

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

5.1 Overview

We know that the square of a real number is always non-negative e.g. $(4)^2 = 16$ and $(-4)^2 = 16$. Therefore, square root of 16 is ± 4 . What about the square root of a negative number? It is clear that a negative number can not have a real square root. So we need to extend the system of real numbers to a system in which we can find out the square roots of negative numbers. Euler (1707 - 1783) was the first mathematician to introduce the symbol i (iota) for positive square root of -1 i.e., $i = \sqrt{-1}$.

5.1.1 Imaginary numbers

Square root of a negative number is called an imaginary number., for example,

$$\sqrt{-9} = \sqrt{-1} \sqrt{9} = i3, \sqrt{-7} = \sqrt{-1} \sqrt{7} = i\sqrt{7}$$

5.1.2 Integral powers of i

$$i = \sqrt{-1}, i^2 = -1, i^3 = i^2 i = -i, i^4 = (i^2)^2 = (-1)^2 = 1.$$

To compute i^n for $n > 4$, we divide n by 4 and write it in the form $n = 4m + r$, where m is quotient and r is remainder ($0 \leq r \leq 4$)

Hence

$$i^n = i^{4m+r} = (i^4)^m \cdot (i)^r = (1)^m (i)^r = i^r$$

For example,

$$(i)^{39} = i^{4 \times 9 + 3} = (i^4)^9 \cdot (i)^3 = i^3 = -i$$

and

$$\begin{aligned} (i)^{-435} &= i^{-(4 \times 108 + 3)} = (i)^{-(4 \times 108)} \cdot (i)^{-3} \\ &= \frac{1}{(i^4)^{108}} \cdot \frac{1}{(i)^3} = \frac{i}{(i)^4} = i \end{aligned}$$

- (i) If a and b are positive real numbers, then

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{-1} \sqrt{a} \times \sqrt{-1} \sqrt{b} = i\sqrt{a} \times i\sqrt{b} = -\sqrt{ab}$$

- (ii) $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ if a and b are positive or at least one of them is negative or zero. However, $\sqrt{a} \sqrt{b} \neq \sqrt{ab}$ if a and b , both are negative.

5.1.3 Complex numbers

- A number which can be written in the form $a + ib$, where a, b are real numbers and $i = \sqrt{-1}$ is called a complex number.
- If $z = a + ib$ is the complex number, then a and b are called real and imaginary parts, respectively, of the complex number and written as $\text{Re}(z) = a$, $\text{Im}(z) = b$.
- Order relations “greater than” and “less than” are not defined for complex numbers.
- If the imaginary part of a complex number is zero, then the complex number is known as purely real number and if real part is zero, then it is called purely imaginary number, for example, 2 is a purely real number because its imaginary part is zero and $3i$ is a purely imaginary number because its real part is zero.

5.1.4 Algebra of complex numbers

- Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are said to be equal if $a = c$ and $b = d$.
- Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then $z_1 + z_2 = (a + c) + i(b + d)$.

5.1.5 Addition of complex numbers satisfies the following properties

- As the sum of two complex numbers is again a complex number, the set of complex numbers is closed with respect to addition.
- Addition of complex numbers is commutative, i.e., $z_1 + z_2 = z_2 + z_1$
- Addition of complex numbers is associative, i.e., $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- For any complex number $z = x + iy$, there exist 0, i.e., $(0 + 0i)$ complex number such that $z + 0 = 0 + z = z$, known as identity element for addition.
- For any complex number $z = x + iy$, there always exists a number $-z = -a - ib$ such that $z + (-z) = (-z) + z = 0$ and is known as the additive inverse of z .

5.1.6 Multiplication of complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$, be two complex numbers. Then

$$z_1 \cdot z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

- As the product of two complex numbers is a complex number, the set of complex numbers is closed with respect to multiplication.
- Multiplication of complex numbers is commutative, i.e., $z_1 \cdot z_2 = z_2 \cdot z_1$
- Multiplication of complex numbers is associative, i.e., $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$

4. For any complex number $z = x + iy$, there exists a complex number 1, i.e., $(1 + 0i)$ such that
 $z \cdot 1 = 1 \cdot z = z$, known as identity element for multiplication.
5. For any non zero complex number $z = x + iy$, there exists a complex number $\frac{1}{z}$ such that $z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$, i.e., multiplicative inverse of $a + ib = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.
6. For any three complex numbers z_1, z_2 and z_3 ,

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$
and
$$(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$$
i.e., for complex numbers multiplication is distributive over addition.

5.1.7 Let $z_1 = a + ib$ and $z_2 (\neq 0) = c + id$. Then

$$z_1 \div z_2 = \frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + i \frac{(bc-ad)}{c^2+d^2}$$

5.1.8 Conjugate of a complex number

Let $z = a + ib$ be a complex number. Then a complex number obtained by changing the sign of imaginary part of the complex number is called the conjugate of z and it is denoted by $\bar{z}$, i.e., $\bar{z} = a - ib$.

Note that additive inverse of z is $-a - ib$ but conjugate of z is $a - ib$.

We have :

1. $\overline{(\bar{z})} = z$
2. $z + \bar{z} = 2 \operatorname{Re}(z)$, $z - \bar{z} = 2i \operatorname{Im}(z)$
3. $z = \bar{z}$, if z is purely real.
4. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
5. $z \cdot \bar{z} = \{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2$.
6. $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$, $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$
7. $\overline{(z_1 \cdot z_2)} = (\bar{z}_1)(\bar{z}_2)$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\bar{z}_1)}{(\bar{z}_2)}$ ($\bar{z}_2 \neq 0$)

5.1.9 Modulus of a complex number

Let $z = a + ib$ be a complex number. Then the positive square root of the sum of square of real part and square of imaginary part is called modulus (absolute value) of z and it

is denoted by $|z|$ i.e., $|z| = \sqrt{a^2 + b^2}$

In the set of complex numbers $z_1 > z_2$ or $z_1 < z_2$ are meaningless but

$$|z_1| > |z_2| \text{ or } |z_1| < |z_2|$$

are meaningful because $|z_1|$ and $|z_2|$ are real numbers.

5.1.10 Properties of modulus of a complex number

1. $|z| = 0 \Leftrightarrow z = 0$ i.e., $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) = 0$
2. $|z| = |\bar{z}| = |-z|$
3. $-|z| \leq \operatorname{Re}(z) \leq |z|$ and $-|z| \leq \operatorname{Im}(z) \leq |z|$
4. $z \bar{z} = |z|^2$, $|z^2| = |\bar{z}|^2$
5. $|z_1 z_2| = |z_1| \cdot |z_2|$, $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ ($z_2 \neq 0$)
6. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2)$
7. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1 \bar{z}_2)$
8. $|z_1 + z_2| \leq |z_1| + |z_2|$
9. $|z_1 - z_2| \geq |z_1| - |z_2|$
10. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

In particular:

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

11. As stated earlier multiplicative inverse (reciprocal) of a complex number $z = a + ib$ ($\neq 0$) is

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

5.2 Argand Plane

A complex number $z = a + ib$ can be represented by a unique point P (a, b) in the cartesian plane referred to a pair of rectangular axes. The complex number $0 + 0i$ represent the origin O ($0, 0$). A purely real number a , i.e., $(a + 0i)$ is represented by the point ($a, 0$) on x - axis. Therefore, x-axis is called real axis. A purely imaginary number

ib , i.e., $(0 + ib)$ is represented by the point $(0, b)$ on y -axis. Therefore, y -axis is called imaginary axis.

Similarly, the representation of complex numbers as points in the plane is known as **Argand diagram**. The plane representing complex numbers as points is called complex plane or Argand plane or Gaussian plane.

If two complex numbers z_1 and z_2 be represented by the points P and Q in the complex plane, then

$$|z_1 - z_2| = PQ$$

5.2.1 Polar form of a complex number

Let P be a point representing a non-zero complex number $z = a + ib$ in the Argand plane. If OP makes an angle θ with the positive direction of x -axis, then $z = r(\cos\theta + i\sin\theta)$ is called the polar form of the complex number, where

$r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$. Here θ is called argument or amplitude of z and we write it as $\arg(z) = \theta$.

The unique value of θ such that $-\pi \leq \theta \leq \pi$ is called the principal argument.

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

5.2.2 Solution of a quadratic equation

The equations $ax^2 + bx + c = 0$, where a, b and c are numbers (real or complex, $a \neq 0$) is called the general quadratic equation in variable x . The values of the variable satisfying the given equation are called roots of the equation.

The quadratic equation $ax^2 + bx + c = 0$ with real coefficients has two roots given by $\frac{-b + \sqrt{D}}{2a}$ and $\frac{-b - \sqrt{D}}{2a}$, where $D = b^2 - 4ac$, called the discriminant of the equation.

Notes

1. When $D = 0$, roots of the quadratic equation are real and equal. When $D > 0$, roots are real and unequal. Further, if $a, b, c \in \mathbf{Q}$ and D is a perfect square, then the roots of the equation are rational and unequal, and if $a, b, c \in \mathbf{Q}$ and D is not a perfect square, then the roots are irrational and occur in pair.

When $D < 0$, roots of the quadratic equation are non real (or complex).

2. Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, then sum of the roots

$$(\alpha + \beta) = \frac{-b}{a} \text{ and the product of the roots } (\alpha \cdot \beta) = \frac{c}{a}.$$

3. Let S and P be the sum of roots and product of roots, respectively, of a quadratic equation. Then the quadratic equation is given by $x^2 - Sx + P = 0$.

5.2 Solved Examples

Short Answer Type

Example 1 Evaluate : $(1 + i)^6 + (1 - i)^3$

Solution $(1 + i)^6 = \{(1 + i)^2\}^3 = (1 + i^2 + 2i)^3 = (1 - 1 + 2i)^3 = 8i^3 = -8i$

and $(1 - i)^3 = 1 - i^3 - 3i + 3i^2 = 1 + i - 3i - 3 = -2 - 2i$

Therefore, $(1 + i)^6 + (1 - i)^3 = -8i - 2 - 2i = -2 - 10i$

Example 2 If $(x + iy)^{\frac{1}{3}} = a + ib$, where $x, y, a, b \in \mathbb{R}$, show that $\frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$

Solution $(x + iy)^{\frac{1}{3}} = a + ib$

$$\Rightarrow x + iy = (a + ib)^3$$

$$\begin{aligned} \text{i.e., } x + iy &= a^3 + i^3 b^3 + 3iab(a + ib) \\ &= a^3 - ib^3 + i3a^2b - 3ab^2 \\ &= a^3 - 3ab^2 + i(3a^2b - b^3) \end{aligned}$$

$$\Rightarrow x = a^3 - 3ab^2 \text{ and } y = 3a^2b - b^3$$

$$\text{Thus } \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2$$

$$\text{So, } \frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - 3a^2 + b^2 = -2a^2 - 2b^2 = -2(a^2 + b^2).$$

Example 3 Solve the equation $z^2 = \bar{z}$, where $z = x + iy$

Solution $z^2 = \bar{z} \Rightarrow x^2 - y^2 + i2xy = x - iy$

$$\text{Therefore, } x^2 - y^2 = x \quad \dots (1) \quad \text{and} \quad 2xy = -y \quad \dots (2)$$

From (2), we have $y = 0$ or $x = -\frac{1}{2}$

When $y = 0$, from (1), we get $x^2 - x = 0$, i.e., $x = 0$ or $x = 1$.

When $x = -\frac{1}{2}$, from (1), we get $y^2 = \frac{1}{4} + \frac{1}{2}$ or $y^2 = \frac{3}{4}$, i.e., $y = \pm \frac{\sqrt{3}}{2}$.

Hence, the solutions of the given equation are

$$0 + i0, 1 + i0, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Example 4 If the imaginary part of $\frac{2z+1}{iz+1}$ is -2 , then show that the locus of the point representing z in the argand plane is a straight line.

Solution Let $z = x + iy$. Then

$$\begin{aligned} \frac{2z+1}{iz+1} &= \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix} \\ &= \frac{\{(2x+1)+i2y\}}{\{(1-y)+ix\}} \times \frac{\{(1-y)-ix\}}{\{(1-y)-ix\}} \\ &= \frac{(2x+1-y)+i(2y-2y^2-2x^2-x)}{1+y^2-2y+x^2} \end{aligned}$$

Thus
$$\operatorname{Im}\left(\frac{2z+1}{iz+1}\right) = \frac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2}$$

But
$$\operatorname{Im}\left(\frac{2z+1}{iz+1}\right) = -2 \quad (\text{Given})$$

So
$$\frac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2} = -2$$

$\Rightarrow 2y - 2y^2 - 2x^2 - x = -2 - 2y^2 + 4y - 2x^2$
i.e., $x + 2y - 2 = 0$, which is the equation of a line.

Example 5 If $|z^2 - 1| = |z|^2 + 1$, then show that z lies on imaginary axis.

Solution Let $z = x + iy$. Then $|z^2 - 1| = |z|^2 + 1$

$$\begin{aligned} \Rightarrow & \quad |x^2 - y^2 - 1 + i2xy| = |x + iy|^2 + 1 \\ \Rightarrow & \quad (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2 \\ \Rightarrow & \quad 4x^2 = 0 \quad \text{i.e.,} \quad x = 0 \end{aligned}$$

Hence z lies on y -axis.

Example 6 Let z_1 and z_2 be two complex numbers such that $\bar{z}_1 + i\bar{z}_2 = 0$ and $\arg(z_1 z_2) = \pi$. Then find $\arg(z_1)$.

Solution Given that $\bar{z}_1 + i\bar{z}_2 = 0$

$$\begin{aligned} \Rightarrow & \quad z_1 = iz_2, \text{ i.e., } z_2 = -iz_1 \\ \text{Thus} & \quad \arg(z_1 z_2) = \arg z_1 + \arg(-iz_1) = \pi \\ \Rightarrow & \quad \arg(-iz_1^2) = \pi \\ \Rightarrow & \quad \arg(-i) + \arg(z_1^2) = \pi \\ \Rightarrow & \quad \arg(-i) + 2\arg(z_1) = \pi \\ \Rightarrow & \quad \frac{-\pi}{2} + 2\arg(z_1) = \pi \\ \Rightarrow & \quad \arg(z_1) = \frac{3\pi}{4} \end{aligned}$$

Example 7 Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$.

Then show that $\arg(z_1) - \arg(z_2) = 0$.

Solution Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

where $r_1 = |z_1|$, $\arg(z_1) = \theta_1$, $r_2 = |z_2|$, $\arg(z_2) = \theta_2$.

We have, $|z_1 + z_2| = |z_1| + |z_2|$

$$\begin{aligned} &= |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)| = r_1 + r_2 \\ &= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = (r_1 + r_2)^2 \Rightarrow \cos(\theta_1 - \theta_2) = 1 \\ &\Rightarrow \theta_1 - \theta_2 \text{ i.e. } \arg z_1 = \arg z_2 \end{aligned}$$

Example 8 If z_1, z_2, z_3 are complex numbers such that

$$|z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1, \text{ then find the value of } |z_1 + z_2 + z_3|.$$

Solution $|z_1| = |z_2| = |z_3| = 1$

$$\Rightarrow |z_1|^2 = |z_2|^2 = |z_3|^2 = 1$$

$$\Rightarrow z_1 \bar{z}_1 = z_2 \bar{z}_2 = z_3 \bar{z}_3 = 1$$

$$\Rightarrow \bar{z}_1 = \frac{1}{z_1}, \bar{z}_2 = \frac{1}{z_2}, \bar{z}_3 = \frac{1}{z_3}$$

Given that $\left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1$

$$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1, \text{ i.e., } |\overline{z_1 + z_2 + z_3}| = 1$$

$$\Rightarrow |z_1 + z_2 + z_3| = 1$$

Example 9 If a complex number z lies in the interior or on the boundary of a circle of radius 3 units and centre $(-4, 0)$, find the greatest and least values of $|z+1|$.

Solution Distance of the point representing z from the centre of the circle is $|z - (-4 + i0)| = |z+4|$.

According to given condition $|z+4| \leq 3$.

$$\text{Now } |z+1| = |z+4-3| \leq |z+4| + |-3| \leq 3+3=6$$

Therefore, greatest value of $|z+1|$ is 6.

Since least value of the modulus of a complex number is zero, the least value of $|z+1|=0$.

Example 10 Locate the points for which $3 < |z| < 4$

Solution $|z| < 4 \Rightarrow x^2 + y^2 < 16$ which is the interior of circle with centre at origin and radius 4 units, and $|z| > 3 \Rightarrow x^2 + y^2 > 9$ which is exterior of circle with centre at origin and radius 3 units. Hence $3 < |z| < 4$ is the portion between two circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.

Example 11 Find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$, when $x = -2 - \sqrt{3}i$

Solution $x + 2 = -\sqrt{3}i \Rightarrow x^2 + 4x + 7 = 0$

Therefore
$$\begin{aligned} 2x^4 + 5x^3 + 7x^2 - x + 41 &= (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6 \\ &= 0 \times (2x^2 - 3x + 5) + 6 = 6. \end{aligned}$$

Example 12 Find the value of P such that the difference of the roots of the equation $x^2 - Px + 8 = 0$ is 2.

Solution Let α, β be the roots of the equation $x^2 - Px + 8 = 0$

Therefore $\alpha + \beta = P$ and $\alpha \cdot \beta = 8$.

$$\text{Now } \alpha - \beta = \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$\text{Therefore } 2 = \pm \sqrt{P^2 - 32}$$

$$\Rightarrow P^2 - 32 = 4, \text{ i.e., } P = \pm 6.$$

Example 13 Find the value of a such that the sum of the squares of the roots of the equation $x^2 - (a - 2)x - (a + 1) = 0$ is least.

Solution Let α, β be the roots of the equation

Therefore, $\alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$

$$\begin{aligned} \text{Now } \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (a - 2)^2 + 2(a + 1) \\ &= (a - 1)^2 + 5 \end{aligned}$$

Therefore, $\alpha^2 + \beta^2$ will be minimum if $(a - 1)^2 = 0$, i.e., $a = 1$.

Long Answer Type

Example 14 Find the value of k if for the complex numbers z_1 and z_2 ,

$$|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1 - |z_1|^2)(1 - |z_2|^2)$$

Solution

$$\begin{aligned} \text{L.H.S.} &= |1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 \\ &= (1 - \bar{z}_1 z_2)(\overline{1 - \bar{z}_1 z_2}) - (z_1 - z_2)(\overline{z_1 - z_2}) \\ &= (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= 1 + z_1 \bar{z}_1 z_2 \bar{z}_2 - z_1 \bar{z}_1 - z_2 \bar{z}_2 \\ &= 1 + |z_1|^2 \cdot |z_2|^2 - |z_1|^2 - |z_2|^2 \\ &= (1 - |z_1|^2)(1 - |z_2|^2) \end{aligned}$$

$$\text{R.H.S.} = k(1 - |z_1|^2)(1 - |z_2|^2)$$

$$\Rightarrow k = 1$$

Hence, equating LHS and RHS, we get $k = 1$.

Example 15 If z_1 and z_2 both satisfy $z + \bar{z} = 2|z - 1|$ $\arg(z_1 - z_2) = \frac{\pi}{4}$, then find $\text{Im}(z_1 + z_2)$.

Solution Let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

$$\text{Then } z + \bar{z} = 2|z - 1|$$

$$\Rightarrow (x + iy) + (x - iy) = 2|x - 1 + iy|$$

$$\Rightarrow 2x = 1 + y^2 \quad \dots (1)$$

Since z_1 and z_2 both satisfy (1), we have

$$2x_1 = 1 + y_1^2 \quad \dots \text{ and } 2x_2 = 1 + y_2^2$$

$$\Rightarrow 2(x_1 - x_2) = (y_1 + y_2)(y_1 - y_2)$$

$$\Rightarrow 2 = (y_1 + y_2) \left(\frac{y_1 - y_2}{x_1 - x_2} \right) \quad \dots (2)$$

$$\text{Again } z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$\text{Therefore, } \tan \theta = \frac{y_1 - y_2}{x_1 - x_2}, \text{ where } \theta = \arg(z_1 - z_2)$$

$$\Rightarrow \tan \frac{\pi}{4} = \frac{y_1 - y_2}{x_1 - x_2} \quad \left(\text{since } \theta = \frac{\pi}{4} \right)$$

$$\text{i.e., } 1 = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\text{From (2), we get } 2 = y_1 + y_2, \text{ i.e., } \text{Im}(z_1 + z_2) = 2$$

Objective Type Questions

Example 16 Fill in the blanks:

- (i) The real value of ' a ' for which $3i^3 - 2a^2 + (1 - a)i + 5$ is real is _____.
- (ii) If $|z| = 2$ and $\arg(z) = \frac{\pi}{4}$, then $z =$ _____.
- (iii) The locus of z satisfying $\arg(z) = \frac{\pi}{3}$ is _____.
- (iv) The value of $(-\sqrt{-1})^{4n-3}$, where $n \in \mathbf{N}$, is _____.

- (v) The conjugate of the complex number $\frac{1-i}{1+i}$ is _____.
- (vi) If a complex number lies in the third quadrant, then its conjugate lies in the _____.
- (vii) If $(2+i)(2+2i)(2+3i) \dots (2+ni) = x+iy$, then $5.8.13 \dots (4+n^2) =$ _____.

Solution

- (i) $3i^3 - 2ai^2 + (1-a)i + 5 = -3i + 2a + 5 + (1-a)i$
 $= 2a + 5 + (-a-2)i$, which is real if $-a-2 = 0$ i.e. $a = -2$.
- (ii) $z = |z| \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 2 \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = \sqrt{2}(1+i)$
- (iii) Let $z = x+iy$. Then its polar form is $z = r(\cos \theta + i \sin \theta)$, where $\tan \theta = \frac{y}{x}$ and

θ is arg (z). Given that $\theta = \frac{\pi}{3}$. Thus.

$$\tan \frac{\pi}{3} = \frac{y}{x} \Rightarrow y = \sqrt{3}x, \text{ where } x > 0, y > 0.$$

Hence, locus of z is the part of $y = \sqrt{3}x$ in the first quadrant except origin.

(iv) Here $(-\sqrt{-1})^{4n-3} = (-i)^{4n-3} = (-i)^{4n} (-i)^{-3} = \frac{1}{(-i)^3}$
 $= \frac{1}{-i^3} = \frac{1}{i} = \frac{i}{i^2} = -i$

(v) $\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1+i^2-2i}{1-i^2} = \frac{1-1-2i}{1+1} = -i$

Hence, conjugate of $\frac{1-i}{1+i}$ is i .

- (vi) Conjugate of a complex number is the image of the complex number about the x -axis. Therefore, if a number lies in the third quadrant, then its image lies in the second quadrant.

(vii) Given that $(2+i)(2+2i)(2+3i) \dots (2+ni) = x+iy$... (1)

$$\Rightarrow (\overline{2+i})(\overline{2+2i})(\overline{2+3i}) \dots (\overline{2+ni}) = (\overline{x+iy}) = (x-iy)$$

i.e., $(2-i)(2-2i)(2-3i) \dots (2-ni) = x-iy$... (2)

Multiplying (1) and (2), we get 5.8.13 ... $(4 + n^2) = x^2 + y^2$.

Example 17 State true or false for the following:

- (i) Multiplication of a non-zero complex number by i rotates it through a right angle in the anti-clockwise direction.
- (ii) The complex number $\cos\theta + i \sin\theta$ can be zero for some θ .
- (iii) If a complex number coincides with its conjugate, then the number must lie on imaginary axis.
- (iv) The argument of the complex number $z = (1 + i\sqrt{3})(1 + i)(\cos\theta + i \sin\theta)$ is $\frac{7\pi}{12} + \theta$
- (v) The points representing the complex number z for which $|z+1| < |z-1|$ lies in the interior of a circle.
- (vi) If three complex numbers z_1, z_2 and z_3 are in A.P., then they lie on a circle in the complex plane.
- (vii) If n is a positive integer, then the value of $i^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3}$ is 0.

Solution

- (i) True. Let $z = 2 + 3i$ be complex number represented by OP. Then $iz = -3 + 2i$, represented by OQ, where if OP is rotated in the anticlockwise direction through a right angle, it coincides with OQ.
- (ii) False. Because $\cos\theta + i\sin\theta = 0 \Rightarrow \cos\theta = 0$ and $\sin\theta = 0$. But there is no value of θ for which $\cos\theta$ and $\sin\theta$ both are zero.
- (iii) False, because $x + iy = x - iy \Rightarrow y = 0 \Rightarrow$ number lies on x -axis.
- (iv) True, $\arg(z) = \arg(1 + i\sqrt{3}) + \arg(1 + i) + \arg(\cos\theta + i\sin\theta)$

$$\frac{\pi}{3} + \frac{\pi}{4} + \theta = \frac{7\pi}{12} + \theta$$
- (v) False, because $|x+iy+1| < |x+iy-1|$

$$\Rightarrow (x+1)^2 + y^2 < (x-1)^2 + y^2 \text{ which gives } 4x < 0.$$
- (vi) False, because if z_1, z_2 and z_3 are in A.P., then $z_2 = \frac{z_1 + z_3}{2} \Rightarrow z_2$ is the midpoint of z_1 and z_3 , which implies that the points z_1, z_2, z_3 are collinear.
- (vii) True, because $i^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3}$

$$= i^n (1 + i + i^2 + i^3) = i^n (1 + i - 1 - i)$$

$$= i^n (0) = 0$$

Example 18 Match the statements of column A and B.

Column A

Column B

- | | |
|---|--------------------------------------|
| (a) The value of $1+i^2 + i^4 + i^6 + \dots i^{20}$ is | (i) purely imaginary complex number |
| (b) The value of i^{-1097} is | (ii) purely real complex number |
| (c) Conjugate of $1+i$ lies in | (iii) second quadrant |
| (d) $\frac{1+2i}{1-i}$ lies in | (iv) Fourth quadrant |
| (e) If $a, b, c \in \mathbb{R}$ and $b^2 - 4ac < 0$, then the roots of the equation $ax^2 + bx + c = 0$ are non real (complex) and | (v) may not occur in conjugate pairs |
| (f) If $a, b, c \in \mathbb{R}$ and $b^2 - 4ac > 0$, and $b^2 - 4ac$ is a perfect square, then the roots of the equation $ax^2 + bx + c = 0$ | (vi) may occur in conjugate pairs |

Solution

- (a) $\Leftrightarrow$ (ii), because $1 + i^2 + i^4 + i^6 + \dots + i^{20}$
 $= 1 - 1 + 1 - 1 + \dots + 1 = 1$ (which is purely a real complex number)
- (b) $\Leftrightarrow$ (i), because $i^{-1097} = \frac{1}{(i)^{1097}} = \frac{1}{i^{4 \times 274 + 1}} = \frac{1}{\{(i)^4\}^{274} (i)} = \frac{1}{i} = \frac{i}{i^2} = -i$
 which is purely imaginary complex number.
- (c) $\Leftrightarrow$ (iv), conjugate of $1 + i$ is $1 - i$, which is represented by the point $(1, -1)$ in the fourth quadrant.
- (d) $\Leftrightarrow$ (iii), because $\frac{1+2i}{1-i} = \frac{1+2i}{1-i} \times \frac{1+i}{1+i} = \frac{-1+3i}{2} = -\frac{1}{2} + \frac{3}{2}i$, which is
 represented by the point $\left(-\frac{1}{2}, \frac{3}{2}\right)$ in the second quadrant.
- (e) $\Leftrightarrow$ (vi), If $b^2 - 4ac < 0 = D < 0$, i.e., square root of D is a imaginary
 number, therefore, roots are $x = \frac{-b \pm \text{Imaginary Number}}{2a}$, i.e., roots are in
 conjugate pairs.

(f) $\Leftrightarrow$ (v), Consider the equation $x^2 - (5 + \sqrt{2})x + 5\sqrt{2} = 0$, where $a = 1$,
 $b = -(5 + \sqrt{2})$, $c = 5\sqrt{2}$, clearly $a, b, c \in \mathbb{R}$.

$$\text{Now } D = b^2 - 4ac = \{-(5 + \sqrt{2})\}^2 - 4 \cdot 1 \cdot 5\sqrt{2} = (5 - \sqrt{2})^2.$$

Therefore $x = \frac{5 + \sqrt{2} \pm 5 - \sqrt{2}}{2} = 5, \sqrt{2}$ which do not form a conjugate pair.

Example 19 What is the value of $\frac{i^{4n+1} - i^{4n-1}}{2}$?

Solution i , because
$$\frac{i^{4n+1} - i^{4n-1}}{2} = \frac{i^{4n}i - i^{4n}i^{-1}}{2}$$

$$= \frac{i - \frac{1}{i}}{2} = \frac{i^2 - 1}{2i} = \frac{-2}{2i} = i$$

Example 20 What is the smallest positive integer n , for which $(1 + i)^{2n} = (1 - i)^{2n}$?

Solution $n = 2$, because $(1 + i)^{2n} = (1 - i)^{2n} = \left(\frac{1+i}{1-i}\right)^{2n} = 1$

$\Rightarrow (i)^{2n} = 1$ which is possible if $n = 2$ ($\because i^4 = 1$)

Example 21 What is the reciprocal of $3 + \sqrt{7}i$?

Solution Reciprocal of $z = \frac{\bar{z}}{|z|^2}$

Therefore, reciprocal of $3 + \sqrt{7}i = \frac{3 - \sqrt{7}i}{16} = \frac{3}{16} - \frac{\sqrt{7}i}{16}$

Example 22 If $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, then find the quadrant in which

$\left(\frac{z_1}{z_2}\right)$ lies.

Solution
$$\frac{z_1}{z_2} = \frac{\sqrt{3} + i\sqrt{3}}{\sqrt{3} + i} = \left(\frac{3 + \sqrt{3}}{4}\right) + \left(\frac{3 - \sqrt{3}}{4}\right)i$$

which is represented by a point in first quadrant.

Example 23 What is the conjugate of $\frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$?

Solution Let

$$\begin{aligned} z &= \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}} \times \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} + \sqrt{5-12i}} \\ &= \frac{5+12i+5-12i+2\sqrt{25+144}}{5+12i-5+12i} \\ &= \frac{3}{2i} = \frac{3i}{-2} = 0 - \frac{3}{2}i \end{aligned}$$

Therefore, the conjugate of $z = 0 + \frac{3}{2}i$

Example 24 What is the principal value of amplitude of $1 - i$?

Solution Let θ be the principle value of amplitude of $1 - i$. Since

$$\tan \theta = -1 \Rightarrow \tan \theta = \tan\left(-\frac{\pi}{4}\right) \Rightarrow \theta = -\frac{\pi}{4}$$

Example 25 What is the polar form of the complex number $(i^{25})^3$?

Solution $z = (i^{25})^3 = (i)^{75} = i^{4 \times 18 + 3} = (i^4)^{18} (i)^3$
 $= i^3 = -i = 0 - i$

Polar form of $z = r(\cos \theta + i \sin \theta)$

$$\begin{aligned} &= 1 \left\{ \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \right\} \\ &= \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \end{aligned}$$

Example 26 What is the locus of z , if amplitude of $z - 2 - 3i$ is $\frac{\pi}{4}$?

Solution Let $z = x + iy$. Then $z - 2 - 3i = (x - 2) + i(y - 3)$

Let θ be the amplitude of $z - 2 - 3i$. Then $\tan \theta = \frac{y-3}{x-2}$

$$\Rightarrow \tan \frac{\pi}{4} = \frac{y-3}{x-2} \left(\text{since } \theta = \frac{\pi}{4} \right)$$

$$\Rightarrow 1 = \frac{y-3}{x-2} \text{ i.e. } x - y + 1 = 0$$

Hence, the locus of z is a straight line.

Example 27 If $1 - i$, is a root of the equation $x^2 + ax + b = 0$, where $a, b \in \mathbf{R}$, then find the values of a and b .

Solution Sum of roots $\frac{-a}{1} = (1 - i) + (1 + i) \Rightarrow a = -2$.

(since non real complex roots occur in conjugate pairs)

Product of roots, $\frac{b}{1} = (1 - i)(1 + i) \Rightarrow b = 2$

Choose the correct options out of given four options in each of the Examples from 28 to 33 (M.C.Q.).

Example 28 $1 + i^2 + i^4 + i^6 + \dots + i^{2n}$ is

- (A) positive (B) negative
(C) 0 (D) can not be evaluated

Solution (D), $1 + i^2 + i^4 + i^6 + \dots + i^{2n} = 1 - 1 + 1 - 1 + \dots (-1)^n$

which can not be evaluated unless n is known.

Example 29 If the complex number $z = x + iy$ satisfies the condition $|z + 1| = 1$, then z lies on

- (A) x -axis
(B) circle with centre $(1, 0)$ and radius 1
(C) circle with centre $(-1, 0)$ and radius 1
(D) y -axis

Solution (C), $|z + 1| = 1 \Rightarrow |(x + 1) + iy| = 1$

$$\Rightarrow (x + 1)^2 + y^2 = 1$$

which is a circle with centre $(-1, 0)$ and radius 1.

Example 30 The area of the triangle on the complex plane formed by the complex numbers z , $-iz$ and $z + iz$ is:

- (A) $|z|^2$ (B) $|\bar{z}|^2$
(C) $\frac{|z|^2}{2}$ (D) none of these

Solution (C), Let $z = x + iy$. Then $-iz = y - ix$. Therefore,

$$z + iz = (x - y) + i(x + y)$$

$$\text{Required area of the triangle} = \frac{1}{2}(x^2 + y^2) = \frac{|z|^2}{2}$$

Example 31 The equation $|z + 1 - i| = |z - 1 + i|$ represents a

- (A) straight line (B) circle
(C) parabola (D) hyperbola

Solution (A), $|z + 1 - i| = |z - 1 + i|$

$$\Rightarrow |z - (-1 + i)| = |z - (1 - i)|$$

$\Rightarrow PA = PB$, where A denotes the point $(-1, 1)$, B denotes the point $(1, -1)$ and P denotes the point (x, y)

$\Rightarrow z$ lies on the perpendicular bisector of the line joining A and B and perpendicular bisector is a straight line.

Example 32 Number of solutions of the equation $z^2 + |z|^2 = 0$ is

- (A) 1 (B) 2
(C) 3 (D) infinitely many

Solution (D), $z^2 + |z|^2 = 0, z \neq 0$

$$\Rightarrow x^2 - y^2 + i2xy + x^2 + y^2 = 0$$

$$\Rightarrow 2x^2 + i2xy = 0 \Rightarrow 2x(x + iy) = 0$$

$$\Rightarrow x = 0 \text{ or } x + iy = 0 \text{ (not possible)}$$

Therefore, $x = 0$ and $z \neq 0$

So y can have any real value. Hence infinitely many solutions.

Example 33 The amplitude of $\sin \frac{\pi}{5} + i(1 - \cos \frac{\pi}{5})$ is

- (A) $\frac{2\pi}{5}$ (B) $\frac{\pi}{5}$ (C) $\frac{\pi}{15}$ (D) $\frac{\pi}{10}$

Solution (D), Here $r \cos \theta = \sin \left(\frac{\pi}{5} \right)$ and $r \sin \theta = 1 - \cos \frac{\pi}{5}$

$$\text{Therefore, } \tan \theta = \frac{1 - \cos \frac{\pi}{5}}{\sin \frac{\pi}{5}} = \frac{2 \sin^2 \left(\frac{\pi}{10} \right)}{2 \sin \left(\frac{\pi}{10} \right) \cdot \cos \left(\frac{\pi}{10} \right)}$$

$$\Rightarrow \tan \theta = \tan \left(\frac{\pi}{10} \right) \text{ i.e., } \theta = \frac{\pi}{10}$$

5.3 EXERCISE

Short Answer Type

1. For a positive integer n , find the value of $(1 - i)^n \left(1 - \frac{1}{i} \right)^n$.
2. Evaluate $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $n \in \mathbf{N}$.
3. If $\left(\frac{1+i}{1-i} \right)^3 - \left(\frac{1-i}{1+i} \right)^3 = x + iy$, then find (x, y) .
4. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.
5. If $\left(\frac{1-i}{1+i} \right)^{100} = a + ib$, then find (a, b) .
6. If $a = \cos \theta + i \sin \theta$, find the value of $\frac{1+a}{1-a}$.
7. If $(1 + i)z = (1 - i)\bar{z}$, then show that $z = -i\bar{z}$.
8. If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in \mathbf{R}$, represents a circle.
9. If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.
10. Show that the complex number z , satisfying the condition $\arg \left(\frac{z-1}{z+1} \right) = \frac{\pi}{4}$ lies on a circle.
11. Solve the equation $|z| = z + 1 + 2i$.

Long Answer Type

12. If $|z+1| = z+2(1+i)$, then find z .
13. If $\arg(z-1) = \arg(z+3i)$, then find $x-1:y$, where $z = x+iy$.
14. Show that $\left| \frac{z-2}{z-3} \right| = 2$ represents a circle. Find its centre and radius.
15. If $\frac{z-1}{z+1}$ is a purely imaginary number ($z \neq -1$), then find the value of $|z|$.
16. z_1 and z_2 are two complex numbers such that $|z_1|=|z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.
17. If $|z_1|=1$ ($z_1 \neq -1$) and $z_2 = \frac{z_1-1}{z_1+1}$, then show that the real part of z_2 is zero.
18. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$.
19. If $|z_1|=|z_2| = \dots = |z_n|=1$, then show that $|z_1+z_2+z_3+\dots+z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \dots + \frac{1}{z_n} \right|$.
20. If for complex numbers z_1 and z_2 , $\arg(z_1) - \arg(z_2) = 0$, then show that $|z_1-z_2| = |z_1|-|z_2|$.
21. Solve the system of equations $\operatorname{Re}(z^2) = 0$, $|z|=2$.
22. Find the complex number satisfying the equation $z + \sqrt{2}|z+1| + i = 0$.
23. Write the complex number $z = \frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$ in polar form.
24. If z and w are two complex numbers such that $|zw|=1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.

Objective Type Questions

25. Fill in the blanks of the following

- (i) For any two complex numbers z_1, z_2 and any real numbers a, b ,
 $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \dots$
- (ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is
- (iii) The number $\frac{(1-i)^3}{1-i^3}$ is equal to
- (iv) The sum of the series $i + i^2 + i^3 + \dots$ upto 1000 terms is
- (v) Multiplicative inverse of $1 + i$ is
- (vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_2 = \dots$
- (vii) $\arg(z) + \arg(\bar{z})$ ($z \neq 0$) is
- (viii) If $|z+4| \leq 3$, then the greatest and least values of $|z+1|$ are and
- (ix) If $\left| \frac{z-2}{z+2} \right| = \frac{\pi}{6}$, then the locus of z is
- (x) If $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$, then $z = \dots$

26. State True or False for the following :

- (i) The order relation is defined on the set of complex numbers.
- (ii) Multiplication of a non zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.
- (iii) For any complex number z the minimum value of $|z| + |z-1|$ is 1.
- (iv) The locus represented by $|z-1| = |z-i|$ is a line perpendicular to the join of (1, 0) and (0, 1).
- (v) If z is a complex number such that $z \neq 0$ and $\operatorname{Re}(z) = 0$, then $\operatorname{Im}(z^2) = 0$.
- (vi) The inequality $|z-4| < |z-2|$ represents the region given by $x > 3$.

(vii) Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1 - z_2) = 0$.

(viii) 2 is not a complex number.

27. Match the statements of Column A and Column B.

Column A

Column B

- | | |
|---|--|
| (a) The polar form of $i + \sqrt{3}$ is | (i) Perpendicular bisector of segment joining $(-2, 0)$ and $(2, 0)$ |
| (b) The amplitude of $-1 + \sqrt{-3}$ is | (ii) On or outside the circle having centre at $(0, -4)$ and radius 3. |
| (c) If $ z + 2 = z - 2 $, then locus of z is | (iii) $\frac{2\pi}{3}$ |
| (d) If $ z + 2i = z - 2i $, then locus of z is | (iv) Perpendicular bisector of segment joining $(0, -2)$ and $(0, 2)$. |
| (e) Region represented by $ z + 4i \geq 3$ is | (v) $2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$ |
| (f) Region represented by $ z + 4 \leq 3$ is | (vi) On or inside the circle having centre $(-4, 0)$ and radius 3 units. |
| (g) Conjugate of $\frac{1+2i}{1-i}$ lies in | (vii) First quadrant |
| (h) Reciprocal of $1 - i$ lies in | (viii) Third quadrant |

28. What is the conjugate of $\frac{2-i}{(1-2i)^2}$?

29. If $|z_1| = |z_2|$, is it necessary that $z_1 = z_2$?

30. If $\frac{(a^2 + 1)^2}{2a - i} = x + iy$, what is the value of $x^2 + y^2$?

31. Find z if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.

32. Find $\left| (1+i) \frac{(2+i)}{(3+i)} \right|$

33. Find principal argument of $(1 + i\sqrt{3})^2$.

34. Where does z lie, if $\left| \frac{z-5i}{z+5i} \right| = 1$.

Choose the correct answer from the given four options indicated against each of the Exercises from 35 to 50 (M.C.Q)

35. $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for:

(A) $x = n\pi$

(B) $x = \left(n + \frac{1}{2}\right) \frac{\pi}{2}$

(C) $x = 0$

(D) No value of x

36. The real value of α for which the expression $\frac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely real is :

(A) $(n+1)\frac{\pi}{2}$

(B) $(2n+1)\frac{\pi}{2}$

(C) $n\pi$

(D) None of these, where $n \in \mathbb{N}$

37. If $z = x + iy$ lies in the third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant if

(A) $x > y > 0$

(B) $x < y < 0$

(C) $y < x < 0$

(D) $y > x > 0$

38. The value of $(z+3)(\bar{z}+3)$ is equivalent to

(A) $|z+3|^2$

(B) $|z-3|$

(C) z^2+3

(D) None of these

39. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

(A) $x = 2n+1$

(B) $x = 4n$

(C) $x = 2n$

(D) $x = 4n + 1$, where $n \in \mathbb{N}$

40. A real value of x satisfies the equation $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta$ ($\alpha, \beta \in \mathbf{R}$)
if $\alpha^2 + \beta^2 =$
(A) 1 (B) -1 (C) 2 (D) -2
41. Which of the following is correct for any two complex numbers z_1 and z_2 ?
(A) $|z_1 z_2| = |z_1| |z_2|$ (B) $\arg(z_1 z_2) = \arg(z_1) \cdot \arg(z_2)$
(C) $|z_1 + z_2| = |z_1| + |z_2|$ (D) $|z_1 + z_2| \geq |z_1| - |z_2|$
42. The point represented by the complex number $2 - i$ is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is:
(A) $1 + 2i$ (B) $-1 - 2i$ (C) $2 + i$ (D) $-1 + 2i$
43. Let $x, y \in \mathbf{R}$, then $x + iy$ is a non real complex number if:
(A) $x = 0$ (B) $y = 0$ (C) $x \neq 0$ (D) $y \neq 0$
44. If $a + ib = c + id$, then
(A) $a^2 + c^2 = 0$ (B) $b^2 + c^2 = 0$
(C) $b^2 + d^2 = 0$ (D) $a^2 + b^2 = c^2 + d^2$
45. The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right| = 1$ lies on
(A) circle $x^2 + y^2 = 1$ (B) the x -axis
(C) the y -axis (D) the line $x + y = 1$.
46. If z is a complex number, then
(A) $|z^2| > |z|^2$ (B) $|z^2| = |z|^2$
(C) $|z^2| < |z|^2$ (D) $|z^2| \geq |z|^2$
47. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if
(A) $z_2 = \bar{z}_1$ (B) $z_2 = \frac{1}{z_1}$
(C) $\arg(z_1) = \arg(z_2)$ (D) $|z_1| = |z_2|$

48. The real value of θ for which the expression $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number is:

(A) $n\pi + \frac{\pi}{4}$

(B) $n\pi + (-1)^n \frac{\pi}{4}$

(C) $2n\pi \pm \frac{\pi}{2}$

(D) none of these.

49. The value of $\arg(x)$ when $x < 0$ is:

(A) 0

(B) $\frac{\pi}{2}$

(C) π

(D) none of these

50. If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is

(A) $\frac{|z|}{2}$

(B) $|z|$

(C) $2|z|$

(D) none of these.

