

Vector Algebra

Vector and Its Related Concepts

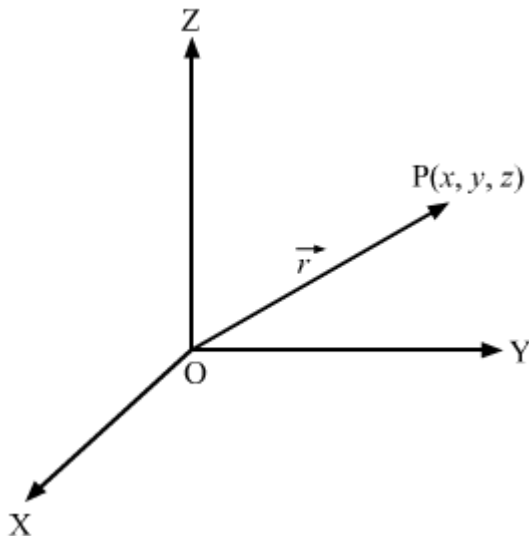
Vector

- The quantity that involves only magnitude (a value) is called a scalar quantity.
Example: Length, mass, time, distance, etc.
- The quantity that involves both magnitude and direction is called a vector.
Example: Acceleration, momentum, force, etc.
- Vector is represented as a directed line segment (line segment whose direction is given by means of an arrowhead).
- In the following figure, line segment AB is directed towards B.



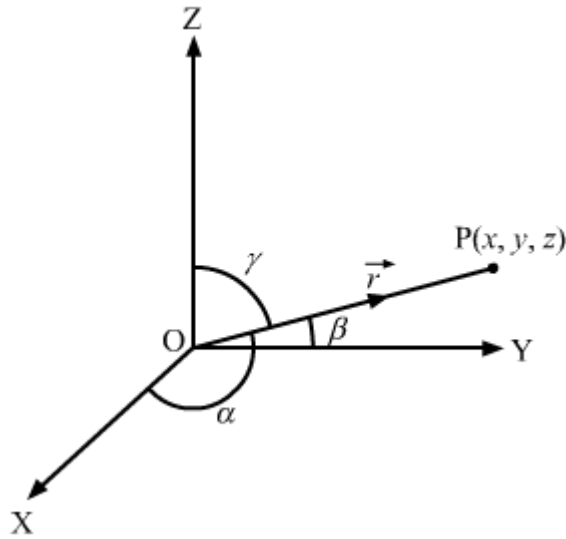
Hence, the vector representing directed line segment AB is $\overrightarrow{AB}$ or simply $\vec{a}$. Here, the arrow indicates the direction of AB. In $\overrightarrow{AB}$, A is called the initial point and B is called the terminal point.

- The position vector of a point P in space having coordinates (x, y, z) with respect to origin O $(0, 0, 0)$ is given by $\overrightarrow{OP}$ or $\vec{r}$.



- Here, the magnitude of $\vec{r}$ i.e., $|\vec{r}|$ is given by $\sqrt{x^2 + y^2 + z^2}$.

- If a position vector $\vec{r}$ of point P (x, y, z) makes angles α, β , and γ with the positive directions of x-axis, y-axis and z-axis respectively, then these angles are called direction angles.

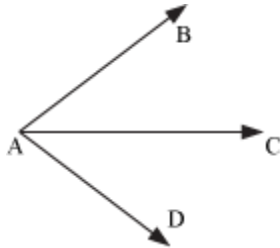


- The cosine values of direction angles are called direction cosines of $\vec{r}$. This means that direction cosines (d.c.s.) of $\vec{r}$ are $\cos \alpha, \cos \beta$, and $\cos \gamma$. We may write the d.c.s of $\vec{r}$ as l, m, n where $l = \cos \alpha, m = \cos \beta$ and $n = \cos \gamma$.
- The direction ratios of $\vec{r}$ will be lr, mr , and nr . We may write the direction ratios (d.r.s.) of $\vec{r}$ as a, b, c , where $a = lr, b = mr$ and $c = nr$.
- If l, m, n are the d.c.s. of a position vector $\vec{r}$, then $l^2 + m^2 + n^2 = 1$

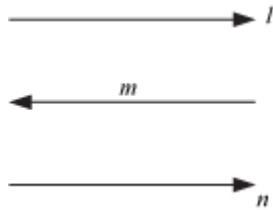
Types of Vectors

- A vector whose initial and terminal points coincide is called a zero vector or a null vector.
- It is represented as $\vec{0}$.
- A zero vector cannot be assigned in a definite direction since its magnitude is zero or it may be regarded as having any direction.
- The vector $\overrightarrow{PP}, \overrightarrow{AA}$, etc. represents a zero vector.
- A vector whose magnitude is unity or 1 unit is called a unit vector.
- A unit vector in the direction of a position vector $\vec{r}$ is given as $\hat{r}$.

- Two or more vectors having the same initial point are called co-initial vectors.
- In the following figure, vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are called initial vectors as each vector has the same initial point i.e., A.



- Two or more vectors are said to be collinear if they are parallel to the same line irrespective of their magnitudes and directions.
- In the following figure, $\vec{l}$, $\vec{m}$ and $\vec{n}$ are collinear vectors.



- Two vectors are said to be equal if they have the same magnitude and direction regardless of the positions of their initial points.
- For two equal vectors $\vec{a}$ and $\vec{b}$, we write $\vec{a} = \vec{b}$.
- A vector whose magnitude is the same as that of a given vector but whose direction is opposite to that of the given vector is called the negative of the given vector.
- The negative vector of $\overrightarrow{PQ}$ is $\overrightarrow{QP}$ and it is written as $\overrightarrow{PQ} = -\overrightarrow{QP}$.

Solved Examples

Example 1

Find the direction cosines and direction ratios of the position vector of point P(8, -4, 1).

Solution:

Let O be the origin. The position vector of point P(8, -4, 1) with respect to origin will be $\overrightarrow{OP}$.

$$\therefore r = \sqrt{x^2 + y^2 + z^2} = \sqrt{(8)^2 + (-4)^2 + (1)^2} = 9$$

The direction cosines of $\overrightarrow{OP}$ are

$$l, m, n = \frac{x}{r}, \frac{y}{r}, \frac{z}{r} = \frac{8}{9}, \frac{-4}{9}, \frac{1}{9}$$

The direction ratios of $\vec{r}$ are

$$lr, mr, nr = \frac{8}{9} \times 9, -\frac{4}{9} \times 9, \frac{1}{9} \times 9 = 8, -4, 1$$

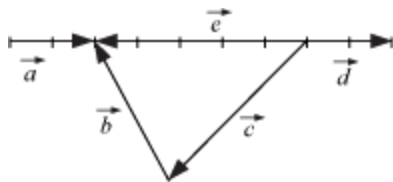
Example 2

In the following figure, which of the vectors are

(i) Collinear

(ii) Equal

(iii) Co-initial



Solution:

(i) Collinear vectors: $\vec{a}$, $\vec{e}$ and $\vec{d}$

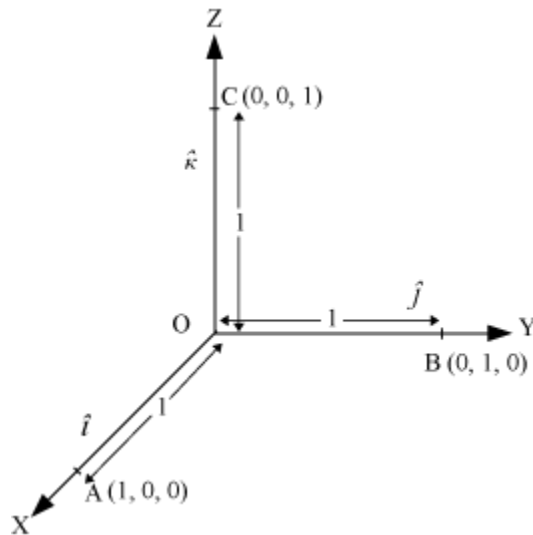
(ii) Equal vectors: $\vec{a}$ and $\vec{d}$

(iii) Co-initial vectors, $\vec{b}$, $\vec{c}$ and $\vec{e}$.

Component Form of a Vector

Key Concept0073

- In a space, the unit vectors along the x-axis, y-axis and z-axis are denoted by $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively.



It can be noted that $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$

- The position vector ($\vec{r}$) of any point (x, y, z) in a space is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
- The scalar components of $\vec{r}$ are x, y and z .
- The vector components of $\vec{r}$ are $x\hat{i}, y\hat{j}$ and $z\hat{k}$, which are along the x, y and z -axes respectively.
- If the position vector ($\vec{r}$) of a point is $x\hat{i} + y\hat{j} + z\hat{k}$, then $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$
- Two vectors $\vec{a}$ and $\vec{b}$ given by $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are equal if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$.

Solved Examples

Example 1

Find the value of x if $5|\vec{a}| = 3|\vec{b}|$, where $\vec{a} = 7\hat{i} + (x-2)\hat{j} + 4\hat{k}$ and $\vec{b} = 9\hat{i} - 12\hat{j} + (x+2)\hat{k}$.

Solution:

We have

$$\vec{a} = 7\hat{i} + (x-2)\hat{j} + 4\hat{k}$$

$$\vec{b} = 9\hat{i} - 12\hat{j} + (x+2)\hat{k}$$

$$\therefore |\vec{a}| = \sqrt{(7)^2 + (x-2)^2 + (4)^2} = \sqrt{x^2 - 4x + 69}, \quad |\vec{b}| = \sqrt{(9)^2 + (-12)^2 + (x+2)^2} = \sqrt{x^2 + 4x + 229}$$

It is given that

$$5|\vec{a}| = 3|\vec{b}|$$

$$\Rightarrow 25|\vec{a}|^2 = 9|\vec{b}|^2$$

$$\Rightarrow 25(x^2 - 4x + 69) = 9(x^2 + 4x + 229)$$

$$\Rightarrow 25x^2 - 100x + 1725 = 9x^2 + 36x + 2061$$

$$\Rightarrow 16x^2 - 136x - 336 = 0$$

$$\Rightarrow 2x^2 - 17x - 42 = 0$$

$$\Rightarrow 2x^2 + 4x - 21x - 42 = 0$$

$$\Rightarrow 2x(x+2) - 21(x+2) = 0$$

$$\Rightarrow (x+2)(2x-21) = 0$$

$$\Rightarrow (x+2) = 0 \text{ or } (2x-21) = 0$$

$$\Rightarrow x = -2 \text{ or } x = \frac{21}{2}$$

Thus, the required value of x is -2 or $\frac{21}{2}$.

Example 2

Find the values of x, y and z if $\vec{a} = \vec{b}$, where $\vec{a} = (x+y)\hat{i} + (7-2x)\hat{j} + (z-2)\hat{k}$
and $\vec{b} = (2y-1)\hat{i} + (2-y)\hat{j} + 2\hat{k}$.

Solution:

It is given that

$$\vec{a} = (x+y)\hat{i} + (7-2x)\hat{j} + (z-2)\hat{k}$$

$$\vec{b} = (2y-1)\hat{i} + (2-y)\hat{j} + 2\hat{k}$$

$$\vec{a} = \vec{b}$$

$$\Rightarrow x+y = 2y-1, 7-2x = 2-y \text{ and } z-2 = 2$$

$$\Rightarrow x-y+1 = 0 \dots (1)$$

$$2x-y-5 = 0 \dots (2)$$

$$z = 4 \dots (3)$$

On subtracting (2) from (1), we obtain

$$-x + 6 = 0$$

$$\Rightarrow x = 6$$

On substituting $x = 6$ in equation (1), we obtain

$$6 - y + 1 = 0$$

$$\Rightarrow y = 7$$

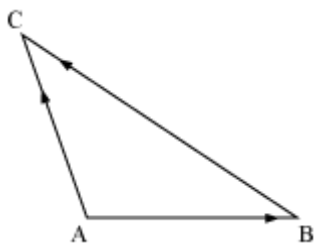
Hence, $x = 6, y = 7$, and $z = 4$.

Addition and Difference of vectors

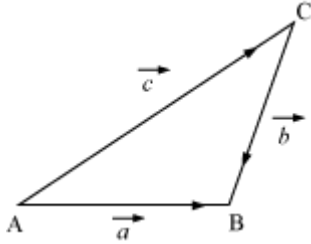
Triangle Law and Parallelogram Law of Vector Addition

- Let points A, B, C form a triangle. If one person goes from A to B (represented by vector $\overrightarrow{AB}$) and B to C (represented by $\overrightarrow{BC}$), then the net displacement of the person from A to C ($\overrightarrow{AC}$) is given by $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

This is known as the triangle law of vector addition.



- To add two vectors $\vec{a}$ and $\vec{b}$, they are positioned in such a manner that the initial point of one coincides with the terminal point of the other.
- In the following figure, the initial point of $\vec{b}$ and the final point of $\vec{c}$ coincide at C.



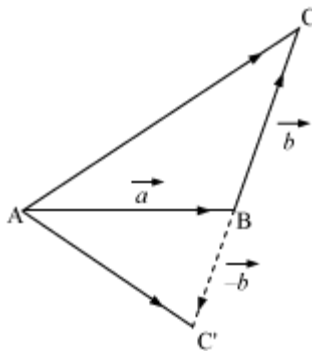
Hence, we write: $\vec{c} + \vec{b} = \vec{a}$

- In the following figure,

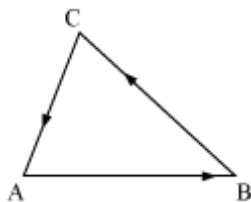
$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$$

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$$

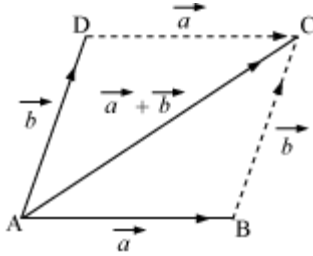
Here, the vector $\overrightarrow{AC'}$ is said to represent the difference between vectors $\vec{a}$ and $\vec{b}$.



- When the sides of a triangle are taken in order, then their resultant vector is $\vec{0}$ because the initial and terminal points coincide.
- In $\triangle ABC$, if $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ are in the same order, then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$



- If two vectors $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram in magnitude and direction, then their sum i.e., $\vec{a} + \vec{b}$ represents the magnitude and direction of the vector through their common point. This is known as the parallelogram law of vector addition.



Addition and Difference of Vectors

If $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$, then

$$\vec{a} + \vec{b} = (a_1 + b_1) \hat{i} + (a_2 + b_2) \hat{j} + (a_3 + b_3) \hat{k}$$

$$\vec{a} - \vec{b} = (a_1 - b_1) \hat{i} + (a_2 - b_2) \hat{j} + (a_3 - b_3) \hat{k}$$

Here, $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are called the sum and difference of vectors $\vec{a}$ and $\vec{b}$ respectively.

- Properties of vector addition
- Commutative Law: For any two vectors $\vec{a}$ and $\vec{b}$,
 $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associative Law: For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$
 $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- The existence of additive identity: For any vector $\vec{a}$, we find $-\vec{a}$ (negative of vector $\vec{a}$) such that
 $\vec{a} + (-\vec{a}) = 0$
Here, $(-\vec{a})$ is called the additive inverse of $\vec{a}$.

Solved Examples

Example 1

If $\vec{a} = 9\hat{i} - 3\hat{j} - 3\hat{k}$ and $\vec{b} = -\hat{i} + 7\hat{j} + 2\hat{k}$, then find the value of $\|\vec{a} - \vec{b}\| - 2\|\vec{a} + \vec{b}\|$.

Solution:

We have, $\vec{a} = 9\hat{i} - 3\hat{j} - 3\hat{k}$ and $\vec{b} = -\hat{i} + 7\hat{j} + 2\hat{k}$.
 $\therefore \vec{a} + \vec{b} = [9 + (-1)]\hat{i} + [(-3) + 7]\hat{j} + [(-3) + (2)]\hat{k}$
 $= 8\hat{i} + 4\hat{j} - \hat{k}$

And, $\vec{a} - \vec{b} = [9 - (-1)]\hat{i} + [(-3) - 7]\hat{j} + [(-3) - 2]\hat{k}$
 $= 10\hat{i} - 10\hat{j} - 5\hat{k}$

Hence, $|\vec{a} + \vec{b}| = \sqrt{(8)^2 + (4)^2 + (-1)^2} = 9$

And, $|\vec{a} - \vec{b}| = \sqrt{(10)^2 + (-10)^2 + (-5)^2} = 15$

Hence, $\left| |\vec{a} - \vec{b}| - 2|\vec{a} + \vec{b}| \right| = |15 - 2 \times 9|$
 $= |15 - 18|$
 $= |-3|$
 $= 3$

Example 2

Find $x + 5y$ if $|\vec{a} + \vec{b}| = 6\sqrt{10}$ and $\vec{a} = x\hat{i} + 3x\hat{j} + \sqrt{xy}\hat{k}$ and $\vec{b} = 9y\hat{i} + 13y\hat{j} + \sqrt{xy}\hat{k}$ such that both x and y are positive integers.

Solution:

We have:

$$\vec{a} = x\hat{i} + 3x\hat{j} + \sqrt{xy}\hat{k}$$

$$\text{and } \vec{b} = 9y\hat{i} + 13y\hat{j} + \sqrt{xy}\hat{k}$$

$$\text{Hence, } \vec{a} + \vec{b} = (x + 9y)\hat{i} + (3x + 13y)\hat{j} + 2\sqrt{xy}\hat{k}$$

$$\begin{aligned} \therefore |\vec{a} + \vec{b}| &= \sqrt{(x + 9y)^2 + (3x + 13y)^2 + (2\sqrt{xy})^2} \\ &= \sqrt{(x^2 + 81y^2 + 18xy) + (9x^2 + 169y^2 + 78xy) + 4xy} \\ &= \sqrt{10(x^2 + 25y^2 + 10xy)} \\ &= \sqrt{10}\sqrt{(x + 5y)^2} \\ &= \sqrt{10}(x + 5y) \end{aligned}$$

It is given that

$$\begin{aligned} |\vec{a} + \vec{b}| &= 6\sqrt{10} \\ \Rightarrow \sqrt{10}(x + 5y) &= 6\sqrt{10} \\ \Rightarrow x + 5y &= 6 \end{aligned}$$

Multiplication of a Vector with a Scalar

Key Concepts:

- If λ is any scalar and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is any vector, then the multiplication of scalar λ with this vector $\vec{a}$, denoted by $\lambda\vec{a}$, is given by:

$$\lambda\vec{a} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

- For example: $5(2\hat{i} - \hat{j} + 4\hat{k}) = (5 \times 2)\hat{i} - 5\hat{j} + (5 \times 4)\hat{k} = 10\hat{i} - 5\hat{j} + 20\hat{k}$

- If $\vec{a}$ is any vector and λ is any scalar, then $|\lambda\vec{a}| = |\lambda||\vec{a}|$.

- The unit vector in the direction of vector $\vec{a}$ is denoted by $\hat{a}$ and it is given by: $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$

- If $\vec{a} = 9\hat{i} - 8\hat{j} - 12\hat{k}$, then $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$

$$= \frac{1}{\sqrt{(9)^2 + (-8)^2 + (-12)^2}}(9\hat{i} - 8\hat{j} - 12\hat{k})$$

$$= \frac{1}{17}(9\hat{i} - 8\hat{j} - 12\hat{k})$$

$$= \frac{9}{17}\hat{i} - \frac{8}{17}\hat{j} - \frac{12}{17}\hat{k}$$

- Two vectors $\vec{a}$ and $\vec{b}$ are said to be collinear vectors, if there exists a scalar λ such that $\vec{b} = \lambda\vec{a}$

- In this case, if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}$ and $\vec{b}$ are collinear,
provided $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then a_1, a_2, a_3 are called direction ratios of $\vec{a}$.

- $\frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|}$ are called direction cosines of $\vec{a}$.

- Some properties of multiplication of a scalar with a vector:
If $\vec{a}$ and $\vec{b}$ are any two vectors and k and m are any scalars, then

- $(k+m)\vec{a} = k\vec{a} + m\vec{a}$

- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

- $k(m\vec{a}) = (km)\vec{a}$

Solved Examples:

Example 1:

If $\vec{a} = 5\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - \hat{k}$, then find the value of λ , such that

$$\lambda \left| \frac{(2\vec{a} + 3\vec{b})}{|\vec{a} + \vec{b}|} \right| + \mu |2\sqrt{35}\hat{a} + \sqrt{6}\hat{b}| = 4\sqrt{2} + 3\sqrt{10}$$

Solution:

$$\vec{a} = 5\hat{i} - 3\hat{j} + \hat{k}$$

$$\vec{b} = -2\hat{i} + \hat{j} - \hat{k}$$

$$\therefore |\vec{a}| = \sqrt{(5)^2 + (-3)^2 + (1)^2} = \sqrt{35}$$

$$|\vec{b}| = \sqrt{(-2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$$

Now,

$$2\vec{a} + 3\vec{b} = 2(5\hat{i} - 3\hat{j} + \hat{k}) + 3(-2\hat{i} + \hat{j} - \hat{k}) = (10-6)\hat{i} + (-6+3)\hat{j} + (2-3)\hat{k} = 4\hat{i} - 3\hat{j} - \hat{k}$$

$$\vec{a} + \vec{b} = (5\hat{i} - 3\hat{j} + \hat{k}) + (-2\hat{i} + \hat{j} - \hat{k}) = (5-2)\hat{i} + (-3+1)\hat{j} + (1-1)\hat{k} = 3\hat{i} - 2\hat{j}$$

$$\therefore |\vec{a} + \vec{b}| = \sqrt{(3)^2 + (-2)^2} = \sqrt{13}$$

Therefore,

$$\begin{aligned}\left| \frac{2\vec{a}+3\vec{b}}{|\vec{a}+\vec{b}|} \right| &= \left| \frac{4\hat{i}-3\hat{j}-\hat{k}}{\sqrt{13}} \right| = \left| \frac{1}{\sqrt{13}} \right| |4\hat{i}-3\hat{j}-\hat{k}| = \frac{1}{\sqrt{13}} \sqrt{(4)^2 + (-3)^2 + (-1)^2} \\ &= \frac{1}{\sqrt{13}} \sqrt{26} = \sqrt{2}\end{aligned}$$

Then,

$$\begin{aligned}2\sqrt{35}\hat{a} + \sqrt{6}\hat{b} &= 2\sqrt{35} \times \frac{1}{|\vec{a}|} \vec{a} + \sqrt{6} \times \frac{1}{|\vec{b}|} \vec{b} = 2\sqrt{35} \times \frac{1}{\sqrt{35}} (5\hat{i} - 3\hat{j} + \hat{k}) + \sqrt{6} \times \frac{1}{\sqrt{6}} (-2\hat{i} + \hat{j} - \hat{k}) \\ &= 2(5\hat{i} - 3\hat{j} + \hat{k}) + (-2\hat{i} + \hat{j} - \hat{k}) \\ &= (10 - 2)\hat{i} + (-6 + 1)\hat{j} + (2 - 1)\hat{k} \\ &= 8\hat{i} - 5\hat{j} + \hat{k} \\ \therefore |2\sqrt{35}\hat{a} + \sqrt{6}\hat{b}| &= \sqrt{(8)^2 + (-5)^2 + (1)^2} = \sqrt{90} = 3\sqrt{10}\end{aligned}$$

It is given that,

$$\begin{aligned}\lambda \left| \frac{2\vec{a}+3\vec{b}}{|\vec{a}+\vec{b}|} \right| + \mu |2\sqrt{35}\hat{a} + \sqrt{6}\hat{b}| &= 4\sqrt{2} + 3\sqrt{10} \\ \Rightarrow \lambda\sqrt{2} + \mu \times 3\sqrt{10} &= 4\sqrt{2} + 3\sqrt{10}\end{aligned}$$

Comparing the co-efficients of $\sqrt{2}$ and $\sqrt{10}$, we obtain $\lambda = 4$ and $3\mu = 3$

Thus, $\lambda = 4$ and $\mu = 1$

Example 2:

If for a vector $\vec{a}$, $\hat{a} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + x\hat{k}$ and $|\vec{a}| = 12$, then find $\vec{a}$.

Solution:

Since $\hat{a} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + x\hat{k}$ is a unit vector, we must have $|\vec{a}| = 1$

$$\begin{aligned}
|\vec{a}| &= 1 \\
\Rightarrow \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2 + x^2} &= 1 \\
\Rightarrow \sqrt{\frac{1}{2} + \frac{1}{3} + x^2} &= 1 \\
\Rightarrow \frac{5}{6} + x^2 &= 1 \\
\Rightarrow x^2 &= \frac{1}{6} \\
\Rightarrow x &= \pm \frac{1}{\sqrt{6}} \\
\therefore \hat{a} &= \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{6}}\hat{k}
\end{aligned}$$

We know that,

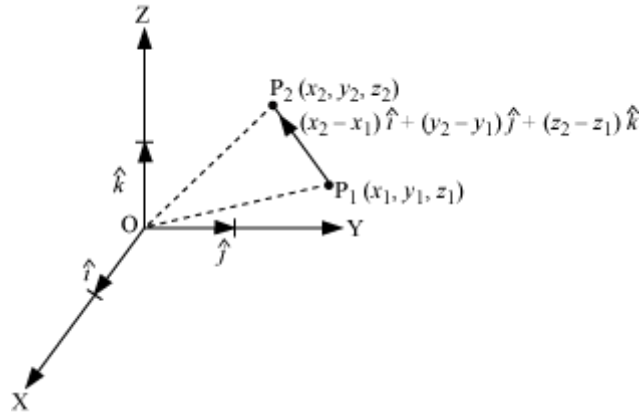
$$\begin{aligned}
\hat{a} &= \frac{1}{|\vec{a}|}\vec{a} \\
\therefore \vec{a} &= |\vec{a}|\hat{a} \\
\Rightarrow \vec{a} &= 12\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{6}}\hat{k}\right) = 6\sqrt{2}\hat{i} - 4\sqrt{3}\hat{j} \pm 2\sqrt{6}\hat{k}
\end{aligned}$$

Vector Joining Two Points and Section Formula

Vector Joining Two Points

- The vector joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, represented as $\overrightarrow{P_1P_2}$, is calculated as

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$



- The magnitude of $\overline{P_1P_2}$ is given by $|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Section Formula

- If point R (position vector $\vec{r}$) lies on the vector $\overline{PQ}$ joining two points P (position vector $\vec{a}$) and Q (position vector $\vec{b}$) such that R divides $\overline{PQ}$ in the ratio $m:n$ $\left[\text{i.e. } \frac{\overline{PR}}{\overline{RQ}} = \frac{m}{n} \right]$
 - Internally, then $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$
 - Externally, then $\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$
- If the position vectors of points P, Q and R are $\vec{a}, \vec{b}$ and $\vec{r}$ respectively such that R is the mid-point of $\overline{PQ}$, then $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$.

Solved Examples

Example 1

Using the concept of vectors, find the area of ΔABC formed by the vertices

$$A(-2\hat{i} - 4\hat{j} + 2\hat{k}), B(-3\hat{i} - 3\hat{j} + 4\hat{k}) \text{ and } C(2\hat{j} + 3\hat{k}).$$

Solution:

We have

$$\overrightarrow{AB} = [-3 - (-2)]\hat{i} + [(-3) - (-4)]\hat{j} + (4 - 2)\hat{k}$$

$$= -\hat{i} + \hat{j} + 2\hat{k}$$

$$\overrightarrow{BC} = [0 - (-3)]\hat{i} + [2 - (-3)]\hat{j} + (3 - 4)\hat{k}$$

$$= 3\hat{i} + 5\hat{j} - \hat{k}$$

$$\overrightarrow{CA} = (-2 - 0)\hat{i} + (-4 - 2)\hat{j} + (2 - 3)\hat{k}$$

$$= -2\hat{i} - 6\hat{j} - \hat{k}$$

$$\therefore |\overrightarrow{AB}| = \sqrt{(-1)^2 + (1)^2 + (2)^2} = \sqrt{6}$$

$$|\overrightarrow{BC}| = \sqrt{(3)^2 + (5)^2 + (-1)^2} = \sqrt{35}$$

$$|\overrightarrow{CA}| = \sqrt{(-2)^2 + (-6)^2 + (-1)^2} = \sqrt{41}$$

We know that, $(\sqrt{41})^2 = (\sqrt{35})^2 + (\sqrt{6})^2$

$$\Rightarrow |\overrightarrow{CA}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{AB}|^2$$

Hence, ΔABC is right-angled at B.

Thus, area $(\Delta ABC) = \frac{1}{2} BC \cdot AB$

$$= \frac{1}{2} \sqrt{35} \cdot \sqrt{6}$$

$$= \frac{1}{2} \sqrt{210} \text{ sq.units}$$

Example 2

If P is the mid-point of the line segment joining the points $A(3\hat{i} - 6\hat{j} + 5\hat{k})$ and $B(-7\hat{i} + \hat{k})$, Q is the mid-point of the line segment joining the points $C(-\hat{i} - 2\hat{j} + 3\hat{k})$ and $D(\hat{i} + 5\hat{k})$, and R is a point on $\overrightarrow{PQ}$ such that it divides PQ externally in the ratio 2:3, then find the position vector of point R .

Solution:

It is given that P is the mid-point of the line segment joining the points $A(3\hat{i} - 6\hat{j} + 5\hat{k})$ and $B(-7\hat{i} + \hat{k})$.

$\therefore$ Position vector of $P(\vec{p})$ is given by

$$\begin{aligned}\vec{p} &= \frac{(3\hat{i} - 6\hat{j} + 5\hat{k}) + (-7\hat{i} + \hat{k})}{2} \\ &= \frac{-4\hat{i} - 6\hat{j} + 6\hat{k}}{2} \\ &= -2\hat{i} - 3\hat{j} + 3\hat{k}\end{aligned}$$

It is also given that Q is the mid-point of the line segment joining the points $C(-\hat{i} - 2\hat{j} + 3\hat{k})$ and $D(\hat{i} + 5\hat{k})$.

$$\begin{aligned}\therefore \text{Position vector of } Q(\vec{q}) &= \frac{(-\hat{i} - 2\hat{j} + 3\hat{k}) + (\hat{i} + 5\hat{k})}{2} \\ &= \frac{-2\hat{j} + 8\hat{k}}{2} \\ &= -\hat{j} + 4\hat{k}\end{aligned}$$

Now, R divides line segment PQ externally in the ratio 2:3.

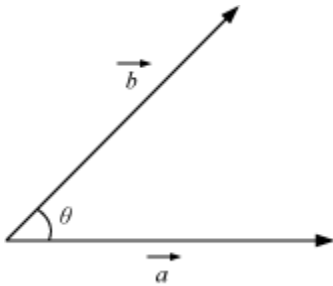
$\therefore$ Position vector $\vec{r}$ of point R is given by

$$\begin{aligned}\vec{r} &= \frac{2\vec{q} - 3\vec{p}}{2-3} \\ &= \frac{2(-\hat{j} + 4\hat{k}) - 3(-2\hat{i} - 3\hat{j} + 3\hat{k})}{2-3} \\ &= \frac{-2\hat{j} + 8\hat{k} + 6\hat{i} + 9\hat{j} - 9\hat{k}}{-1} \\ &= -6\hat{i} - 7\hat{j} + \hat{k}\end{aligned}$$

Scalar (or Dot) Product of Vectors and Projection of a Vector on a Line

Scalar (or Dot) Product of Vectors

- The scalar (or dot) product of two non-zero vectors $\vec{a}$ and $\vec{b}$ (denoted by $\vec{a} \cdot \vec{b}$) is given by the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$.



- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- Some observations on scalar product of $\vec{a}$ and $\vec{b}$:
- $\vec{a} \cdot \vec{b}$ is a real number
- The angle between $\vec{a}$ and $\vec{b}$ is given by

$$\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \right) \text{ or } \theta = \cos^{-1} \left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}} \right)$$

$$\text{where, } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

- If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \perp \vec{b}$
- If the angle between $\vec{a}$ and $\vec{b}$ is 0, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$
- If the angle between $\vec{a}$ and $\vec{b}$ is π , then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$
- $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1, \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$
- Some important properties of scalar product:

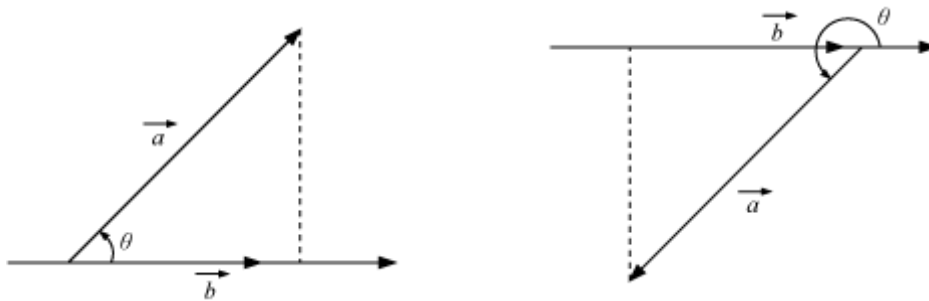
- Commutativity: If $\vec{a}$ and $\vec{b}$ are two vectors, then $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Distributivity: If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors, then $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- If $\vec{a}$ and $\vec{b}$ are any two vectors and λ be any scalar, then $(\lambda \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda \vec{b}) = \lambda (\vec{a} \cdot \vec{b})$
- (Triangle inequality): For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$
- (Cauchy-Schwarz inequality): For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$

Projection of a Vector on a Line

- If $\hat{b}$ is the unit vector along $\vec{b}$, then the projection of a vector $\vec{a}$ on $\vec{b}$ is given by

$$\vec{a} \cdot \hat{b} \text{ or } \vec{a} \cdot \left(\frac{\vec{b}}{|\vec{b}|} \right) \text{ or } \frac{1}{|\vec{b}|} (\vec{a} \cdot \vec{b}) \text{ or } \frac{|\vec{a}| |\vec{b}| \cos \theta}{|\vec{b}|} \text{ or } |\vec{a}| \cos \theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ measured anti-clockwise, $0 \leq \theta < 2\pi$.



- Some observations on projection of vector $\vec{a}$ on $\vec{b}$:
- If $\theta = 0$, then projection of $\vec{a}$ on $\vec{b}$ is $|\vec{a}| \cos 0 = |\vec{a}|$
- If $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, then projection of $\vec{a}$ on $\vec{b}$ is $|\vec{a}| \cos \frac{\pi}{2}$ (or $|\vec{a}| \cos \frac{3\pi}{2}$) = 0
- If $\theta = \pi$, then projection of $\vec{a}$ on $\vec{b}$ is $\vec{a} \cos \pi = \vec{a}(-1) = -\vec{a}$

Solved Examples

Example 1

If vectors $\vec{a} = 2\hat{i} - (x-2)\hat{j} - \hat{k}$ and $\vec{b} = (x-1)\hat{i} - 3\hat{j} + 3x\hat{k}$ are perpendicular to each other, then find the value of x .

Also, find $\frac{|\vec{a} + \vec{b}|}{|\vec{a}| + |\vec{b}|}$.

Solution:

The given vectors are

$$\vec{a} = 2\hat{i} - (x-2)\hat{j} - \hat{k}$$

$$\vec{b} = (x-1)\hat{i} - 3\hat{j} + 3x\hat{k}$$

Since $\vec{a} \perp \vec{b}$, $\vec{a} \cdot \vec{b} = 0$,

$$\Rightarrow [2\hat{i} - (x-2)\hat{j} - \hat{k}] \cdot [(x-1)\hat{i} - 3\hat{j} + 3x\hat{k}] = 0$$

$$\Rightarrow 2(x-1) + 3(x-2) - 3x = 0$$

$$\Rightarrow 2x - 8 = 0$$

$$\Rightarrow x = 4$$

$$\vec{a} = 2\hat{i} - 2\hat{j} - \hat{k}$$

$$\vec{b} = 3\hat{i} - 3\hat{j} + 12\hat{k}$$

$$\therefore \vec{a} + \vec{b} = 5\hat{i} - 5\hat{j} + 11\hat{k}$$

$$|\vec{a}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

$$|\vec{b}| = \sqrt{(3)^2 + (-3)^2 + (12)^2} = 9\sqrt{2}$$

$$|\vec{a} + \vec{b}| = \sqrt{(5)^2 + (-5)^2 + (11)^2} = 3\sqrt{19}$$

$$\text{Thus, } \frac{|\vec{a} + \vec{b}|}{|\vec{a}| + |\vec{b}|} = \frac{3\sqrt{19}}{3 + 9\sqrt{2}} = \frac{\sqrt{19}}{1 + 3\sqrt{2}}$$

Example 2

For two vectors $\vec{a}$ and $\vec{b}$, the product of the projection of $\vec{a}$ on $\vec{b}$ with the projection of $\vec{b}$ on $\vec{a}$ is half of the dot product of $\vec{a}$ and $\vec{b}$. Find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then, the projection $\vec{a}$ on $\vec{b}$ is $\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b})$.

The projection of $\vec{b}$ on $\vec{a}$ is $\frac{1}{|\vec{a}|}(\vec{a} \cdot \vec{b})$.

According to the given condition,

$$\begin{aligned} \left[\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) \right] \left[\frac{1}{|\vec{a}|}(\vec{a} \cdot \vec{b}) \right] &= \frac{1}{2}(\vec{a} \cdot \vec{b}) \\ \Rightarrow \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} &= \frac{1}{2} \\ \Rightarrow \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{a}||\vec{b}|} &= \frac{1}{2} \\ \Rightarrow \cos\theta &= \frac{1}{2} \\ \Rightarrow \theta &= \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \end{aligned}$$

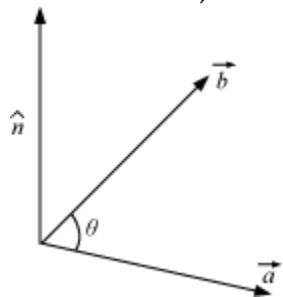
Thus, the angle between the vectors $\vec{a}$ and $\vec{b}$ is 60° .

Vector Product of Vectors

- The vector product (cross product) of two non-zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$ and $\hat{n}$ is a unit vector, which is perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$, and $\hat{n}$ form a right hand system (i.e., the system moves in the

direction of $\hat{n}$, when it is rotated from $\vec{a}$ to $\vec{b}$)



- Some observations of vector product of $\vec{a}$ and $\vec{b}$:

(a) $\vec{a} \times \vec{b}$ is a vector.

(b) If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined. In this case, we define $\vec{a} \times \vec{b}$ as 0.

(c) If $|\vec{a} \times \vec{b}|$ are non-zero vectors such that $\vec{a} \parallel \vec{b}$ or $\vec{a}$ and $\vec{b}$ are collinear, then $\vec{a} \times \vec{b} = 0$. In particular, $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{a} \times (-\vec{a}) = \vec{0}$

(d) If $\theta = \frac{\pi}{2}$, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\hat{n}$

(e) For mutually perpendicular unit vectors $\hat{i}, \hat{j}$, and $\hat{k}$,

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$$

(f) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(g) The angle between $\vec{a}$ and $\vec{b}$ in terms of vector product is given as

$$\sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

(h) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(i) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given as $\frac{1}{2}|\vec{a} \times \vec{b}|$.

(j) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given as $|\vec{a} \times \vec{b}|$.

- Distributive property of vector product over addition – If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are any three vectors and λ be a scalar, then

$$(i) \quad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$(ii) \quad \lambda (\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b})$$

Solved Examples

Example 1:

For what integral value of x , $0 \leq x \leq 5$, the area of the parallelogram whose adjacent sides are determined by the vectors, $\vec{a} = \hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 7\hat{i} + (1-x)\hat{j} - x\hat{k}$, is $15\sqrt{2}$ square units? Also, find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

The adjacent sides of the parallelogram are determined by the vectors

$$\vec{a} = \hat{i} - 3\hat{j} - \hat{k} \text{ and}$$

$$\vec{b} = 7\hat{i} + (1-x)\hat{j} - x\hat{k}$$

$\therefore$ Area of the parallelogram $|\vec{a} \times \vec{b}|$

Now,

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -1 \\ 7 & 1-x & -x \end{vmatrix} = (3x+1-x)\hat{i} + (-7+x)\hat{j} + (1-x+21)\hat{k} \\ &= (2x-1)\hat{i} + (-7+x)\hat{j} + (22-x)\hat{k} \end{aligned}$$

Therefore, area of the parallelogram is

$$\begin{aligned}
|\vec{a} \times \vec{b}| &= \sqrt{(2x+1)^2 + (-7+x)^2 + (22-x)^2} \\
|\vec{a} \times \vec{b}| &= \sqrt{4x^2 + 4x + 1 + 49 + x^2 - 14x + 484 + x^2 - 44x} \\
&\Rightarrow 15\sqrt{2} = \sqrt{6x^2 - 54x + 534} \\
&\Rightarrow 450 = 6x^2 - 54x + 534 \\
&\Rightarrow 6x^2 - 54x + 84 = 0 \\
&\Rightarrow x^2 - 9x + 14 = 0 \\
&\Rightarrow (x-2)(x-7) = 0 \\
&\Rightarrow x = 2, x = 7
\end{aligned}$$

Since $0 \leq x \leq 5$,

$$x = 2$$

Now,

$$\vec{a} = \hat{i} - 3\hat{j} - \hat{k}$$

$$\text{And, } \vec{b} = 7\hat{i} - \hat{j} - 2\hat{k}$$

$$\therefore |\vec{a}| = \sqrt{(1)^2 + (-3)^2 + (-1)^2} = \sqrt{11}$$

$$|\vec{b}| = \sqrt{(7)^2 + (-1)^2 + (-2)^2} = 3\sqrt{6}$$

$$\therefore \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{15\sqrt{2}}{\sqrt{11} \times 3\sqrt{6}} = \frac{5}{\sqrt{33}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{5}{\sqrt{33}}\right)$$

Therefore, angle between $\vec{a}$ and $\vec{b}$ is $\sin^{-1}\left(\frac{5}{\sqrt{33}}\right)$.

Example 2:

For two vectors $\vec{a}$ and $\vec{b}$, if $|\vec{a}| = \sqrt{29}$, $|\vec{b}| = \sqrt{13}$, and $|\vec{a} \cdot \vec{b}| = 16$, then find $|\vec{b} \times 2\vec{a}|$.

Solution:

$$\begin{aligned}
|\vec{a} \cdot \vec{b}| &= |\vec{a}| |\vec{b}| |\cos \theta| \\
\Rightarrow |\cos \theta| &= \frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}| |\vec{b}|} = \frac{16}{\sqrt{29} \times \sqrt{13}} = \frac{16}{\sqrt{377}} \\
\therefore |\sin \theta| &= \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \left(\frac{16}{\sqrt{377}}\right)^2} \\
&= \sqrt{1 - \frac{256}{377}} \\
&= \sqrt{\frac{121}{377}} \\
&= \frac{11}{\sqrt{377}}
\end{aligned}$$

Now,

$$\begin{aligned}
|\vec{b} \times 2\vec{a}| &= 2|\vec{b} \times \vec{a}| = 2|\vec{b}| |\vec{a}| |\sin \theta| \\
&= 2 \times \sqrt{13} \times \sqrt{29} \times \frac{11}{\sqrt{377}} = 22
\end{aligned}$$