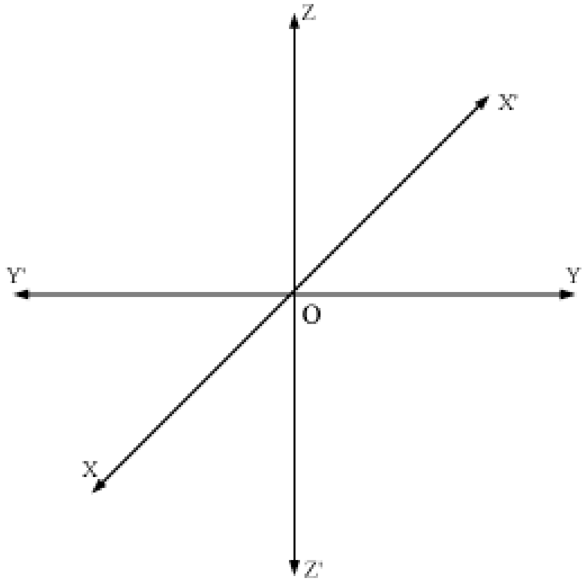
Introduction to Three Dimensional Geometry

Rectangular Coordinate System

- If we draw three planes intersecting at O such that they are mutually perpendicular to each other, then these will intersect along the lines X′OX, Y′OY and Z′OZ. These lines constitute the **rectangular coordinate system** and are respectively known as the *x*, *y*, and *z*-axes.

- Point O is called the **origin** of the coordinate system.



- The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downward in the direction of OZ′ are taken as negative.

- The distances measured to the right of ZX-plane along OY are taken as positive and those measured to the left of ZX-plane along OY′ are taken as negative.

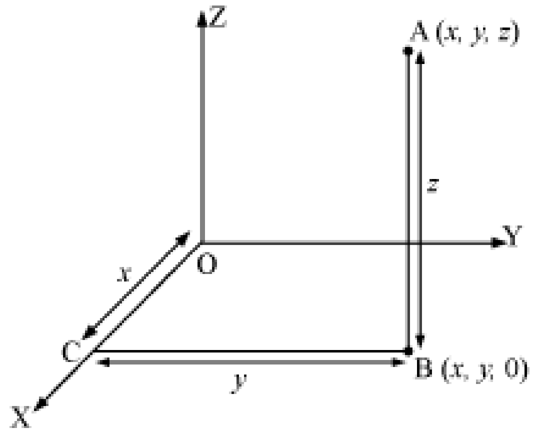
- The distances measured in front of YZ-plane along OX are taken as positive and those measured at the back of YZ-plane along OX′ are taken as negative.

- The planes XOY, YOZ, and ZOX are known as the three **coordinate planes** and are respectively called the XY-plane, the YZ-plane, and the ZX-plane.

- The three coordinate planes divide the space into eight parts known as **octants**. These octants are named as XOYZ, X′OYZ, X′ OY′Z, XOY′Z, XOYZ′, X′OYZ′, X′OY′Z′, and XOY′Z′ and are denoted by I, II, III, IV, V, VI, VII, and VIII respectively.

- If a point A lies in the first octant of a coordinate space, then the lengths of the perpendiculars drawn from point A to the planes XY, YZ and ZX are represented by *x*, *y*, and z respectively and are called the **coordinates** of point A.

This means that the coordinates of point A are (x, y, z). However, if point A would have been in any other quadrant, then the signs of x, y, and z would change accordingly.



- The coordinates of the origin are (0, 0, 0).
- The sign of the coordinates of a point determines the octant in which the point lies. The following table shows the signs of the coordinates in the eight octants.

Octants → / **Coordinates↓**	**I**	**II**	**III**	**IV**	**V**	**VI**	**VII**	**VIII**
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

- The coordinates of a point lying on different axes are as follows:
- The coordinates of a point lying on the x-axis will be of the form (x, 0, 0).

- The coordinates of a point lying on the *y*-axis will be of the form (0, *y*, 0).
- The coordinates of a point lying on the *z*-axis will be of the form (0, 0, *z*).
- The coordinates of a point lying on different planes are as follows:
- The coordinates of a point lying in the XY-plane will be of the form (*x*, *y*, 0).
- The coordinates of a point lying in the YZ-plane will be of the form (0, *y*, *z*).
- The coordinates of a point lying in the ZX-plane will be of the form (*x*, 0, *z*).
- Let's now try and solve the following puzzle to check whether we have understood the basic concepts that we just studied.

Solved Examples

Example 1

State whether the following statements are true or false.

1. The point (−5, 0, 1) lies on the *y*-axis.
2. The point (1, 7, −1) lies in octant V, whereas the point (−10, −8, 6) lies in octant VIII.
3. The *x*, *y*, and *z* coordinates of the point (5, 7, 18) are 5, 7 and 18 respectively.
4. The point (0, 0, 19) lies in the ZX-plane.
5. The point (−2, 11, 8) lies in octant II.

Solution:

1. False. The point (−5, 0, 1) does not lie on the *y*-axis since the point that lies on the *y*-axis is of the form (0, *y*, 0).
2. False. Point (−10, −8, 6) lies in octant III.
3. True.
4. False. A point lying in the ZX-plane is of the form (*x*, 0, *z*).
5. True.

Example 2

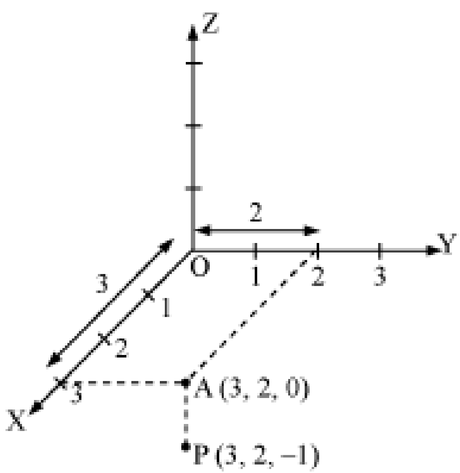
Locate point (3, 2, −1) in a three-dimensional space.

Solution:

We have to locate point (3, 2, −1) in a three-dimensional space. In order to do this, we will first draw the three axes.

Then, starting from the origin, when we move 2 units in the positive *y*-direction and then 3 units in the positive *x*-direction, we will reach at the point A(3, 2, 0).

From point A(3, 2, 0), we move 1 unit in the negative *z*-direction to reach the required point i.e., P(3, 2, −1).



Distance between Two Points in Three-Dimensional Space

- The distance formula that is used for finding the distance between two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) lying in three-dimensional space is given by

AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- Example: The distance between points A(3, 7, 8) and B(5, 0, 1) is given by
$\mathrm{AB} = \sqrt{(5-3)^2 + (0-7)^2 + (1-8)^2} = \sqrt{102}$

Solved Examples

Example 1:

Prove that points (−4, 8, 1), (2, 4, −1) and (−2, 2, 5) are the vertices of an equilateral triangle.

Solution:

Let the given points be A(−4, 8, 1), B(2, 4, −1) and C(−2, 2, 5) respectively.

We know that in an equilateral triangle, the lengths of all the sides are the same.

On using the distance formula, we obtain

AB = $\sqrt{\left[2-(-4)\right]^2+(4-8)^2+(-1-1)^2} = \sqrt{(6)^2+(-4)^2+(-2)^2} = \sqrt{36+16+4} = \sqrt{56}$

BC = $\sqrt{(-2-2)^2+(2-4)^2+\left[5-(-1)\right]^2} = \sqrt{(-4)^2+(-2)^2+(6)^2} = \sqrt{16+4+36} = \sqrt{56}$

CA = $\sqrt{\left[-4-(-2)\right]^2+(8-2)^2+(1-5)^2} = \sqrt{(-2)^2+(6)^2+(-4)^2} = \sqrt{4+36+16} = \sqrt{56}$

Thus,

AB = BC = CA = $\sqrt{56}$ units

Thus, the given vertices are the vertices of an equilateral triangle.

Example 2:

Find a point lying in the XY-plane such that the sum of the squares of its *x* and *y* coordinates is 5. Also, its distance from point (3, 5, 0) is 5 units, while its distance from point (5, −1, 2) is 7 units.

Solution:

The two given points are A(3, 5, 0) and B(5, −1, 2).

We know that any point lying in the XY-plane is of the form (*x*, *y*, 0). Hence, let the required point be C(*x*, *y*, 0).

It is given that the distance between points A and C is 5 units.

On applying the distance formula, we obtain

$$\begin{aligned} AC &= \sqrt{(x-3)^2+(y-5)^2+(0-0)^2} \\ &= \sqrt{x^2+9-6x+y^2+25-10y} = \sqrt{x^2+y^2-6x-10y+34} \end{aligned}$$

Thus,

$$\sqrt{x^2+y^2-6x-10y+34}=5$$
$$\Rightarrow x^2+y^2-6x-10y+34=25$$
$$\Rightarrow x^2+y^2-6x-10y+9=0$$

It is given that $x^2 + y^2 = 5$. Hence,

$5 - 6x - 10y + 9 = 0$

$\Rightarrow -6x - 10y + 14 = 0$

$\Rightarrow 3x + 5y - 7 = 0$ … (1)

It is also given that the distance between points B and C is 7 units.

On applying the distance formula, we obtain

$$\text{BC} = \sqrt{(x-5)^2+(y+1)^2+(0-2)^2}$$
$$= \sqrt{x^2+25-10x+y^2+1+2y+4} = \sqrt{x^2+y^2-10x+2y+30}$$

Thus,

$$\sqrt{x^2+y^2-10x+2y+30}=7$$
$$\Rightarrow x^2+y^2-10x+2y+30=49$$
$$\Rightarrow x^2+y^2-10x+2y-19=0$$

It is given that $x^2 + y^2 = 5$. Hence,

$5 - 10x + 2y - 19 = 0$

$\Rightarrow -10x + 2y - 14 = 0$

$\Rightarrow 5x - y + 7 = 0$ … (2)

On solving equations (1) and (2), we obtain

$x = -1$ and $y = 2$

Thus, the required point is $(-1, 2, 0)$.

Example 3:

Show that points (4, 3, −3), (3, 2, −1), and (2, 1, 1) are collinear.

Solution:

Let the given points be P(4, 3, −3), Q(3, 2, −1), and R(2, 1, 1).

We know that points are said to be collinear if they lie on a line.

On applying the distance formula, we obtain

PQ = $\sqrt{(3-4)^2+(2-3)^2+[-1-(-3)]^2}=\sqrt{(-1)^2+(-1)^2+(2)^2}=\sqrt{1+1+4}=\sqrt{6}$

QR = $\sqrt{(2-3)^2+(1-2)^2+[1-(-1)]^2}=\sqrt{(-1)^2+(-1)^2+(2)^2}=\sqrt{1+1+4}=\sqrt{6}$

RP = $\sqrt{(4-2)^2+(3-1)^2+(-3-1)^2}=\sqrt{(2)^2+(2)^2+(-4)^2}=\sqrt{4+4+16}=\sqrt{24}=2\sqrt{6}$

Thus, PQ + QR = RP.

Hence, the given points i.e., P, Q, and R are collinear.

Section Formula

- The coordinates of the point that divides the line segment joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) internally in the ratio m: n are given by

  $\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$. This formula is known as the section formula.

- The coordinates of a point that divides the line segment joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) externally in the ratio m: n are given by

  $\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n},\frac{mz_2-nz_1}{m-n}\right)$.

- If X is the mid-point of the line segment joining the points A(x_1, y_1, z_1) and B(x_2, y_2, z_2), then X divides AB in the ratio 1:1. Hence, by using the section formula, the coordinates of point X will be given by

$$\left(\frac{x_2+x_1}{2},\frac{y_2+y_1}{2},\frac{z_2+z_1}{2}\right).$$

- If a point R divides the line segment joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) internally in the ratio k : 1, then, by using the section formula, the coordinates of point R will be given by

$$\left(\frac{kx_2+x_1}{k+1},\frac{ky_2+y_1}{k+1},\frac{kz_2+z_1}{k+1}\right).$$

- The coordinates of the centroid of a triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are given by

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right).$$

Solved Examples

Example 1:

The coordinates of the vertices of ΔWXY are W(−12, 5, 6), X(−2, 1, −8) and Y(−1, −6, −7). Does the centroid of ΔWXY lie on the *XZ*-plane?

Solution:

It is given that the coordinates of the vertices of ΔWXY are W(−12, 5, 6), X(−2, 1, −8) and Y(−1, −6, −7).

We know that the coordinates of the centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are given by

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right).$$

Thus, the coordinates of the centroid of ΔWXY are given by

$$\left(\frac{(-12)+(-2)+(-1)}{3},\frac{5+1+(-6)}{3},\frac{6+(-8)+(-7)}{3}\right)$$
$$=\left(\frac{-12-2-1}{3},\frac{5+1-6}{3},\frac{6-8-7}{3}\right)$$
$$=\left(\frac{-15}{3},0,\frac{-9}{3}\right)$$
$$=(-5,0,-3)$$

We also know that any point whose coordinates are of the form (x, 0, z) lies on the *XZ*-plane. The coordinates of the centroid of ΔWXY are (−5, 0, −3).

Hence, the centroid of ΔWXY lies on the *XZ*-plane.

Example 2:

Find the ratio in which the line segment joining the points (23, 16, −19) and (−15, 20, 21) is divided externally by the ZX-plane.

Solution:

Let the given points be A(23, 16, −19) and B(−15, 20, 21).

Let point C(x, y, z) divide the line segment AB externally in the ratio k: 1.

Accordingly, the coordinates of point C are

$$\left(\frac{k(-15)-1(23)}{k-1},\frac{k(20)-1(16)}{k-1},\frac{k(21)-1(-19)}{k-1}\right)$$
$$=\left(\frac{-15k-23}{k-1},\frac{20k-16}{k-1},\frac{21k+19}{k-1}\right)$$

Since point C lies on the ZX-plane, its y-coordinate is zero.

Therefore,

$$\frac{20k-16}{k-1}=0$$
$$\Rightarrow 20k-16=0$$
$$\Rightarrow k=\frac{16}{20}=\frac{4}{5}$$

Thus, the ZX-plane divides AB externally in the ratio 4: 5.

Example 3:

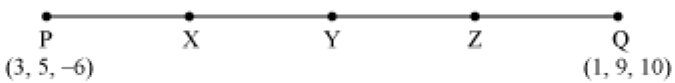
Find the coordinates of the points that divide the line segment joining the points P(3, 5, −6) and Q(1, 9, 10) into four equal parts.

Solution:

The given points are P(3, 5, −6) and Q(1, 9, 10).

Let points X, Y, and Z divide the line segment PQ. Hence,

PX = XY = YZ = ZQ.



From the figure, we can clearly see that point Y is the mid-point of line segment PQ.

Hence, the coordinates of point Y are given by

$$\left(\frac{3+1}{2},\frac{5+9}{2},\frac{-6+10}{2}\right)=\left(\frac{4}{2},\frac{14}{2},\frac{4}{2}\right)=(2,7,2)$$

Again, X is the mid-point of line segment PY. Hence, the coordinates of point X are given by

$$\left(\frac{3+2}{2},\frac{5+7}{2},\frac{-6+2}{2}\right)=\left(\frac{5}{2},\frac{12}{2},\frac{-4}{2}\right)=\left(\frac{5}{2},6,-2\right)$$

Again, Z is the mid-point of line segment YQ. Hence, the coordinates of point Z are given by

$$\left(\frac{2+1}{2},\frac{7+9}{2},\frac{2+10}{2}\right)=\left(\frac{3}{2},\frac{16}{2},\frac{12}{2}\right)=\left(\frac{3}{2},8,6\right)$$

Thus, the coordinates of the points that divide the line segment joining the given points are (2, 7, 2), $\left(\frac{5}{2},6,-2\right)$, and $\left(\frac{3}{2},8,6\right)$.