx} = \cos 2x \frac{d}{dx} \left(e^{x^2-1} \right) + \left(e^{x^2-1} \right) \cdot \frac{d}{dx} (\cos 2x)$$

$$\frac{dy}{dx} = \cos 2x \cdot e^{x^2-1} \cdot 2x + \left(e^{x^2-1} \right) \cdot (-2 \sin 2x)$$

$$\frac{dy}{dx} = 2e^{(x^2-1)} [x \cos 2x - \sin 2x]$$

Example 2

Show that $y = e^{-x} \sin x$ satisfies the equation $\frac{dy}{dx} + y \left(1 - \frac{1}{\tan x} \right) = 0$.

Solution:

$$y = e^{-x} \sin x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(e^{-x} \right) \cdot \sin x + e^{-x} \frac{d}{dx} (\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$$

$$\Rightarrow \frac{dy}{dx} = e^{-x} \cos x - y \quad \dots (1)$$

We have to prove: $\frac{dy}{dx} + y \left(1 - \frac{1}{\tan x} \right) = 0$

Consider L.H.S. as $\frac{dy}{dx} + y \left(1 - \frac{1}{\tan x} \right)$.

$$\frac{dy}{dx} + y - \frac{y}{\tan x}$$

On using (1), we obtain

$$\begin{aligned}
& e^{-x} \cos x - \frac{y}{\tan x} \\
&= e^{-x} \cos x - \frac{e^{-x} \sin x}{\tan x} \\
&= e^{-x} \cos x - e^{-x} \cos x \\
&= 0 \\
&= \text{R.H.S.}
\end{aligned}$$

Thus, the given result is proved.

Logarithmic Functions

Logarithmic Functions

- Let $b > 1$ be a real number. Then, the logarithm of a to base b is x if $b^x = a$. It is denoted by $\log_b a$ i.e., $\log_b a = x$ if $b^x = a$.
- In other words, if $b > 1$, then logarithmic functions are defined from positive real numbers to all real numbers such that

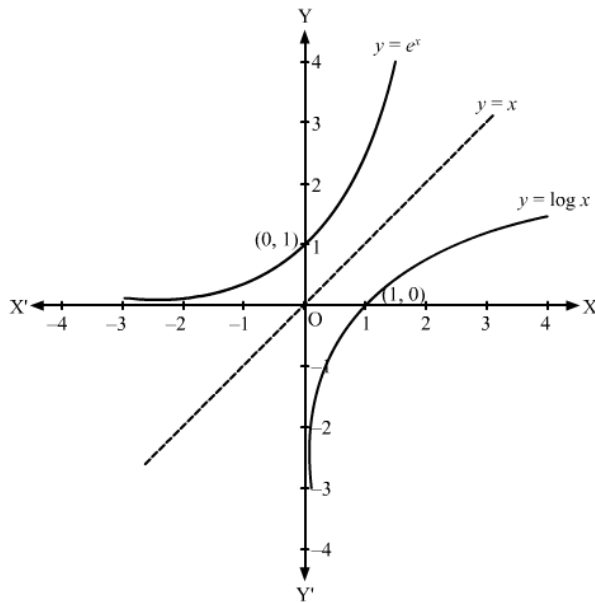
$$\begin{aligned}
& \log_b : \mathbf{R}^+ \rightarrow \mathbf{R} \\
& x \rightarrow \log_b x = y \text{ if } b^y = x
\end{aligned}$$

- Based on their bases, logarithmic functions are of two types.

Base of logarithmic function	Name
Base 10	Common logarithmic
Base e	Natural logarithmic

- Some observations about logarithmic functions are:
- Domain of log function is $\mathbf{R}^+$.

- Range of log function is $\mathbf{R}$.
- Point (1, 0) always lies on the graph of log function.
- Log function is ever increasing.
- For x very near to zero, the value of $\log x$ can be made lesser than any given real number i.e., when x approaches 0, $\log x$ approaches the y -axis.
- The graphs of $y = e^x$ and $y = \log x$ are mirror images of each other, reflected in the line $y = x$.



- Some properties of log functions are:

$$\log_a p = \frac{\log_b p}{\log_b a}$$

- $\log_b pq = \log_b p + \log_b q$
- $\log_b p^2 = \log_b p + \log_b p = 2 \log_b p$
- $\log_b p^n = n \log_b p$

$$\log_b \left(\frac{x}{y} \right) = \log_b (x) - \log_b (y)$$

- $x = e^{\log x}$ for all positive values of x .

- The derivative of $\log x$ with respect to x is $\frac{1}{x}$ i.e., $\frac{d}{dx}(\log x) = \frac{1}{x}$

Solved Examples

Example 1

Find the derivative of $y = \log (\cos^3 2x)$.

Solution:

On using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\log (\cos^3 (2x)) \right) \\ \frac{dy}{dx} &= \frac{1}{\cos^3 (2x)} \cdot \frac{d}{dx} (\cos^3 (2x)) \\ \frac{dy}{dx} &= \frac{1}{\cos^3 (2x)} \cdot 3 \cdot \cos^2 (2x) (-\sin (2x)) 2 \\ \frac{dy}{dx} &= -6 \tan 2x\end{aligned}$$

Example 2

Find the derivative of $y = \log (\log (\log x))$.

Solution:

$$y = \log (\log (\log x))$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\log (\log x)} \cdot \frac{d}{dx} (\log (\log x)) \\ \frac{dy}{dx} &= \frac{1}{\log (\log x)} \cdot \frac{1}{\log x} \cdot \frac{d}{dx} (\log x) \\ \frac{dy}{dx} &= \frac{1}{\log (\log x)} \cdot \frac{1}{\log x} \cdot \frac{1}{x} \\ \therefore \frac{dy}{dx} &= \frac{1}{x \cdot \log x \log (\log x)}\end{aligned}$$

Example 3

If $y = (\log x)^2$, then prove that $x^2 (y')^2 - 4y = 0$.

Solution:

$$\begin{aligned}y &= (\log x)^2 \\ \Rightarrow \frac{dy}{dx} &= 2 \log x \cdot \frac{d}{dx} (\log x) = 2 \log x \cdot \frac{1}{x} = \frac{2}{x} \log x\end{aligned}$$

We have to prove that $x^2(y') - 4y = 0$

Consider L.H.S.

$$x^2(2x \log x)^2 - 4(\log x)^2 x^2 = 2x^2 \log x^2 - 4 \log x^2 \\ \Rightarrow 4(\log x)^2 - 4(\log x)^2 = 0 = \text{R.H.S.}$$

Thus, the given result is proved.

Logarithmic Differentiation

- If $y = f(x) = [u(x)]^{v(x)}$, then by taking logarithm (to base e) on both sides and then differentiating using chain rule, we obtain

$$\log y = v(x) \log u(x) \\ \frac{1}{y} \frac{dy}{dx} = v(x) \frac{1}{u(x)} u'(x) + v'(x) \log u(x) \\ \frac{dy}{dx} = y \left[\frac{v(x)}{u(x)} \cdot u'(x) + v'(x) \log u(x) \right] \text{ provided } f(x) > 0 \text{ and } u(x) > 0$$

This process is called logarithms differentiation.

- If $y = a^x$, where $a > 0$, then $\frac{dy}{dx} = a^x \log a$

Solved Examples

Example 1:

If $y = 10^{x^2}$, find $\frac{dy}{dx}$.

Solution:

Taking logarithm on both sides, we obtain

$$\log y = x^2 \log 10$$

Differentiating on both sides with respect to x ,

$$\frac{1}{y} \frac{dy}{dx} = 2x \log 10$$

$$\frac{dy}{dx} = 2(\log 10)xy = 2(\log 10)x10^{x^2}$$

Example 2:

Find the derivative of y with respect to x , if $y = (\sin x)^x$

Solution:

Taking logarithm on both sides, we obtain

$$\begin{aligned} \log y &= \log (\sin x)^x \\ &= x \log (\sin x) \quad \left[\log (a^n) = n \log a \right] \end{aligned}$$

Differentiating both sides with respect to x ,

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{1}{\sin x} \cdot \cos x + \log (\sin x) \\ \Rightarrow \frac{dy}{dx} &= (\sin x)^x [\log (\sin x) + x \cot x] \end{aligned}$$

Example 3:

Find the derivative y' of the function y given by $y = \sqrt{\frac{(x-2)(x+4)}{(x+1)(x+5)}}$

Solution:

Taking logarithms on both sides, we obtain

$$\log y = \frac{1}{2} \log \left[\frac{(x-2)(x+4)}{(x+1)(x+5)} \right]$$

Now, we know that $\log \frac{a}{b} = \log a - \log b$ and $\log ab = \log a + \log b$

$$\therefore \log y = \frac{1}{2} [\log (x-2) + \log (x+4) - \log (x+1) - \log (x+5)]$$

Differentiating both sides with respect to x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x-2} + \frac{1}{x+4} - \frac{1}{x+1} - \frac{1}{x+5} \right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x-2} + \frac{1}{x+4} - \frac{1}{x+1} - \frac{1}{x+5} \right] \sqrt{\frac{(x-2)(x+4)}{(x+1)(x+5)}}$$

Derivatives of Functions in Parametric Form

Differentiation of Functions in Parametric Form

- Parametric equations are of the form $x = f(t)$ and $y = g(t)$, where t is called a parameter. These equations are used for establishing a relationship between two variables with the help of a third variable.
- The functions in parametric form can be differentiated using chain rule.
- If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} \quad \left[\frac{dx}{dt} \neq 0 \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{g'(t)}{f'(t)} \quad [f'(t) \neq 0]$$

Solved Examples

Example 1

Find $\frac{dy}{dx}$ in terms of the parameter $x = \frac{1}{s}, y = \frac{s+1}{s-1}$.

Solution:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{ds}\right)}{\left(\frac{dx}{ds}\right)}$$

$$\frac{dy}{ds} = \frac{d}{ds} \left(\frac{s+1}{s-1} \right) = \frac{(s-1)(1) - (s+1)(1)}{(s-1)^2} = \frac{s-1-s-1}{(s-1)^2} = -\frac{2}{(s-1)^2}$$

$$\frac{dx}{ds} = -\frac{1}{s^2}$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{2}{(s-1)^2}}{-\frac{1}{s^2}} = \frac{2s^2}{(s-1)^2}$$

$$\therefore \frac{dy}{dx} = 2 \left(\frac{s}{s-1} \right)^2$$

Example 2

What is the value of $\frac{dy}{dx}$ in terms of x and y if $y = e^u - e^{-u}$ and $x = e^u + e^{-u}$?

Solution:

$$\frac{dy}{du} = e^u - (-e^{-u}) = e^u + e^{-u}$$

$$\frac{dx}{du} = e^u + (-e^{-u}) = e^u - e^{-u}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{du}\right)}{\left(\frac{dx}{du}\right)} = \frac{e^u + e^{-u}}{e^u - e^{-u}} = \frac{x}{y}$$

$$\therefore \frac{dy}{dx} = \frac{x}{y}$$

Example 3

What is the value of $\frac{dy}{dx}$ at $\alpha = 60^\circ$ if $x = \cos \alpha$ and $y = \sin 2\alpha$?

Solution:

$$\frac{dy}{d\alpha} = 2 \cos 2\alpha$$

$$\frac{dx}{d\alpha} = -\sin \alpha$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 \cos 2\alpha}{-\sin \alpha}$$

$$\text{At } \alpha = 60^\circ, \frac{dy}{dx} = \frac{2 \cos 120^\circ}{-\sin 60^\circ} = -\frac{2 \cos 60^\circ}{-\frac{\sqrt{3}}{2}} = \frac{-2 \times \frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

Second Order Derivatives

Second Order Derivatives

- If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$

If $f'(x)$ is a differentiable function of x , then the derivative of $f'(x)$ exists and is called the second order derivative of $y = f(x)$ with respect to x .

- Notation of the second order derivatives depends on the notations of the original functions and first order derivative.

Original	First order derivative	Second order derivative
y	y'	y''
	Dy	D^2y
	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$

	y_1	y_2
$f(x)$	$f'(x)$	$f''(x)$
	$\frac{df}{dx}$	$\frac{d^2 f}{dx^2}$

Solved Examples

Example 1:

If $y = \frac{a}{x} + b$, then prove that $\frac{d^2 y}{dx^2} + \frac{2}{x} \left(\frac{dy}{dx} \right) = 0$

Solution:

$$y = \frac{a}{x} + b$$

$$\Rightarrow \frac{dy}{dx} = -\frac{a}{x^2}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{2a}{x^3}$$

To prove: $\frac{d^2 y}{dx^2} + \frac{2}{x} \left(\frac{dy}{dx} \right) = 0$

$$\text{L.H.S.} = \frac{2a}{x^3} + \frac{2}{x} \left(\frac{-a}{x^2} \right) = \frac{2a}{x^3} - \frac{2a}{x^3} = 0 = \text{R.H.S.}$$

Hence proved.

Example 2:

Find the second order derivative of $y = \cos^{-1}(x^2)$

Solution:

$$y = \cos^{-1}(x^2)$$

$$\frac{dy}{dx} = \frac{-1(2x)}{\sqrt{1-x^4}} = \frac{-2x}{\sqrt{1-x^4}}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\sqrt{1-x^4}(-2) - (-2x)\left(-\frac{1}{2}\right)\frac{(-4x^3)}{(1-x^4)^{\frac{3}{2}}}}{(1-x^4)} = \frac{-2\sqrt{1-x^4} + \frac{4x^4}{(1-x^4)^{\frac{3}{2}}}}{(1-x^4)} \\ &= \frac{-2(1-x^4)^{\frac{1}{2}} + 4x^4}{(1-x^4)^{\frac{5}{2}}} = \frac{(-2 - 2x^8 + 4x^4) + 4x^4}{(1-x^4)^{\frac{5}{2}}} = \frac{-2(1+x^8-4x^4)}{(1-x^4)^{\frac{5}{2}}} \end{aligned}$$

Rolle's Theorem

- Let $f: [a, b] \rightarrow \mathbf{R}$ be continuous in $[a, b]$ and differentiable in (a, b) such that $f(a) = f(b)$, where a and b are some real numbers.

Then, there exists some c in (a, b) such that $f'(c) = 0$.

- The tangent to graph of f where the slope becomes zero at any point on the graph of graph $y = f(x)$ is the claim of Rolle's theorem.
- Example:

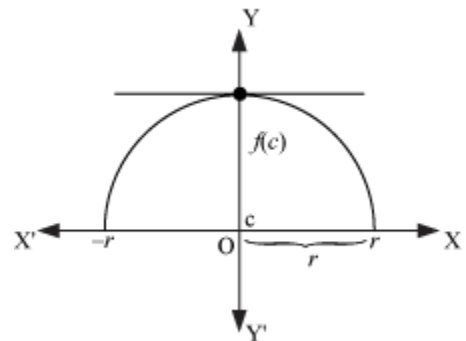
$$f(x) = \sqrt{r^2 - x^2}, r > 0, x \in [-r, r]$$

$f(x)$ is continuous in $[-r, r]$ and differentiable in $(-r, r)$.

$$\text{Also, } f(-r) = f(r)$$

$\therefore$ Rolle's theorem is applicable for $c = 0$.

$$\text{Here, } f'(c) = f'(0) = 0$$

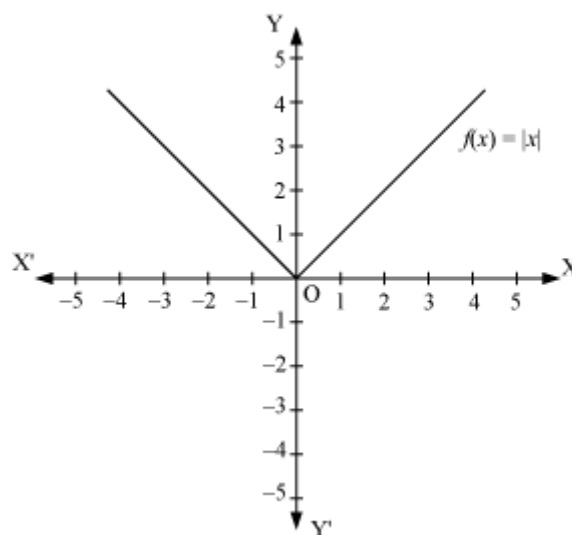


$$f(x) = |x|, x \in [-1, 1]$$

f is continuous in $[-1, 1]$ and $f(-1) = f(1)$.

However, f is not differentiable in $(-1, 1)$ as it is not differentiable at $x = 0$.

$\therefore$ Rolle's theorem is not applicable.



Solved Examples

Example 1

Verify Rolle's theorem for the function $f(x) = x^2 - 5x + 4$ in the interval $[1, 4]$.

Solution:

The given function is $f(x) = x^2 - 5x + 4$.

(i) $f(x)$ is continuous in $[1, 4]$ as f is a polynomial function.

(ii) $f(x)$ is differentiable in $(1, 4)$ as f is a polynomial function.

(iii) $f(1) = 1^2 - 5(1) + 4 = 0$, $f(4) = 4^2 - 5(4) + 4 = 0$

$\therefore f(1) = f(4)$

$\therefore$ The hypothesis of Rolle's theorem is satisfied.

Thus, there exists $c \in (1, 4)$ such that $f'(c) = 0$.

$$f'(x) = 2x - 5$$

$$f'(c) = 0$$

$$\Rightarrow 2c - 5 = 0$$

$$\Rightarrow c = \frac{5}{2} \in (1, 4)$$

Thus, Rolle's theorem is verified.

Example 2

Verify Rolle's theorem for the function $f(x) = 1 + \sin^2 x$ in the interval $[0, \pi]$.

Solution:

The given function is $f(x) = 1 + \sin^2 x$, as f is a trigonometric function.

(i) f is continuous in $[0, \pi]$ as f is a trigonometric function.

(ii) f is differentiable in $(0, \pi)$ as f is a trigonometric function.

(iii) $f(0) = 1 + \sin^2 0 = 1, f(\pi) = 1 + \sin^2 \pi = 1$

$$\therefore f(0) = f(\pi)$$

$\therefore$ The hypothesis of Rolle's theorem is satisfied.

Thus, there exists $c \in (0, \pi)$ such that $f'(c) = 0$.

$$f'(c) = 0$$

$$\Rightarrow 2 \sin c \cos c = 0$$

$$[f'(x) = 2 \sin x \cos x]$$

$$\Rightarrow \sin 2c = 0$$

$$\Rightarrow 2c = n\pi, n \in \mathbf{Z}$$

$$\Rightarrow c = \frac{n\pi}{2}, n \in \mathbf{Z}$$

$$\text{For } n = 1, c = \frac{\pi}{2} \in (0, \pi)$$

$$\therefore f'(c) = 0 \text{ for } c = \frac{\pi}{2} \in (0, \pi)$$

Thus, Rolle's theorem is verified.

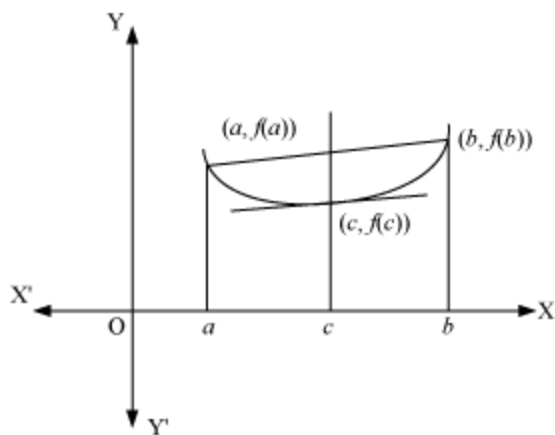
Mean Value Theorem

- Let $f: [a, b] \rightarrow \mathbf{R}$ be a continuous function in $[a, b]$ and differentiable in (a, b) . Then, there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

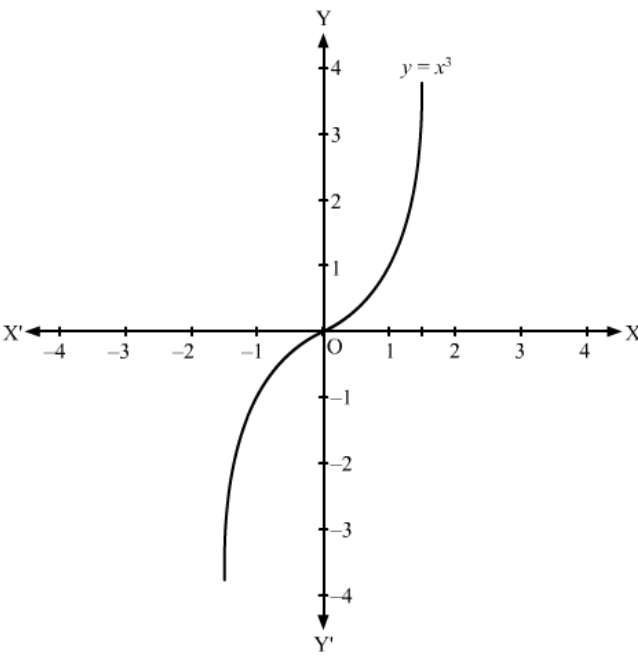
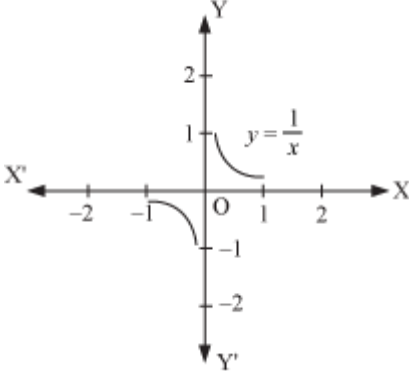
- Some observations about mean value theorem:
- Mean Value Theorem (MVT) is an extension of Rolle's theorem.
- $f'(c)$ is the slope of the tangent to the curve $y = f(x)$ at point $(c, f(c))$.

- $\frac{f(b) - f(a)}{b - a}$ is the slope of the secant drawn between $(a, f(a))$ and $(b, f(b))$. This can be diagrammatically represented as



- Mean Value Theorem can be described by an example as

Function	Graph

Consider $f(x) = x^3, x \in [3, 6]$. f is continuous in $[3, 6]$ and differentiable in $(3, 6)$. Thus, there exists $c \in (3, 6)$ such $f'(c) = \frac{f(b) - f(a)}{b - a}, \quad b = 6, \quad a = 3$ that $f'(x) = 3x^2$ $f'(c) = \frac{f(b) - f(a)}{b - a}$ $\Rightarrow 3c^2 = \frac{6^3 - 3^3}{3}$ $\Rightarrow c^2 = \frac{189}{9} = 21$ $\Rightarrow c = \sqrt{21} \in (3, 6)$	
Consider $f(x) = \frac{1}{x}, x \in [-1, 1]$. The function f does not exist at $x = 0$. Thus, it is not continuous in $[-1, 1]$. Hence, the hypothesis of mean value theorem is not satisfied. Thus, mean value theorem cannot be applied to the given function.	

Solved Examples

Example 1

Discuss the applicability of mean value theorem for the function $f(x) = |\sin x|$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution:

$$f(x) = |\sin x|, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{Let } h(x) = \sin x, g(x) = |x|$$

$$(g \circ h)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)$$

Since h and g are continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, f is continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Now, $h(x) = \sin x$ is differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

However, $g(x) = |x|$ is not differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Thus, $f(x)$ is not differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore$ The hypothesis of mean value theorem is not satisfied.

Thus, mean value theorem is not applicable for the given function.

Example 2

Use the mean value theorem to prove that for any two real numbers a and b $|\cos a - \cos b| \leq |a - b|$.

Solution:

Let $f(x) = \cos x$ in $[a, b]$

It is clear that $f(x) = \cos x$ is continuous in $[a, b]$ and differentiable in (a, b) .

Therefore, the hypothesis of mean value theorem is satisfied.

Thus, by mean value theorem, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

$$\begin{aligned}
 f'(c) &= \frac{f(b) - f(a)}{b - a} \\
 \Rightarrow -\sin c &= \frac{\cos b - \cos a}{b - a} & [f'(x) = -\sin x] \\
 \Rightarrow |-\sin c| &= \left| \frac{\cos b - \cos a}{b - a} \right|
 \end{aligned}$$

We know that $|-\sin c| \leq 1$.

$$\begin{aligned}
 \therefore \left| \frac{\cos b - \cos a}{b - a} \right| &\leq 1 \\
 |\cos b - \cos a| &\leq |b - a| \\
 \text{or, } |\cos a - \cos b| &\leq |a - b|
 \end{aligned}$$

Hence proved.