

Limits and Derivatives

Limit of a Function Using Intuitive Approach

- For a function $f(x)$, if for x closes to a implies that $f(x)$ closes to l , then l is called the **limit** of function $f(x)$ at a .
- l is the limit of function $f(x)$ is written as $\lim_{x \rightarrow a} f(x) = l$ [read as “limit of $f(x)$ is l , when x tends to a ” or “for $x \rightarrow a$ (x tends to a), $f(x) \rightarrow l$ ($f(x)$ tends to l)”]
- If $f(x) = x^3 - 2$, then for x very close to 3, $f(x)$ will be very close to 25. This can be written as $\lim_{x \rightarrow 3} (x^3 - 2) = 25$. So, limiting value of $x^3 - 2$ at x closes to 3 is 25.

Solved Examples

Example 1:

For $f(x) = x(a - 3x)$, find the value of a at which the limits of function $f(x)$ when x tends to 4 and when it tends to 5 are the same?

Solution:

It is given that

$$f(x) = x(a - 3x)$$

$$\Rightarrow f(x) = ax - 3x^2$$

The limit of function $f(x)$ when x tends to 4 is calculated as follows:

x	3.9	3.95	3.99	3.999	4.001	4.01	4.05	4.1
$f(x)$	$3.9a - 45.63$	$3.95a - 46.8075$	$3.99a - 47.7603$	$3.999a - 47.976003$	$4.001a - 48.024003$	$4.01a - 48.2403$	$4.05a - 49.2075$	$4.1a - 50.43$

$$\therefore \lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} (ax - 3x^2) = 4a - 48$$

The limit of function $f(x)$ when x tends to 5 is calculated as follows:

x	4.9	4.95	4.99	4.999	5.001	5.01	5.05	5.1
$f(x)$	$4.9a - 72.03$	$4.95a - 73.5075$	$4.99a - 74.7003$	$4.999a - 74.970003$	$5.001a - 75.030003$	$5.01a - 75.3003$	$5.05a - 76.5075$	$5.1a - 78.03$

$$\therefore \lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} (ax - 3x^2) = 5a - 75$$

We have to find the particular value of a at which the limits of function $f(x)$ when x tends to 4 and when it tends to 5 are equal.

$$\therefore \lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 5} f(x)$$

$$\Rightarrow 4a - 48 = 5a - 75$$

$$\Rightarrow a = 27$$

Thus, the limiting values of $f(x) = x(a - 3x)$ when x tends to 4 and 5 are equal for $a = 27$.

Example 2:

Show that the limit value of $g(y) = [2y - 5]$ does not exist when y tends to 2.

Solution:

The given function is

$$g(y) = [2y - 5].$$

Clearly, $g(y)$ is a greatest integer function

$$\text{Hence, } g(y) = \begin{cases} a-1, & \text{for } a-1 < g(y) < a \\ a, & \text{for } a \leq g(y) < a+1 \end{cases}$$

Where, a is an integer

The limit of $g(y)$ when y tends to 2 is calculated as follows:

y	1.9	1.95	1.99	1.999	2.001	2.01	2.05	2.1
g(y)	-2	-2	-2	-2	-1	-1	-1	-1

We may observe that

$$\text{Left hand limit of the function} = \lim_{y \rightarrow 2^-} g(y) = -2 \quad \text{whereas the right hand limit} = \lim_{y \rightarrow 2^+} g(y) = -1$$

Since the left hand and the right hand limits of the function are not equal, the given function does not have a limiting value.

Example 3:

For what real and complex values of b , $\lim_{t \rightarrow b} v(t) \neq v(b)$, where $v(t) = \frac{(t^4 - 16)(t^2 - 16)}{(t^3 - 1)(2t^2 - t - 28)}$?

Solution:

We know that if a function $v(t)$ is defined at $t = b$, then $\lim_{t \rightarrow b} v(t) = v(b)$, else not.

Since $\lim_{t \rightarrow b} v(t) \neq v(b)$, we need to find the value of b , i.e., t , where $v(t)$ does not exist.

This is only possible, if

$$\begin{aligned}(t^3 - 1)(2t^2 - t - 28) &= 0 \\ \Rightarrow (t - 1)(t^2 + t + 1)(2t^2 - 8t + 7t - 28) &= 0 \\ \Rightarrow (t - 1)(t^2 + t + 1)[2t(t - 4) + 7(t - 4)] &= 0 \\ \Rightarrow (t - 1)(t^2 + t + 1)(t - 4)(2t + 7) &= 0 \\ \Rightarrow t = 1 \text{ or } 4 \text{ or } \frac{-7}{2} \text{ or } \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} \\ \Rightarrow t = 1 \text{ or } 4 \text{ or } \frac{-7}{2} \text{ or } \frac{-1 \pm i\sqrt{3}}{2}\end{aligned}$$

So, for $b = 1, 4, \frac{-7}{2}$ as real values and $b = \frac{-1 \pm i\sqrt{3}}{2}$ as the complex

values, $\lim_{t \rightarrow b} v(t) \neq v(b)$, where $v(t) = \frac{(t^4 - 16)(t^2 - 16)}{(t^3 - 1)(2t^2 - t - 28)}$.

Limit of a Polynomial and a Rational Function

Algebra of Limits

- If f and g are two functions such that both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

- $$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

The limit of the sum of two functions is the sum of the limits of the functions.

$$\lim_{x \rightarrow 4} \left(x^{\frac{5}{2}} + x^{\frac{3}{2}} \right) = \lim_{x \rightarrow 4} x^{\frac{5}{2}} + \lim_{x \rightarrow 4} x^{\frac{3}{2}} = 4^{\frac{5}{2}} + 4^{\frac{3}{2}} = 32 + 8 = 40$$

For example,

- $$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

The limit of the difference between two functions is the difference between the limits of the functions.

$$\lim_{x \rightarrow 4} \left(x^{\frac{5}{2}} - x^{\frac{3}{2}} \right) = \lim_{x \rightarrow 4} x^{\frac{5}{2}} - \lim_{x \rightarrow 4} x^{\frac{3}{2}} = 4^{\frac{5}{2}} - 4^{\frac{3}{2}} = 32 - 8 = 24$$

For example,

- $$\lim_{x \rightarrow a} [f(x).g(x)] = \lim_{x \rightarrow a} f(x). \lim_{x \rightarrow a} g(x)$$

The limit of the product of two functions is the product of the limits of the functions.

$$\lim_{x \rightarrow 4} \left(x^{\frac{5}{2}} . x^{\frac{3}{2}} \right) = \lim_{x \rightarrow 4} x^{\frac{5}{2}} . \lim_{x \rightarrow 4} x^{\frac{3}{2}} = 4^{\frac{5}{2}} \times 4^{\frac{3}{2}} = 32 \times 8 = 256$$

For example,

- $$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ where } \lim_{x \rightarrow a} g(x) \neq 0$$

The limit of the quotient of the two functions is the quotient of the limits of the functions, where the denominator is not zero.

$$\lim_{x \rightarrow 4} \frac{x^{\frac{5}{2}}}{x^{\frac{3}{2}}} = \frac{\lim_{x \rightarrow 4} x^{\frac{5}{2}}}{\lim_{x \rightarrow 4} x^{\frac{3}{2}}} = \frac{4^{\frac{5}{2}}}{4^{\frac{3}{2}}} = \frac{32}{8} = 4$$

For example,

- $$\lim_{x \rightarrow a} [k.f(x)] = k \lim_{x \rightarrow a} f(x), \text{ where } k \text{ is a constant}$$

The limit of the product of a constant and a function is the product of the constant and the limit of that function.

$$\lim_{x \rightarrow 4} \left(\frac{9}{2} x^{\frac{5}{2}} \right) = \frac{9}{2} \lim_{x \rightarrow 4} x^{\frac{5}{2}} = \frac{9}{2} \times 4^{\frac{5}{2}} = \frac{9}{2} \times 32 = 144$$

For example,

Limit of a Polynomial Function

$$p(x) = \sum_{i=0}^n a_i x^i$$

- A function $p(x)$ is said to be a polynomial function if $p(x) = 0$ or R and $a_r \neq 0$ for some whole number r .
- The limit of a polynomial function $p(x)$ at $x = a$ is given by $\lim_{x \rightarrow a} p(x) = p(a)$

For example, the value of $\lim_{m \rightarrow n+3} (3m^3 - 9m^2n + 9mn^2 - 3n^3 - m + n - 80)$ can be calculated as follows:

$$\begin{aligned} & \lim_{m \rightarrow n+3} (3m^3 - 9m^2n + 9mn^2 - 3n^3 - m + n - 80) \\ &= \lim_{m \rightarrow n+3} [3(m^3 - 3m^2n + 3mn^2 - n^3) - (m - n) - 80] \\ &= \lim_{m \rightarrow n+3} [3(m - n)^3 - (m - n) - 80] \\ &= [3(3)^3 - (3) - 80] \\ &= 81 - 3 - 80 \\ &= -2 \end{aligned}$$

Limit of a Rational Function

$$p(x) = \frac{q(x)}{r(x)}$$

- A function $p(x)$ is said to be a rational function if $p(x) = \frac{q(x)}{r(x)}$, where $q(x)$ and $r(x)$ are polynomials such that $r(x) \neq 0$.

$$p(x) = \frac{q(x)}{r(x)}$$

- The limit of a rational function $p(x)$ of the form $\frac{q(x)}{r(x)}$ at $x = a$ is given by

$$\lim_{x \rightarrow a} p(x) = \frac{q(a)}{r(a)}$$

- For example, to find the value of $\lim_{x \rightarrow 64} \frac{\sqrt{x} + 7}{\sqrt[3]{x} + 2}$, we may proceed as follows.

$$\lim_{x \rightarrow 64} \frac{\sqrt{x} + 7}{\sqrt[3]{x} + 2} = \frac{\sqrt{64} + 7}{\sqrt[3]{64} + 2} = \frac{8 + 7}{4 + 2} = \frac{15}{6} = \frac{5}{2}$$

- For any positive integer n , $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$

- To know its proof, let us look at the following video.

- For example, $\lim_{y \rightarrow 0} \frac{(y+5)^4 - 625}{y}$ can be calculated as follows.

$$\begin{aligned} \lim_{y \rightarrow 0} \frac{(y+5)^4 - 625}{y} &= \lim_{y+5 \rightarrow 5} \frac{(y+5)^4 - 5^4}{(y+5) - 5} && (y \rightarrow 0 \text{ shows that } y+5 \rightarrow 5) \\ &= 4 \times 5^{4-1} \\ &= 500 \end{aligned}$$

Solved Examples

Example 1:

Find the values of a and b if

$$\lim_{n \rightarrow \infty} \frac{3a.(n+5)! - 2b.(n+4)!}{b.(n+5)! + a.(n+4)!} = -2 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{(a+2b).(n+1)! - b.(n-1)!}{(2a-b+1).(n+1)! - a.(n-1)!} = \frac{-1}{2}$$

Also, show that $\lim_{x \rightarrow 1} \frac{a+2b}{x} = \lim_{x \rightarrow \frac{-3}{2}} \frac{b-a}{x^2-1}$.

Solution:

We have $\lim_{n \rightarrow \infty} \frac{3a.(n+5)! - 2b.(n+4)!}{b.(n+5)! + a.(n+4)!} = -2$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{[3a.(n+5) - 2b](n+4)!}{[b.(n+5) + a](n+4)!} = -2$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{3an + 15a - 2b}{bn + 5b + a} = -2$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n \left(3a + \frac{15a - 2b}{n} \right)}{n \left(b + \frac{5b + a}{n} \right)} = -2$$

$$\Rightarrow \frac{\lim_{n \rightarrow \infty} \left(3a + \frac{15a - 2b}{n} \right)}{\lim_{n \rightarrow \infty} \left(b + \frac{5b + a}{n} \right)} = -2$$

$$\Rightarrow \frac{3a + 0}{b + 0} = -2$$

$$\Rightarrow 3a = -2b$$

$$\Rightarrow a = \frac{-2b}{3}$$

... (1)

We also have $\lim_{n \rightarrow \infty} \frac{(a+2b).(n+1)! - b.(n-1)!}{(2a-b+1).(n+1)! - a.(n-1)!} = \frac{-1}{2}$

$$\begin{aligned}
&\Rightarrow \lim_{n \rightarrow \infty} \frac{[(a+2b).n(n+1)-b](n-1)!}{[(2a-b+1).n(n+1)-a](n-1)!} = \frac{-1}{2} \\
&\Rightarrow \lim_{n \rightarrow \infty} \frac{(a+2b)n^2 + (a+2b)n - b}{(2a-b+1)n^2 + (2a-b+1)n - a} = \frac{-1}{2} \\
&\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2 \left[(a+2b) + \frac{(a+2b)}{n} - \frac{b}{n^2} \right]}{n^2 \left[(2a-b+1) + \frac{(2a-b+1)}{n} - \frac{a}{n^2} \right]} = \frac{-1}{2} \\
&\Rightarrow \frac{\lim_{n \rightarrow \infty} \left[(a+2b) + \frac{(a+2b)}{n} - \frac{b}{n^2} \right]}{\lim_{n \rightarrow \infty} \left[(2a-b+1) + \frac{(2a-b+1)}{n} - \frac{a}{n^2} \right]} = \frac{-1}{2} \\
&\Rightarrow \frac{a+2b}{2a-b+1} = \frac{-1}{2} \\
&\Rightarrow \frac{\frac{-2b}{3} + 2b}{2 \times \frac{-2b}{3} - b + 1} = \frac{-1}{2} \quad \text{[Using equation (1)]} \\
&\Rightarrow \frac{4b}{-7b+3} = \frac{-1}{2} \\
&\Rightarrow 8b = 7b - 3 \\
&\Rightarrow b = -3
\end{aligned}$$

Substituting the value of b in equation (1), we obtain

$$a = 2$$

Hence, $a = 2$ and $b = -3$

Now,

$$\lim_{x \rightarrow 1} \frac{a+2b}{x} = \frac{2+2(-3)}{1} = -4$$

and

$$\lim_{x \rightarrow \frac{-3}{2}} \frac{b-a}{x^2-1} = \frac{(-3)-2}{\left(\frac{-3}{2}\right)^2-1} = \frac{-5}{\frac{5}{4}} = -4$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{a+2b}{x} = \lim_{x \rightarrow \frac{-3}{2}} \frac{b-a}{x^2-1}.$$

Example 2:

Find the value of n , such that $\lim_{a \rightarrow b-3} \frac{(a-b)^{2n}-9^n}{(a-b)^{3n}+27^n} = -\frac{2}{729}$, where n is an odd number.

Solution:

$$\lim_{a \rightarrow b-3} \frac{(a-b)^{2n}-9^n}{(a-b)^{3n}+27^n} = -\frac{2}{729}$$

$$\Rightarrow \lim_{a-b \rightarrow -3} \frac{(a-b)^{2n}-3^{2n}}{(a-b)^{3n}+3^{3n}} = -\frac{2}{729} \quad (a \rightarrow b-3 \Rightarrow a-b \rightarrow -3)$$

$$\Rightarrow \lim_{a-b \rightarrow -3} \frac{(a-b)^{2n}-(-3)^{2n}}{(a-b)^{3n}-(-3)^{3n}} = -\frac{2}{729} \quad (\text{Since } n \text{ is an odd number, } (-3)^{2n} = 3^{2n} \text{ and } (-3)^{3n} = -3^{3n})$$

$$\Rightarrow \frac{\lim_{a-b \rightarrow -3} \frac{(a-b)^{2n}-(-3)^{2n}}{(a-b)-(-3)}}{\lim_{a-b \rightarrow -3} \frac{(a-b)^{3n}-(-3)^{3n}}{(a-b)-(-3)}} = -\frac{2}{729}$$

$$\Rightarrow \frac{2n(-3)^{2n-1}}{3n(-3)^{3n-1}} = -\frac{2}{729}$$

$$\Rightarrow \frac{2}{3(-3)^n} = -\frac{2}{729}$$

$$\Rightarrow (-3)^n = \frac{-729}{3}$$

$$\Rightarrow (-3)^n = -243 = (-3)^5$$

$$\Rightarrow n = 5$$

Example 3:

Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{4+x^3} - \sqrt{4+x}}{\sqrt{9+x^7} - \sqrt{9+x}}$

Solution:

$$\lim_{x \rightarrow 0} \frac{\sqrt{4+x^3} - \sqrt{4+x}}{\sqrt{9+x^7} - \sqrt{9+x}} = \frac{0}{0} \text{ form}$$

Hence,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{4+x^3} - \sqrt{4+x}}{\sqrt{9+x^7} - \sqrt{9+x}} \\ &= \lim_{x \rightarrow 0} \left(\left(\sqrt{4+x^3} - \sqrt{4+x} \right) \times \frac{1}{\sqrt{9+x^7} - \sqrt{9+x}} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\left(\sqrt{4+x^3} - \sqrt{4+x} \right) \left(\sqrt{4+x^3} + \sqrt{4+x} \right)}{\sqrt{4+x^3} + \sqrt{4+x}} \times \frac{\sqrt{9+x^7} + \sqrt{9+x}}{\left(\sqrt{9+x^7} - \sqrt{9+x} \right) \left(\sqrt{9+x^7} + \sqrt{9+x} \right)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{(4+x^3) - (4+x)}{\sqrt{4+x^3} + \sqrt{4+x}} \times \frac{\sqrt{9+x^7} + \sqrt{9+x}}{(9+x^7) - (9+x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{x(x^2-1)}{\sqrt{4+x^3} + \sqrt{4+x}} \times \frac{\sqrt{9+x^7} + \sqrt{9+x}}{x(x^6-1)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{x^2-1}{x^6-1} \times \frac{\sqrt{9+x^7} + \sqrt{9+x}}{\sqrt{4+x^3} + \sqrt{4+x}} \right) \\ &= \lim_{x \rightarrow 0} \frac{x^2-1}{x^6-1} \times \lim_{x \rightarrow 0} \frac{\sqrt{9+x^7} + \sqrt{9+x}}{\sqrt{4+x^3} + \sqrt{4+x}} \\ &= \frac{-1}{-1} \times \frac{3+3}{2+2} \\ &= \frac{3}{2} \end{aligned}$$

Limits of Trigonometric Functions

- Let f and g be two real-valued functions with the same domain, such that $f(x) \leq g(x)$ for all x in the domain of definition. For some a , if both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$.
- For example, we know that $x^2 \leq x^3$, for $x \in \mathbf{R}$ and $x \geq 1$. So, for any $a \in \mathbf{R}$ and $a \geq 1$, $\lim_{x \rightarrow a} x^2 \leq \lim_{x \rightarrow a} x^3$.
- Two important limits are

- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$
- To know the proofs of the above two formulae, let us look at the following video.

Solved Examples

Example 1:

$$\lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} \sin\left(\frac{\pi}{2} - x\right) + \sin(\pi + x)}{3\pi\left(\frac{\pi}{3} - x\right)}$$

Evaluate

Solution

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} \sin\left(\frac{\pi}{2} - x\right) + \sin(\pi + x)}{3\pi\left(\frac{\pi}{3} - x\right)} &= \lim_{\frac{\pi}{3} - x \rightarrow 0} \frac{\sqrt{3} \cos x - \sin x}{3\pi\left(\frac{\pi}{3} - x\right)} \\ &= \frac{1}{3\pi} \cdot \lim_{\frac{\pi}{3} - x \rightarrow 0} \frac{2 \cdot \left[\frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x \right]}{\frac{\pi}{3} - x} \\ &= \frac{2}{3\pi} \cdot \lim_{\frac{\pi}{3} - x \rightarrow 0} \frac{\left[\sin \frac{\pi}{3} \cos x - \cos \frac{\pi}{3} \sin x \right]}{\frac{\pi}{3} - x} \\ &= \frac{2}{3\pi} \cdot \lim_{\frac{\pi}{3} - x \rightarrow 0} \frac{\sin\left(\frac{\pi}{3} - x\right)}{\frac{\pi}{3} - x} \\ &= \frac{2}{3\pi} \times 1 \\ &= \frac{2}{3\pi} \end{aligned}$$

Example 2:

If $\lim_{x \rightarrow 0} \frac{\cos 4x - \sin\left(\frac{\pi}{2} + 5x\right)}{x^2} = \frac{3a+b}{2}$ and $\lim_{x \rightarrow 0} \frac{\sin\left(\frac{\pi}{4} + 5x\right) - \sin\left(\frac{\pi}{4} + 3x\right)}{x} = \sqrt{4b-5a}$, then find the value of $\sqrt{5a+2b}$.

Solution:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\cos 4x - \sin\left(\frac{\pi}{2} + 5x\right)}{x^2} &= \frac{3a+b}{2} \\
 \Rightarrow \frac{3a+b}{2} &= \lim_{x \rightarrow 0} \frac{\cos 4x - \cos 5x}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin\left(\frac{5x+4x}{2}\right) \sin\left(\frac{5x-4x}{2}\right)}{x^2} \\
 &= 2 \lim_{x \rightarrow 0} \frac{\sin \frac{9x}{2} \cdot \sin \frac{x}{2}}{x^2} \\
 &= 2 \lim_{x \rightarrow 0} \frac{\sin \frac{9x}{2}}{x} \times \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x} \\
 &= 2 \times \frac{9}{2} \lim_{\frac{9x}{2} \rightarrow 0} \frac{\sin \frac{9x}{2}}{\frac{9x}{2}} \times \frac{1}{2} \cdot \lim_{\frac{x}{2} \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \\
 &= 9 \times 1 \times \frac{1}{2} \times 1 \\
 &= \frac{9}{2} \\
 \Rightarrow 3a + b &= 9 \\
 \Rightarrow b &= 9 - 3a \dots (1)
 \end{aligned}$$

It is also given that

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\sin\left(\frac{\pi}{4} + 5x\right) - \sin\left(\frac{\pi}{4} + 3x\right)}{x} &= \sqrt{4b - 5a} \\
\Rightarrow \sqrt{4b - 5a} &= \lim_{x \rightarrow 0} \frac{2 \sin\left[\frac{\left(\frac{\pi}{4} + 5x\right) - \left(\frac{\pi}{4} + 3x\right)}{2}\right] \cdot \cos\left[\frac{\left(\frac{\pi}{4} + 5x\right) + \left(\frac{\pi}{4} + 3x\right)}{2}\right]}{x} \\
&= 2 \lim_{x \rightarrow 0} \frac{\sin x \cdot \cos\left(\frac{\pi}{4} + 4x\right)}{x} \\
&= 2 \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \cos\left(\frac{\pi}{4} + 4x\right) \\
&= 2 \times 1 \times \frac{1}{\sqrt{2}} \\
&= \sqrt{2}
\end{aligned}$$

$$\Rightarrow 4b - 5a = 2$$

From (1), we have

$$4(9 - 3a) - 5a = 2$$

$$36 - 17a = 2$$

$$17a = 34$$

$$a = 2$$

Substituting $a = 2$ in equation (1), we obtain $b = 3$

$$\text{Now, } \sqrt{5a + 2b} = \sqrt{5 \times 2 + 2 \times 3} = \sqrt{16} = 4$$

Derivative of a Function

- Suppose f is a real-valued function and a is a point in the domain of definition. If the limit $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists, then it is called the derivative of f at a . The derivative of f at a is denoted by $f'(a)$.

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

- Suppose f is a real-valued function. The derivative of f {denoted by $f'(x)$ or $\frac{d}{dx}[f(x)]$ } is defined by

$$\frac{d}{dx}[f(x)] = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

This definition of derivative is called the **first principle** of derivative.

- For example, the derivative of $y = (ax - b)^{10}$ is calculated as follows.

We have $y = f(x) = (ax - b)^{10}$; using the first principle of derivative, we obtain

$$\begin{aligned} \frac{dy}{dx} = f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[a(x+h) - b]^{10} - (ax - b)^{10}}{h} \\ &= \lim_{h \rightarrow 0} \frac{[a(x+h) - b - (ax - b)] \cdot \sum_{r=0}^9 [a(x+h) - b]^{9-r} (ax - b)^r}{h} \\ &= \lim_{h \rightarrow 0} \frac{ah}{h} \cdot \lim_{h \rightarrow 0} \sum_{r=0}^9 [a(x+h) - b]^{9-r} (ax - b)^r \\ &= a \sum_{r=0}^9 (ax - b)^{9-r} \cdot (ax - b)^r \\ &= a[(ax - b)^{9-0} \cdot (ax - b)^0 + (ax - b)^{9-1} \cdot (ax - b)^1 + \dots + (ax - b)^{9-9} \cdot (ax - b)^9] \\ &= 10a(ax - b)^9 \end{aligned}$$

Solved Examples

Example 1:

Find the derivative of $f(x) = \operatorname{cosec}^2 2x + \tan^2 4x$. Also, find $f'(x)$ at $x = \frac{\pi}{6}$.

Solution:

The derivative of $f(x) = \operatorname{cosec}^2 2x + \tan^2 4x$ is calculated as follows.

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{\operatorname{cosec}^2 2(x+h) + \tan^2 4(x+h) - [\operatorname{cosec}^2 2(x) + \tan^2 4(x)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{[\operatorname{cosec}^2(2x+2h) - \operatorname{cosec}^2 2x] + [\tan^2(4x+4h) - \tan^2(4x)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{\sin^2(2x+2h)} - \frac{1}{\sin^2 2x} \right) + \left(\frac{\sin^2(4x+4h)}{\cos^2(4x+4h)} - \frac{\sin^2 4x}{\cos^2 4x} \right)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\left(\frac{\sin^2 2x - \sin^2(2x+2h)}{\sin^2 2x \sin^2(2x+2h)} \right) + \left(\frac{\sin^2(4x+4h)\cos^2 4x - \cos^2(4x+4h)\sin^2 4x}{\cos^2 4x \cos^2(4x+4h)} \right)}{h} \\
&= \lim_{h \rightarrow 0} \frac{[\sin 2x - \sin(2x+2h)][\sin 2x + \sin(2x+2h)]}{h \sin^2 2x \sin^2(2x+2h)} \\
&\quad + \lim_{h \rightarrow 0} \frac{[\sin(4x+4h)\cos 4x - \cos(4x+4h)\sin 4x][\sin(4x+4h)\cos 4x + \cos(4x+4h)\sin 4x]}{h \cos^2 4x \cos^2(4x+4h)} \\
&= \lim_{h \rightarrow 0} \frac{2 \cos(2x+h) \sin(-h) \times 2 \sin(2x+h) \cos(-h)}{h \sin^2 2x \sin^2(2x+2h)} + \lim_{h \rightarrow 0} \frac{\sin(4x+4h-4x) \sin(4x+4h+4x)}{h \cos^2 4x \cos^2(4x+4h)} \\
&= -4 \lim_{h \rightarrow 0} \frac{\sin h}{h} \times \lim_{h \rightarrow 0} \frac{\cos(2x+h) \times \sin(2x+h) \cos(h)}{\sin^2 2x \sin^2(2x+2h)} + 4 \lim_{4h \rightarrow 0} \frac{\sin(4h)}{4h} \times \lim_{h \rightarrow 0} \frac{\sin(4x+4h+4x)}{\cos^2 4x \cos^2(4x+4h)} \\
&= -4 \times 1 \times \frac{\cos 2x}{\sin^3 2x} + 4 \times 1 \times \frac{\sin 8x}{\cos^4 4x} \\
&= -4 \cot 2x \operatorname{cosec}^2 2x + \frac{8 \sin 4x \cos 4x}{\cos^4 4x} \\
&= -4 \cot 2x \operatorname{cosec}^2 2x + 8 \tan 4x \sec^2 4x
\end{aligned}$$

At $x = \frac{\pi}{6}$, $f'\left(\frac{\pi}{6}\right)$ is given by

$$\begin{aligned}
f'\left(\frac{\pi}{6}\right) &= -4 \cot\left(\frac{\pi}{3}\right) \operatorname{cosec}^2\left(\frac{\pi}{3}\right) + 8 \tan\left(\frac{2\pi}{3}\right) \sec^2\left(\frac{2\pi}{3}\right) \\
&= -4 \times \frac{1}{\sqrt{3}} \times \left(\frac{2}{\sqrt{3}}\right)^2 + 8(-\sqrt{3}) \times (-2)^2 \\
&= \frac{-16}{3\sqrt{3}} - 32\sqrt{3} \\
&= \frac{-304}{3\sqrt{3}}
\end{aligned}$$

Example 2:

If $y = (ax^2 + x + b)^2$, then find the values of a and b , such that $\frac{dy}{dx} = 4x^2(4x+3) + 2(13x+3)$.

Solution:

It is given that $y = (ax^2 + x + b)^2$

$$\begin{aligned}
 \Rightarrow \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{[a(x+h)^2 + (x+h) + b]^2 - [ax^2 + x + b]^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[a(x+h)^2 + (x+h) + b - (ax^2 + x + b)][a(x+h)^2 + (x+h) + b + (ax^2 + x + b)]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[a(2xh + h^2) + h][a(x+h)^2 + (x+h) + b + (ax^2 + x + b)]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h[a(2x+h) + 1]}{h} \times \lim_{h \rightarrow 0} [a(x+h)^2 + (x+h) + b + (ax^2 + x + b)] \\
 &= (2ax + 1) \times 2(ax^2 + x + b) \\
 &= 4a^2x^3 + 6ax^2 + (4ab + 2)x + 2b \\
 \Rightarrow 4x^2(4x+3) + 2(13x+3) &= 4a^2x^3 + 6ax^2 + (4ab + 2)x + 2b \\
 \Rightarrow 4a^2x^3 + 6ax^2 + (4ab + 2)x + 2b &= 16x^3 + 12x^2 + 26x + 6
 \end{aligned}$$

Comparing the coefficients of x^3 , x^2 , x , and the constant terms of the above expression, we obtain

$$4a^2 = 16, 6a = 12, 4ab + 2 = 26 \text{ and } 2b = 6$$

$$\Rightarrow a = \pm 2, a = 2, b = 3 \text{ and } b = 3$$

$$\Rightarrow a = 2 \text{ and } b = 3$$

Example 3:

What is the derivative of y with respect to x , if $y = \sqrt{\frac{ax+b}{cx-d}}$?

Solution:

It is given that $y = \sqrt{\frac{ax+b}{cx-d}}$

$$\begin{aligned}
\Rightarrow \frac{dy}{dx} &= \frac{d}{dx} \left(\sqrt{\frac{ax+b}{cx-d}} \right) \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{\frac{a(x+h)+b}{c(x+h)-d}} - \sqrt{\frac{ax+b}{cx-d}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{[a(x+h)+b](cx-d)} - \sqrt{[c(x+h)-d][ax+b]}}{h\sqrt{[c(x+h)-d][cx-d]}} \\
&\quad \left(\sqrt{[a(x+h)+b](cx-d)} - \sqrt{[c(x+h)-d][ax+b]} \right) \times \\
&\quad \left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]} \right) \\
&= \lim_{h \rightarrow 0} \frac{h \left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]} \right)}{h \left(\sqrt{[c(x+h)-d][cx-d]} \right) \left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]} \right)} \\
&= \lim_{h \rightarrow 0} \frac{[a(x+h)+b](cx-d) - [c(x+h)-d][ax+b]}{h \left(\sqrt{[c(x+h)-d][cx-d]} \right) \left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]} \right)} \\
&= \lim_{h \rightarrow 0} \frac{h[a(cx-d) - c(ax+b)]}{h \left(\sqrt{[c(x+h)-d][cx-d]} \right) \left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]} \right)} \\
&= \frac{a(cx-d) - c(ax+b)}{\left(\sqrt{[cx-d][cx-d]} \right) \left(\sqrt{(ax+b)(cx-d)} + \sqrt{(cx-d)(ax+b)} \right)} \\
&= \frac{-(ad+bc)}{2(cx-d)\sqrt{(ax+b)(cx-d)}}
\end{aligned}$$

Derivatives of Trigonometric and Polynomial Functions

Derivatives of Trigonometric Functions and Standard Formulas

- $\frac{d}{dx}(\sin x) = \cos x$

- $\frac{d}{dx}(\cos x) = -\sin x$

- $\frac{d}{dx}(x^n) = nx^{n-1}$

For example, $\frac{d}{dx}(x^7) = 7x^{7-1} = 7x^6$

- $\frac{d}{dx}(C) = 0$, where C is a constant

Algebra of Derivatives

- If f and g are two functions such that their derivatives are defined in a common domain, then

- $$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

This means that the derivative of the sum of two functions is the sum of the derivatives of the functions.

For example,

$$\frac{d}{dx}\left(x^{\frac{5}{2}} + x^{\frac{3}{2}}\right) = \frac{d}{dx}\left(x^{\frac{5}{2}}\right) + \frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{5}{2}x^{\frac{5}{2}-1} + \frac{3}{2}x^{\frac{3}{2}-1} = \frac{5}{2}x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}$$

- $$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

This means that the derivative of the difference between two functions is the difference between the derivatives of the function.

For example,

$$\frac{d}{dx}\left(\sin x - x^{\frac{1}{3}}\right) = \frac{d}{dx}(\sin x) - \frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \cos x - \frac{1}{3}x^{\frac{1}{3}-1} = \cos x - \frac{1}{3}x^{-\frac{2}{3}}$$

- $$\frac{d}{dx}[f(x).g(x)] = \frac{d}{dx}f(x).g(x) + f(x).\frac{d}{dx}g(x)$$

This is known as the **product** rule of derivative.
For

example,

$$\frac{d}{dx}(x^3 \cos x) = \frac{d}{dx}(x^3) \cdot \cos x + (x^3) \cdot \frac{d}{dx}(\cos x) = 3x^2 \cos x + x^3(-\sin x) = 3x^2 \cos x - x^3 \sin x$$

- $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}f(x).g(x) - f(x).\frac{d}{dx}g(x)}{[g(x)]^2}, \text{ where } \frac{d}{dx}g(x) \neq 0$$

This is known as the **quotient** rule of derivative.

For example,

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\&= \frac{\frac{d}{dx}(\sin x) \cdot \cos x - \sin x \cdot \frac{d}{dx}(\cos x)}{(\cos x)^2} \\&= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\&= \frac{1}{\cos^2 x} \\&= \sec^2 x\end{aligned}$$

- $\frac{d}{dx}[k \cdot f(x)] = k \frac{d}{dx} f(x)$, where k is a constant

This means that the derivative of the product of a constant and a function is the product of that constant and the derivative of that function.

For example,

$$\begin{aligned}\frac{d}{dx}(\sin 2x) &= \frac{d}{dx}(2 \sin x \cdot \cos x) \\&= 2 \frac{d}{dx}(\sin x \cdot \cos x) \\&= 2 \left(\frac{d}{dx}(\sin x) \cdot \cos x + \sin x \cdot \frac{d}{dx}(\cos x) \right) \\&= 2[\cos x \cdot \cos x + \sin x \cdot (-\sin x)] \\&= 2(\cos^2 x - \sin^2 x) \\&= 2 \cos 2x\end{aligned}$$

Derivative of a Polynomial Function

- A function $p(x)$ is said to be a polynomial function if $p(x) = 0$ or $p(x) = \sum_{r=0}^n a_r x^r$, where $a_r \in \mathbb{R}$ and $a_r \neq 0$ for some whole number r .

- The derivative of a polynomial function $p(x) = \sum_{r=0}^n a_r x^r$ is given by

$$\frac{d}{dx}[p(x)] = \sum_{r=1}^n r a_r x^{r-1}$$

Solved Examples

Example 1:

If $y = \left(\sqrt{\frac{1+\cos 2x}{1-\cos 2x}} + \sqrt{\sec^2 x - 1} \right)^{-1} + (1+x)^n$, then show that $\frac{dy}{dx} - n(1+x)^{n-1} = \cos 2x$.

Solution:

We have

$$\begin{aligned} y &= \left(\sqrt{\frac{1+\cos 2x}{1-\cos 2x}} + \sqrt{\sec^2 x - 1} \right)^{-1} + (1+x)^n \\ &= \left(\sqrt{\frac{\cos^2 x}{\sin^2 x}} + \sqrt{\tan^2 x} \right)^{-1} + \sum_{r=0}^n {}^n C_r x^r \\ &= (\cot x + \tan x)^{-1} + \left(1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 \dots + \frac{n(n-1)\dots 2}{(n-1)!} x^{n-1} + \frac{n(n-1)\dots 1}{n!} x^n \right) \\ &= \left(\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \right)^{-1} + \left(1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 \dots + \frac{n(n-1)\dots 2}{(n-1)!} x^{n-1} + \frac{n(n-1)\dots 1}{n!} x^n \right) \\ &= \left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \right)^{-1} + \left(1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 \dots + \frac{n(n-1)\dots 2}{(n-1)!} x^{n-1} + \frac{n(n-1)\dots 1}{n!} x^n \right) \\ &= \sin x \cdot \cos x + \left(1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 \dots + \frac{n(n-1)\dots 2}{(n-1)!} x^{n-1} + \frac{n(n-1)\dots 1}{n!} x^n \right) \end{aligned}$$

Hence,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x \cdot \cos x) + \frac{d}{dx} \left(1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 \dots + \frac{n(n-1)\dots 2}{(n-1)!} x^{n-1} + \frac{n(n-1)\dots 1}{n!} x^n \right)$$

Now,

$$\begin{aligned}\frac{d}{dx}(\sin x \cdot \cos x) &= \frac{d}{dx}(\sin x) \cdot \cos x + \sin x \cdot \frac{d}{dx}(\cos x) \\ &= \cos x \cdot \cos x + \sin x(-\sin x) \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x\end{aligned}$$

$$\begin{aligned}&\frac{d}{dx}\left(1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \dots + \frac{n(n-1)\dots 2}{(n-1)!}x^{n-1} + \frac{n(n-1)\dots 1}{n!}x^n\right) \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(nx) + \frac{d}{dx}\left(\frac{n(n-1)}{2!}x^2\right) + \frac{d}{dx}\left(\frac{n(n-1)(n-2)}{3!}x^3\right) \dots + \frac{d}{dx}\left(\frac{n(n-1)\dots 2}{(n-1)!}x^{n-1}\right) + \frac{d}{dx}\left(\frac{n(n-1)\dots 1}{n!}x^n\right) \\ &= 0 + n\frac{d}{dx}(x) + \frac{n(n-1)}{2!}\frac{d}{dx}(x^2) + \frac{n(n-1)(n-2)}{3!}\frac{d}{dx}(x^3) + \dots + \frac{n(n-1)\dots 2}{(n-1)!}\frac{d}{dx}(x^{n-1}) + \frac{n(n-1)\dots 1}{n!}\frac{d}{dx}(x^n) \\ &= n + \frac{2n(n-1)}{2!}x + \frac{3n(n-1)(n-2)}{3!}x^2 + \dots + \frac{(n-1)n(n-1)\dots 2}{(n-1)!}(x^{n-2}) + \frac{nn(n-1)\dots 1}{n!}(x^{n-1}) \\ &= n\left(1 + (n-1)x + \frac{(n-1)(n-2)}{2!}x^2 + \dots + \frac{(n-1)(n-2)\dots 2}{(n-2)!}x^{n-2} + \frac{(n-1)(n-2)\dots 1}{(n-1)!}x^{n-1}\right) \\ &= n(1+x)^{n-1}\end{aligned}$$

Hence,

$$\begin{aligned}\frac{dy}{dx} &= \cos 2x + n(1+x)^n \\ \Rightarrow \cos 2x &= \frac{dy}{dx} - n(1+x)^n\end{aligned}$$

Example 2:

Find $\frac{dy}{dx}$ if $y = \frac{2x^7 + 3 + \tan x}{x(\sin x - \cos x)}$.

Solution:

$$\begin{aligned}y &= \frac{2x^7 + 3 + \tan x}{x(\sin x - \cos x)} = \frac{2x^7 + 3 + \tan x}{(x \sin x - x \cos x)} \\ \Rightarrow \frac{dy}{dx} &= \frac{(2x^7 + 3 + \tan x)' \cdot (x \sin x - x \cos x) - (2x^7 + 3 + \tan x) \cdot (x \sin x - x \cos x)'}{(x \sin x - x \cos x)^2} \quad \dots (1) \\ &\because \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}\end{aligned}$$

Now,

$$\begin{aligned}
 (2x^7 + 3 + \tan x)' &= \frac{d}{dx}(2x^7 + 3 + \tan x) \\
 &= 2 \frac{d}{dx}(x^7) + \frac{d}{dx}(3) + \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
 &= 2 \times 7x^6 + 0 + \frac{\frac{d}{dx}(\sin x) \cdot \cos x - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x} \\
 &= 14x^6 + \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
 &= 14x^6 + \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
 &= 14x^6 + \sec^2 x
 \end{aligned}$$

$$\begin{aligned}
 (x \sin x - x \cos x)' &= \frac{d}{dx}(x \sin x - x \cos x) \\
 &= \frac{d}{dx}(x \sin x) - \frac{d}{dx}(x \cos x) \\
 &= \frac{d}{dx}(x) \cdot (\sin x) + x \cdot \frac{d}{dx}(\sin x) - \left(\frac{d}{dx}(x) \cdot (\cos x) + x \cdot \frac{d}{dx}(\cos x) \right) \quad [(uv)' = u'v + uv'] \\
 &= \sin x + x \cos x - [\cos x + x(-\sin x)] \\
 &= (1 + x) \sin x + (x - 1) \cos x
 \end{aligned}$$

On substituting all the values in equation (1), we obtain

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(2x^7 + 3 + \tan x)' \cdot (x \sin x - x \cos x) - (2x^7 + 3 + \tan x) \cdot (x \sin x - x \cos x)'}{(x \sin x - x \cos x)^2} \\
 &= \frac{(14x^6 + \sec^2 x) \cdot (x \sin x - x \cos x) - (2x^7 + 3 + \tan x) \cdot [(1 + x) \sin x + (x - 1) \cos x]}{x^2 (\sin x - \cos x)^2}
 \end{aligned}$$