

Complex Numbers & Quadratic Equations

Complex Numbers

- The square root of -1 is represented by the symbol i . It is read as *iota*.
 $i = \sqrt{-1}$ or $i^2 = -1$
- Any number of the form $a + ib$, where a and b are real numbers, is known as a complex number. A complex number is denoted by z .
 $z = a + ib$
- For the complex number $z = a + ib$, a is the real part and b is the imaginary part. The real and imaginary parts of a complex number are denoted by $\text{Re } z$ and $\text{Im } z$ respectively.
- For complex number $z = a + ib$, $\text{Re } z = a$ and $\text{Im } z = b$
- A complex number is said to be purely real if its imaginary part is equal to zero, while a complex number is said to be purely imaginary if its real part is equal to zero.
- For e.g., 2 is a purely real number and $3i$ is a purely imaginary number.
- Two complex numbers are equal if their corresponding real and imaginary parts are equal.
- Complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.
- Let's now try and solve the following puzzle to check whether we have understood this concept.

Solved Examples

Example 1:

Verify that each of the following numbers is a complex number.

$$3 + \sqrt{-7}, \sqrt{2} + \sqrt{5} \text{ and } 1 - 5i$$

Solution:

$3 + \sqrt{-7}$ can be written as $3 + i\sqrt{7}$, which is of the form $a + ib$. Thus, $3 + \sqrt{-7}$ is a complex number.

$\sqrt{2} + \sqrt{5}$ is not of the form $a + ib$. But it is known that every real number is a complex number.

Thus, $\sqrt{2} + \sqrt{5}$ is a complex number.

$1 - 5i$ is of the form $a + ib$. Thus, $1 - 5i$ is a complex number.

Example 2:

What are the real and imaginary parts of the complex number $-\sqrt{11} - \sqrt{-23}$?

Solution:

The complex number $-\sqrt{11} - \sqrt{-23}$ can be written as $-\sqrt{11} - i\sqrt{23}$, which is of the form $a + ib$.

$$\text{Re } z = a = -\sqrt{11} \text{ and } \text{Im } z = b = -\sqrt{23}$$

Example 3:

For what values of x and y , $z_1 = (x + 1) - 10i$ and $z_2 = 19 + i(y - x)$ represent equal complex numbers?

Solution:

Two complex numbers are equal if their corresponding real and imaginary parts are equal.

For the given complex numbers,

$$x + 1 = 19 \text{ and } y - x = -10$$

$$\Rightarrow x = 18 \text{ and } y - 18 = -10$$

$$\Rightarrow x = 18 \text{ and } y = 8$$

Thus, the values of x and y are 18 and 8 respectively.

Addition and Subtraction of Complex Numbers

- The addition of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is defined as

$$z_1 + z_2 = (a + c) + i(b + d)$$

$$\text{For example: } (4 + 3i) + (-2 + 6i) = (4 - 2) + i(3 + 6) = 2 + 9i$$

- Several properties are exhibited by the addition of complex numbers.
- **Closure law**

The addition of complex numbers satisfies closure property i.e., the sum of two complex numbers is a complex number.

If z_1 and z_2 are any two complex numbers, then $z_1 + z_2$ is also a complex number.

- **Commutative law**

The commutative law holds for the addition of complex numbers.

If z_1 and z_2 are any two complex numbers, then $z_1 + z_2 = z_2 + z_1$.

For example: $z_1 = 3 + 2i$ and $z_2 = -5 + 4i$

$$z_1 + z_2 = (3 + 2i) + (-5 + 4i) = -2 + 6i$$

$$z_2 + z_1 = (-5 + 4i) + (3 + 2i) = -2 + 6i$$

$$\therefore z_1 + z_2 = z_2 + z_1$$

- **Associative law**

The associative law holds for the addition of complex numbers.

If z_1 , z_2 and z_3 are any three complex numbers, then

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

- **Additive identity**

The complex number $(0 + i0)$ is the additive identity. It is denoted by 0.

For every complex number z , $z + 0 = z$

- **Additive inverse**

The complex number $\{-a + i(-b)\}$ is the additive inverse of the complex number $z = a + ib$. The inverse of a complex number z is denoted by $-z$.

Also, $z + (-z) = 0$.

For example: The inverse of the complex number $7 - 3i$ is $-7 + 3i$.

Difference of Complex Numbers

- The difference of complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is defined as
- $$z_1 - z_2 = z_1 + (-z_2) = (a + ib) + \{-(c + id)\}$$
- $$= (a + ib) + \{-c - id\}$$

$$= (a - c) + i(b - d)$$

For example: Let $z_1 = -1 + 3i$ and $z_2 = 7 + 4i$

$$z_1 - z_2 = (-1 + 3i) - (7 + 4i) = (-1 - 7) + i(3 - 4) = -8 - i$$

- **Closure law**

The difference of complex numbers satisfies the closure property i.e., the difference of two complex numbers is a complex number.

If z_1 and z_2 are any two complex numbers, then $z_1 - z_2$ is also a complex number.

Solved Examples

Example1:

If $Z_1 = 3 - i$ and $Z_2 = 1 + 2i$, then write the complex number $(Z_1 + 2Z_2 - 4)$ in the form $a + ib$ and determine the values of a and b .

Solution:

We have $Z_1 = 3 - i$ and $Z_2 = 1 + 2i$

$$Z_1 + 2Z_2 - 4 = (3 - i) + 2(1 + 2i) - 4$$

$$= 3 - i + 2 + 4i - 4$$

$$= 1 + 3i$$

Which is of the form $a + ib$

$$\therefore a = 1 \text{ and } b = 3$$

Example 2:

What is the additive inverse of $(-1 + i\sqrt{3})$?

Solution:

$$\text{Let } Z = -1 + i\sqrt{3}$$

$$\text{Additive inverse of } Z = -(-1 + i\sqrt{3}) = 1 - i\sqrt{3}$$

Multiplication of Complex Numbers

Multiplication of Complex Numbers and Their Properties

- The multiplication of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is defined as

$$z_1 z_2 = (a + ib) \times (c + id)$$

$$= a(c + id) + ib(c + id)$$

$$= ac + iad + ibc + i^2 bd \left[\because i = \sqrt{-1} \Rightarrow i^2 = -1 \right]$$

$$= (ac - bd) + i(ad + bc)$$

$$\therefore \boxed{z_1 z_2 = (ac - bd) + i(ad + bc)}$$

- For example:

$$\text{Let } z_1 = 1 + 2i \text{ and } z_2 = -3 + 4i$$

$$z_1 z_2 = (-3 - 8) + i\{4 + (-6)\} = -11 + i(-2) = -11 - 2i$$

- The multiplication of complex numbers satisfies the following properties:

- Closure law**

The product of two complex numbers is a complex number.

If z_1 and z_2 are any two complex numbers, then $z_1 z_2$ is a complex number.

- Commutative law**

Commutative law holds for the product of complex numbers i.e., for any two complex numbers z_1 and z_2 , $z_1 z_2 = z_2 z_1$

- Associative law**

Associative law holds for the product of complex numbers.

For any three complex numbers z_1 , z_2 and z_3 : $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

- Distributive law**

For any three complex numbers z_1 , z_2 and z_3 :

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

$$(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$$

- **Multiplicative identity**

The complex number $1 + i0$ is the multiplicative identity of the complex number. It is denoted by 1. For any complex number z , $z \times 1 = z$.

- **Multiplicative inverse**

The complex number z_2 is said to be the multiplicative inverse of the complex number z_1 if $z_1z_2 = 1$ (1 is the multiplicative identity). The multiplicative inverse of a complex number z is denoted by z^{-1} .

$$\therefore z \times \frac{1}{z} = 1$$

$$\therefore \frac{1}{z} \text{ is the multiplicative inverse of } z.$$

Multiplicative inverse of the complex number $z = a + ib$ is given by

$$z^{-1} = \frac{1}{z} = \frac{a}{a^2 + b^2} + i \frac{(-b)}{a^2 + b^2}$$

Powers of i

- $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = i^2 \times i = (-1) \times i = -i$$

$$i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$$

$$i^5 = i^4 \times i = 1 \times i = i$$

$$i^6 = i^4 \times i^2 = 1 \times -1 = -1$$

And so on...

- In general, we can write

$$i^{4k} = 1$$

$$i^{4k+1} = i$$

$$i^{4k+2} = -1$$

$$i^{4k+3} = -i$$

Where k is any integer

For example: $i^{39} = i^{36+3} = i^{4 \times 9 + 3}$

It is of the form i^{4k+3} , where $k = 9$

$$\therefore i^{39} = -i$$

Solved Examples

Example 1

Simplify the following:

$$\left[(-i)^{17} + \left(\frac{1}{i} \right)^8 \right]$$

Solution:

$$\begin{aligned}
& (-i)^{17} + \left(\frac{1}{i}\right)^8 \\
&= [(-1) \times i]^{17} + \left(\frac{1}{i}\right)^8 \\
&= (-1)^{17} (i)^{17} + (i^{-1})^8 \\
&= -(i)^{16+1} + (i^8)^{-1} \\
&= -i + [i^{4 \times 2}]^{-1} \quad \left[\because i^{4k+1} = i \right] \\
&= -i + (1)^{-1} \quad \left[\because i^{4k} = 1 \right] \\
&= -i + 1 \\
&= (1 - i)
\end{aligned}$$

Example 2

If $x + iy = (2 + 5i)(7 + i)$, then what are the values of x and y ?

Solution:

$$\begin{aligned}
x + iy &= (2 + 5i)(7 + i) \\
x + iy &= (2 \times 7 - 5 \times 1) + i(2 \times 1 + 5 \times 7) \\
x + iy &= (14 - 5) + i(2 + 35) \\
x + iy &= 9 + i(37)
\end{aligned}$$

On equating the real and imaginary parts, we obtain

$$x = 9 \text{ and } y = 37$$

Example 3

What is the value of $(\sqrt{-25})\left(\sqrt{-\frac{8}{49}}\right)$?

Solution:

We know that $\sqrt{-1} = i$

$$\begin{aligned}\therefore (\sqrt{-25})\left(\sqrt{\frac{-8}{49}}\right) &= (\sqrt{25} \times \sqrt{-1})\left(\sqrt{-1}\sqrt{\frac{8}{49}}\right) \\ &= (5i)\left(\frac{2\sqrt{2}i}{7}\right) \\ &= \left(5 \times \frac{2\sqrt{2}}{7}\right)i \times i \\ &= \frac{10\sqrt{2}}{7}i^2 \\ &= \frac{10\sqrt{2}}{7}(-1) \quad [\because i^2 = -1] \\ &= \frac{-10\sqrt{2}}{7}\end{aligned}$$

Note: Students may make mistakes while solving this question.

We know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. However, when a and b are both negative, then $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$.

Hence, this question cannot be solved as

$$\begin{aligned}(\sqrt{-25})\left(\sqrt{\frac{-8}{49}}\right) &= \sqrt{(-25)\left(-\frac{8}{49}\right)} \\ &= \sqrt{\frac{25 \times 8}{49}} \\ &= \frac{10\sqrt{2}}{7}\end{aligned}$$

Example 4

What is the multiplicative inverse of $5 - 9i$?

Solution:

$$\text{Let } z = a + ib = 5 - 9i$$

Accordingly, $a = 5$ and $b = -9$

We know that

$$\begin{aligned} z^{-1} &= \frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2} \\ &= \frac{5}{5^2 + (-9)^2} + i \frac{9}{5^2 + (-9)^2} \\ &= \frac{5 + 9i}{106} \end{aligned}$$

Thus, $\frac{5 + 9i}{106}$ is the multiplicative inverse of $5 - 9i$.

Division of Complex Numbers

- The division of two complex numbers z_1 and z_2 can be defined as $\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2}$, where $\frac{1}{z_2}$ is the multiplicative inverse of z_2 .

$$\frac{z_1}{z_2} = z_1 \times \text{multiplicative inverse of } z_2$$

- To find the quotient of two complex numbers, find the product of the first number with the multiplicative inverse of the second number.

$$\frac{z_1}{z_2} = \frac{1+i}{2-3i} = (1+i) \left(\frac{1}{2-3i} \right)$$

For example: If $z_1 = 1 + i$ and $z_2 = 2 - 3i$, then

We know that the multiplicative inverse of the complex number $z = a + ib$ is

$$\frac{1}{a+ib} = \frac{a}{a^2 + b^2} + i \frac{(-b)}{a^2 + b^2}$$

given by

$$\therefore \frac{1}{2-3i} = \frac{2}{2^2 + (-3)^2} + i \frac{3}{2^2 + (-3)^2} = \frac{2}{13} + i \frac{3}{13}$$

$$\text{Now, } \frac{z_1}{z_2} = (1+i) \left(\frac{2}{13} + i \frac{3}{13} \right) = \left(\frac{2}{13} - \frac{3}{13} \right) + i \left(\frac{2}{13} + \frac{3}{13} \right) = \frac{-1}{13} + i \frac{5}{13}$$

Solved Examples

Example 1

Write the complex number $\frac{2+\sqrt{3}i}{1-\sqrt{3}i}$ in the form of $a + ib$.

Solution:

$$\begin{aligned}\frac{2+\sqrt{3}i}{1-\sqrt{3}i} &= \frac{2+\sqrt{3}i}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i} \\&= \frac{(2+\sqrt{3}i)(1+\sqrt{3}i)}{(1-\sqrt{3}i)(1+\sqrt{3}i)} \\&= \frac{2+2\sqrt{3}i+\sqrt{3}i+3i^2}{(1)^2-(\sqrt{3}i)^2} \\&= \frac{2+2\sqrt{3}i+\sqrt{3}i+3\times(-1)}{1-3i^2} \quad [i^2 = -1] \\&= \frac{2+3\sqrt{3}i-3}{1-3\times(-1)} \quad [i^2 = -1] \\&= \frac{-1+3\sqrt{3}i}{1+3} \\&= \frac{-1+3\sqrt{3}i}{4} \\&= \frac{-1}{4} + \frac{3\sqrt{3}i}{4}\end{aligned}$$

Identities of Complex Numbers

The identities for complex numbers are the same as the algebraic identities for real numbers. The identities which hold for complex numbers are as follows:

- $(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2$
- $(z_1 - z_2)^2 = z_1^2 + z_2^2 - 2z_1z_2$
- $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

In order to know the proof of the above identities involving squares, let us go through the following video.

- $(z_1 + z_2)^3 = z_1^3 + z_2^3 + 3z_1^2z_2 + 3z_1z_2^2$

- $(z_1 - z_2)^3 = z_1^3 - z_2^3 - 3z_1^2z_2 + 3z_1z_2^2$

Modulus and Conjugate of a Complex Number

Modulus of a Complex Number

- The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and defined as $|z| = \sqrt{a^2 + b^2}$.

- For example: The modulus of the complex number $z = 1 - \sqrt{3}i$ is $|z| = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

- The following results hold true for two complex numbers z_1 and z_2 .

- $|z_1 z_2| = |z_1| |z_2|$

- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$, provided $|z_2| \neq 0$

Conjugate of a Complex Number

- The conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and defined as $\bar{z} = a - ib$.

- For example: The conjugate of the complex number $2 + \sqrt{-5}$ is $\bar{z} = 2 - \sqrt{-5} = 2 - i\sqrt{5}$

- The following results hold true for two complex numbers z_1 and z_2 .

- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

- $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, provided $z_2 \neq 0$

- The modulus of a complex number and the modulus of its conjugate are equal.
 $|z| = |\overline{z}|$

Relation of Multiplicative Inverse with Modulus and Conjugate of a Complex Number

- The multiplicative inverse of a complex number $z = a + ib$ is given by

$$z^{-1} = \frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2}$$

$$\Rightarrow z^{-1} = \frac{a - ib}{a^2 + b^2}$$

$\overline{z} = a - ib$ is the conjugate and $|z| = \sqrt{a^2 + b^2}$ is the modulus of the complex number z .

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

∴

$$\text{Or } z\overline{z} = |z|^2 \quad \left(\because z^{-1} = \frac{1}{z} \right)$$

This is the required relation.

Solved Examples

Example 1:

Determine the conjugate and multiplicative inverse of $3 + \sqrt{7}i$.

Solution:

Let $z = 3 + \sqrt{7}i$

Accordingly, conjugate, $\overline{z} = 3 - \sqrt{7}i$ and $|z|^2 = (3)^2 + (\sqrt{7})^2 = 9 + 7 = 16$

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

Now, the multiplicative inverse is given by

$$z^{-1} = \frac{3 - \sqrt{7}i}{16}$$

Example 2:

What is the conjugate of $\frac{(5+i)(1+2i)}{(3-4i)(1+i)}$?

Solution:

$$\text{Let } z = \frac{(5+i)(1+2i)}{(3-4i)(1+i)}$$

In order to find the conjugate of z , we first write it in the form of $a + ib$.

$$z = \frac{3+11i}{7-i} \quad (\text{By the multiplication of complex numbers})$$

On multiplying the numerator and the denominator with $(7+i)$, we obtain

$$\begin{aligned} z &= \frac{(3+11i) \times (7+i)}{(7-i) \times (7+i)} \\ &= \frac{10+80i}{49+1} \\ &= \frac{10+80i}{50} \\ &= \frac{1+8i}{5} \end{aligned}$$

$$\text{Now, } \bar{z} = \frac{1-8i}{5}$$

Thus, the conjugate of the given complex number is $\frac{1-8i}{5}$.

Example 3:

What is the modulus of $z = (1+i)^{10}$?

Solution:

Modulus, $|z| = |(1+i)^{10}|$

It can be written as

$$|z| = |(1+i)(1+i)^9|$$
$$|z| = |(1+i)| |(1+i)^9| \quad \left(\because |z_1 z_2| = |z_1| |z_2| \right)$$

Continuing in this manner, we can write

$$|z| = |(1+i)| |(1+i)| \dots |(1+i)| \quad (10 \text{ times})$$
$$|z| = |(1+i)|^{10}$$

Now, $|(1+i)| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$|z| = (\sqrt{2})^{10} = 2^5$$

Quadratic Equations with Complex Roots

- Complex numbers are used for finding the roots of a quadratic equation whose discriminant is negative.
- The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } b^2 - 4ac \text{ is the discriminant of the quadratic equation}$$

- If the discriminant i.e., the value under the square root is negative, then the roots of the quadratic equation will be complex numbers.
- For example: For the equation $3x^2 + 7x + 6 = 0$, $a = 3$, $b = 7$ and $c = 6$

$$\therefore \text{Discriminant} = b^2 - 4ac = (7)^2 - 4(3)(6) = 49 - 72 = -23$$

Thus, the roots of the quadratic equation are complex numbers.

Solved Examples

Example 1

Solve the quadratic equation $x^2 - 2\sqrt{3}x + \sqrt{3} + 4 = 0$.

Solution:

The given quadratic equation is $x^2 - 2\sqrt{3}x + \sqrt{3} + 4 = 0$.

The discriminant of this equation is

$$b^2 - 4ac = (-2\sqrt{3})^2 - 4(1)(\sqrt{3} + 4) = 12 - 4\sqrt{3} - 16 = -(4 + 4\sqrt{3})$$

Thus, the solution of the given equation is

$$\frac{-(-2\sqrt{3}) \pm \sqrt{-(4 + 4\sqrt{3})}}{2} = \frac{2\sqrt{3} \pm \sqrt{4 + 4\sqrt{3}} i}{2}$$

Example 2

If the roots of the quadratic equation $ax^2 + bx + c = 0$ are imaginary, then what can we say about the signs of a and c ?

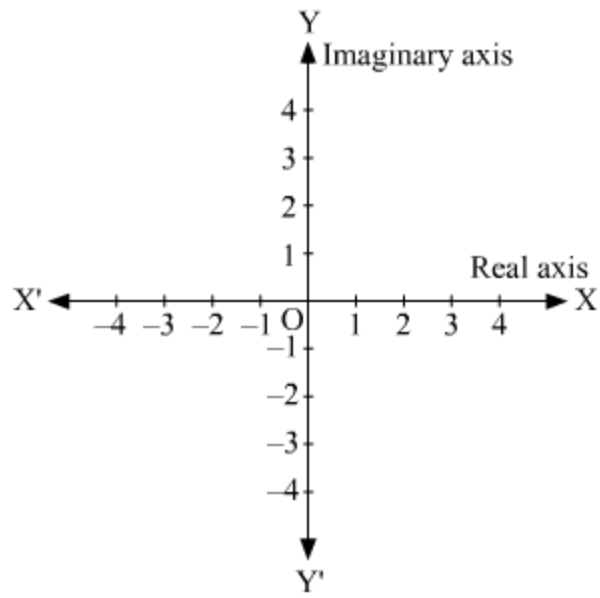
Solution:

The roots of quadratic equation $ax^2 + bx + c = 0$ are imaginary if the discriminant $b^2 - 4ac < 0$.

Here, b^2 is always positive whatever the sign of b is. Hence, the discriminant is negative if the product ac is positive. Thus, a and c must have the same signs.

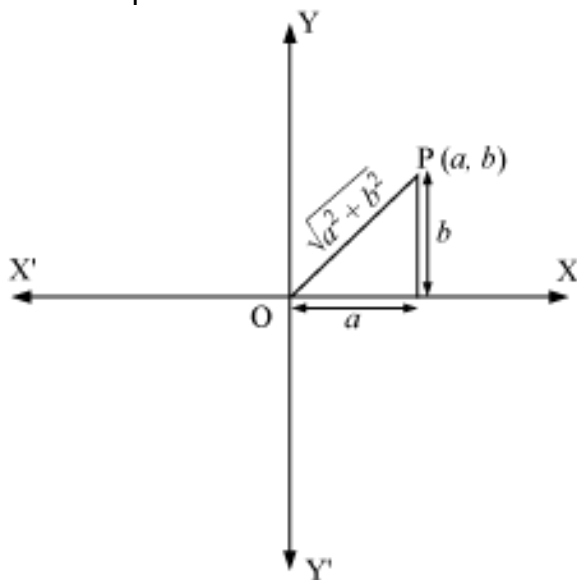
Concept of Argand Plane

- Each complex number represents a unique point on **Argand plane**. An Argand plane is shown in the following figure.



Here, x-axis is known as the **real axis** and y-axis is known as the **imaginary axis**.

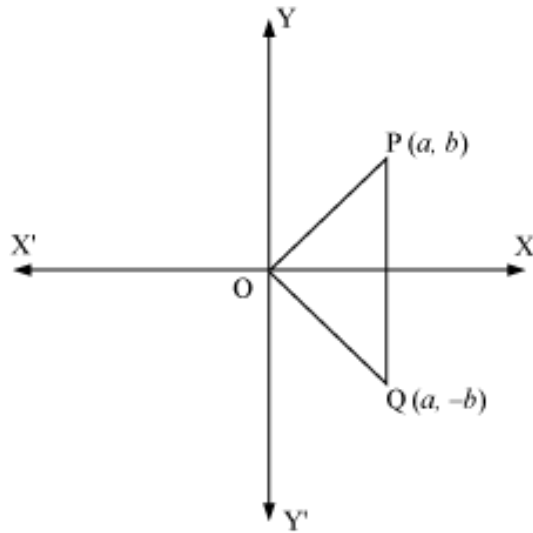
- The complex number $z = a + ib$ can be represented on an Argand plane as



In this figure, $OP = \sqrt{a^2 + b^2} = |z|$

Thus, the modulus of a complex number $z = a + ib$ is the distance between the point $P(x, y)$ and the origin O .

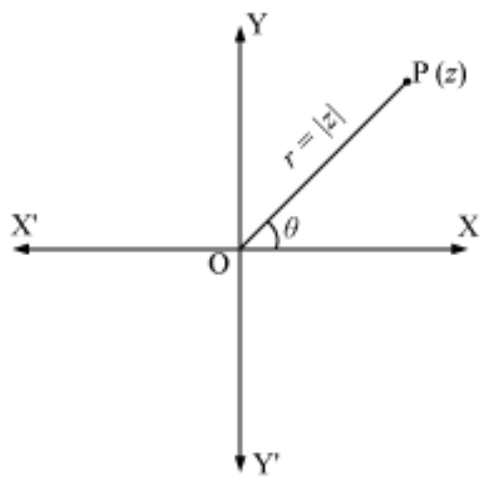
- The conjugate of a complex number $z = a + ib$ is $\bar{z} = a - ib$. z and $\bar{z}$ can be represented by the points $P(a, b)$ and $Q(a, -b)$ on the Argand plane as



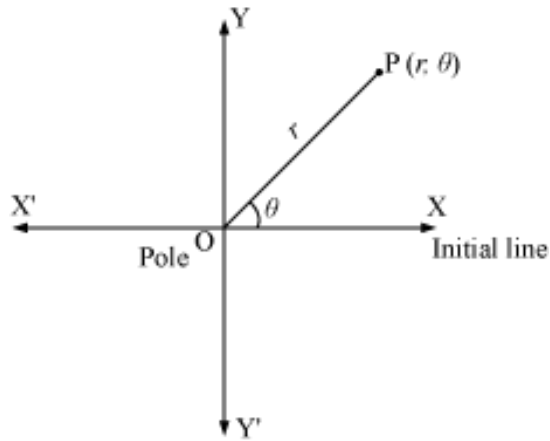
Thus, on the Argand plane, the conjugate of a complex number is the mirror image of the complex number with respect to the real axis.

Polar Representation of Complex Numbers

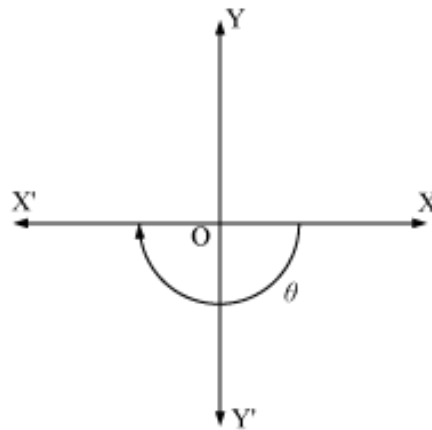
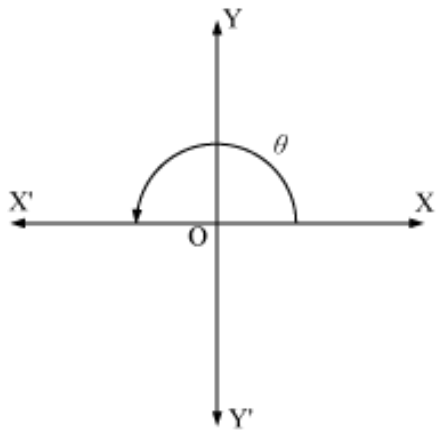
- A complex number $z = a + ib$ can be written in the **polar form** as $z = r (\cos\theta + i \sin\theta)$.
- Here, r is the **modulus** of the complex number and is given by $r = \sqrt{a^2 + b^2}$
- θ is the **argument** of the complex number and is given by $\theta = \tan^{-1} \frac{b}{a}$
- Geometrically, r represents the distance of the point that represents the complex number from the origin, and θ represents the angle formed by the line joining the point and the origin with the positive x-axis.



- The polar coordinates of a complex number z are (r, θ) . The origin is considered as the pole and the positive x-axis is considered as the initial line.



- The value of θ lying in the interval $-\pi < \theta \leq \pi$ is called the **principal argument** of the complex number z . In order to write the polar form of a complex number, we always find the principal argument.
- If θ lies in quadrants I or II, then the argument is found in the anticlockwise direction. If θ lies in quadrants III or IV, then the argument is found in the clockwise direction.



- Let us go through the following video to understand the method of representing a complex number in the polar form.

Solved Examples

Example 1:

Represent the complex number $(\sqrt{3} - i)$ in polar form.

Solution:

Let $z = r(\cos\theta + i\sin\theta)$ be the polar form of the complex number $(\sqrt{3} - i)$.

$$\therefore r\cos\theta = \sqrt{3} \text{ and } r\sin\theta = -1$$

On squaring and adding, we obtain

$$r^2(\cos^2\theta + \sin^2\theta) = (\sqrt{3})^2 + (-1)^2$$

$$r^2 = 3 + 1 = 4$$

$$r = \pm 2$$

$$r = 2 \quad (r \text{ cannot be negative})$$

Now,

$$\cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = -\frac{1}{2}$$

Here, $\cos\theta$ is positive and $\sin\theta$ is negative. Hence, θ lies in quadrant **IV**.

$$\therefore \theta = -\frac{\pi}{6}$$

Thus, the required polar form of the given complex number is

$$2\left\{\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right\}$$

Example 2:

What are the modulus and the argument of the complex number $-\frac{1}{\sqrt{2}}(1+i)$?

Solution:

$$\text{Let } r(\cos\theta + i\sin\theta) = -\frac{1}{\sqrt{2}}(1+i)$$

Which gives,

$$r \cos \theta = \frac{-1}{\sqrt{2}} \quad \text{and} \quad r \sin \theta = \frac{-1}{\sqrt{2}}$$

On squaring and adding, we obtain

$$r^2 (\cos^2 \theta + \sin^2 \theta) = \left(\frac{-1}{\sqrt{2}} \right)^2 + \left(\frac{-1}{\sqrt{2}} \right)^2$$

$$r^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$r = \pm 1$$

$$r = 1 (\because r > 0)$$

$$\text{Now, } \cos \theta = \frac{-1}{\sqrt{2}} \quad \text{and} \quad \sin \theta = \frac{-1}{\sqrt{2}}$$

Here, both $\cos \theta$ and $\sin \theta$ are negative.

Hence, θ lies in quadrant III.

$$\therefore \theta = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

Thus, the modulus and argument of the given complex number are 1 and $-\frac{3\pi}{4}$ respectively.