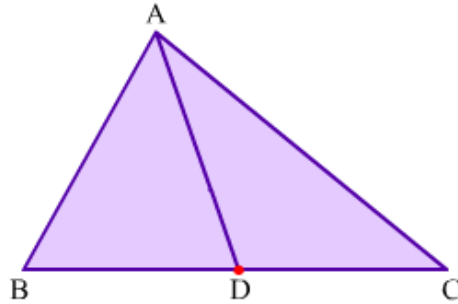


# Triangles

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## Medians of Triangles

Let us consider the following triangle ABC.



In the given figure, A is the vertex of the triangle ABC and  $\overline{BC}$  is the side opposite to vertex A. A line segment  $\overline{AD}$  is drawn joining the point A and the point D, where D is the mid-point of  $\overline{BC}$ .

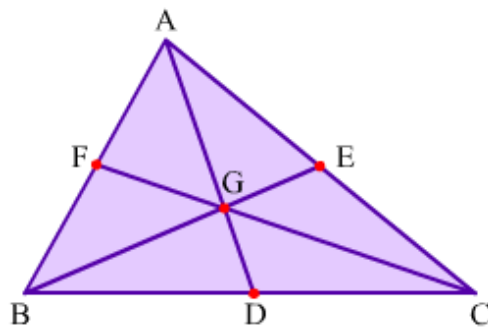
Then, we say that  $\overline{AD}$  is the median of  $\triangle ABC$ .

A median can be defined as follows.

**“The line segment joining any vertex of a triangle to the mid-point of its opposite side is called the median of the triangle.”**

Now we know what a median is, can we tell how many medians can be drawn inside a triangle?

In a triangle, there are three vertices. Therefore, **a triangle can have three medians**, as shown in the following figure.



Here,  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$  are the three medians of  $\triangle ABC$ .

**The medians of a triangle always lie inside the triangle.**

From the figure, it can be observed that the medians  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$  intersect each other at a common point G.

"The point of intersection of the medians is called the **centroid** of the triangle."

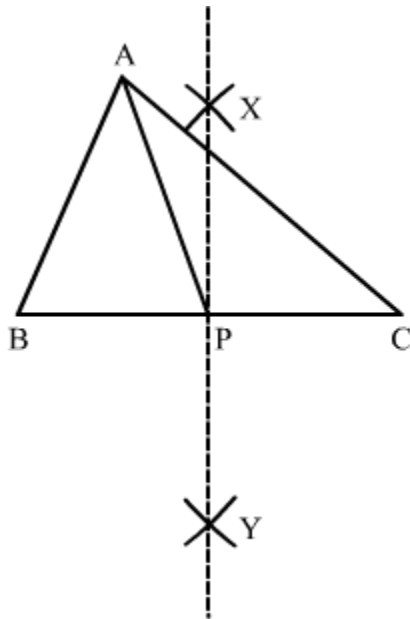
**Thus, medians of a triangle are concurrent.**

**The point where medians intersect each other is known as the point of concurrence.**

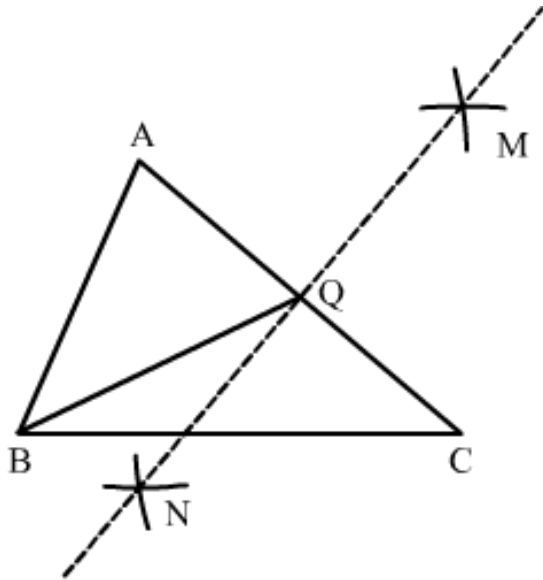
In the above given figure, G is the point of concurrence.

### Construction of Median of Triangle

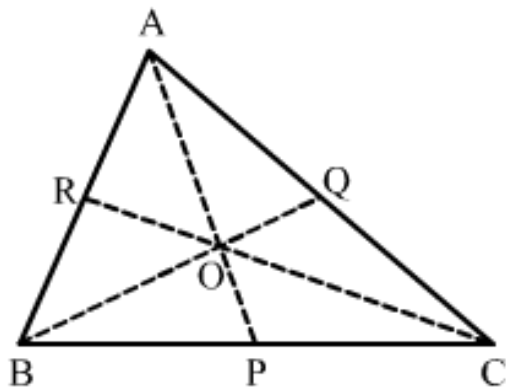
1. Draw a  $\triangle ABC$ .
2. With B and C as centres and radius more than half of BC, draw two arcs intersecting at points X and Y. Join XY thus meeting the line BC at point P.



3. With A and C as centres and radius more than half of AC, draw two arcs intersecting at points M and N. Join MN thus meeting the line AC at point Q.



4. Similarly, draw the perpendicular bisector of line AB meeting AB at point R.
5. Join AP, BQ and CR. Let the meeting point be O.

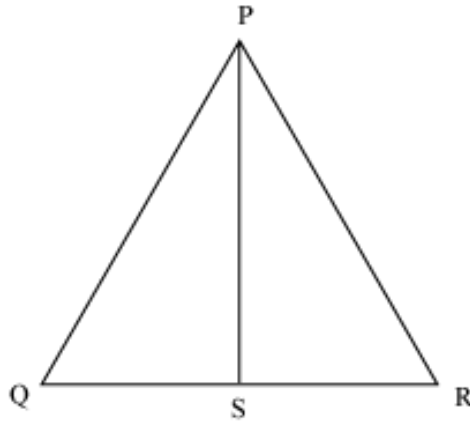


Point O is the centroid of  $\triangle ABC$  and AP, BQ and CR are the medians of sides BC, AC and AB respectively.

Now, let us look at an example.

### Example 1:

In the triangle PQR, PS is a median and the length of  $\overline{SR} = 6.5$  cm. Find the length of  $\overline{QR}$ .



**Solution:**

Here, PS is the median to the side  $\overline{QR}$  and we know that the median connects vertex to the midpoint of other side. Therefore, S is the mid-point of QR.

Therefore,  $\overline{QR} = 2\overline{SR}$

$= 2 \times 6.5 \text{ cm}$

$= 13 \text{ cm}$

**Altitudes of Triangles**

The students of class VII were being taken to a tour to Corbett National Park. They stayed there in a tent. The entrance in the tent was of a triangular shape as shown in the following figure.

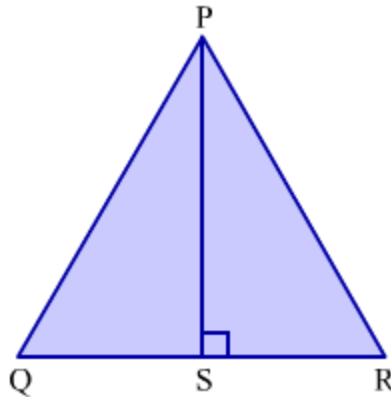


Now can you tell what the height of the tent is?

As we can see in the above figure, the height of the tent is the length of the vertical pole which is standing in the centre of the tent.

Similarly, in any triangle, we can draw a perpendicular which represents its height. The perpendicular representing the **height** of a triangle is called the **altitude** of the triangle.

Look at the triangle PQR below.

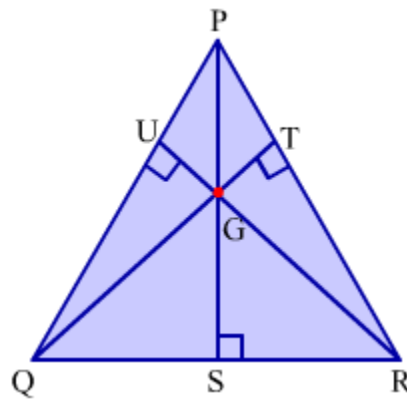


Here, P is a vertex of  $\triangle PQR$  and  $\overline{QR}$  is the opposite side of the vertex P.  $\overline{PS}$  is a perpendicular drawn from P to  $\overline{QR}$ . Line segment  $\overline{PS}$  is called the height or altitude of the triangle.

An altitude can be defined as follows.

**“An altitude of a triangle is the perpendicular drawn from a vertex to the opposite side of the triangle.”**

**Note:** *A triangle can have three altitudes.*



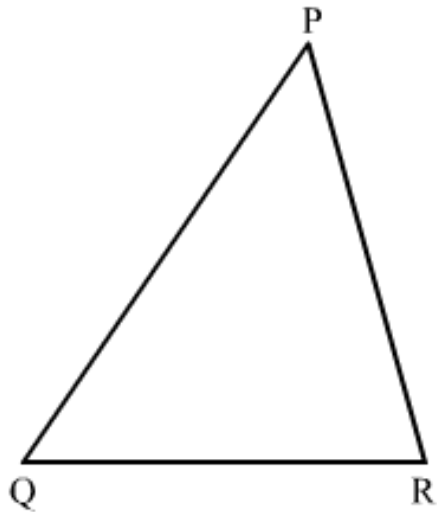
In the above figure,  $\overline{PS}$ ,  $\overline{QT}$ , and  $\overline{RU}$  are the three altitudes of  $\triangle PQR$ .

"The point of intersection of the altitudes is called the **orthocentre** of the triangle."

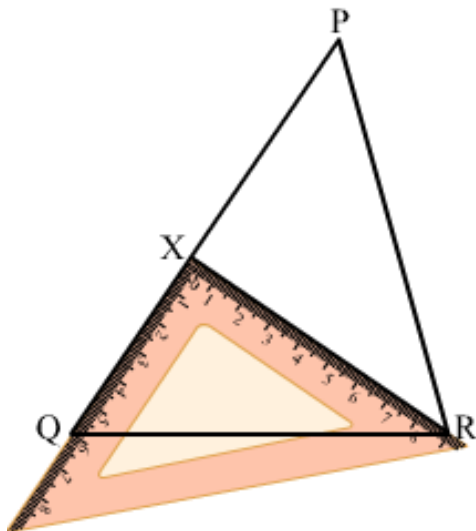
### **Construction of Altitudes of a triangle**

#### **I. Using set-square**

1. Draw a  $\triangle PQR$ .

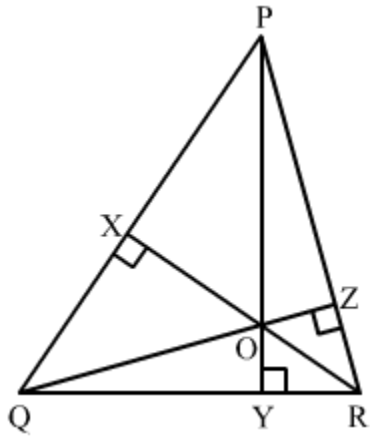


2. From point R, draw a perpendicular on side PQ. Where the perpendicular meets the side PQ, name it as point X. XR is the altitude formed on side PQ.



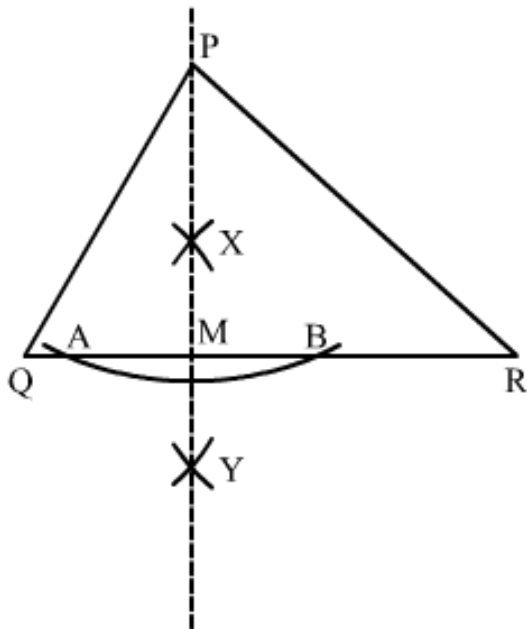
3. In a similar way construct the altitude from point P to side QR and from side Q to line PR.

Thus, the altitudes obtained are XR, QZ and PY.

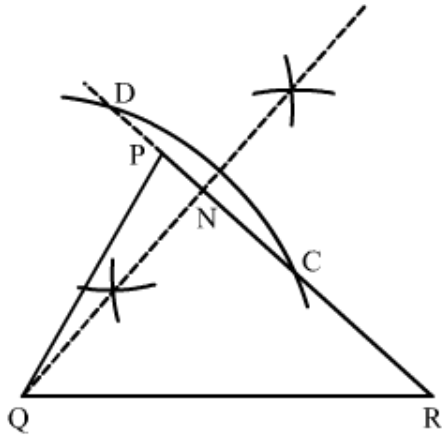


## II. Using Compasses

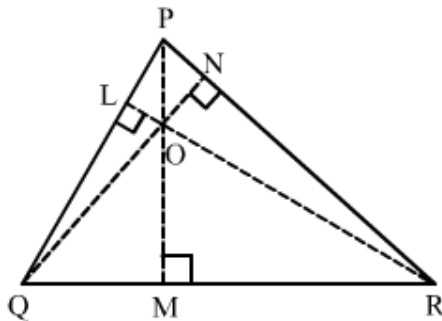
1. Draw a  $\triangle PQR$ .
2. With P as centre, draw an arc on line QR cutting it at points A and B.
3. With A and B as centres, draw two intersecting arcs at points X and Y. Draw a line joining XY cutting the line QR at point M. Join PM.



4. With Q as centre draw an arc on side RP extended to cut it at points C and D. With C and D as centres, draw two intersecting arcs. Let this line intersect PR at point N. Join QN.



5. Similarly, draw altitude from point R on PQ cutting the line PQ at point L.
6. Join PM, RL and QN and name the meeting point of these three altitudes as O.



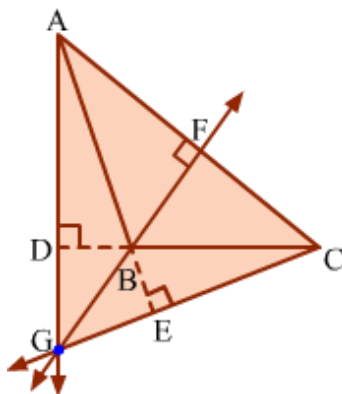
O is called the orthocentre of the  $\triangle PQR$ .

### Remember

*The altitudes of a triangle may not always lie inside it.*

**In an obtuse-angled triangle, the altitude drawn from the vertex of an acute angle lies outside the triangle.** In this case, we have to extend the opposite side of the vertex from which the altitude is drawn.

For example,



In the above figure,  $\triangle ABC$  is an obtuse-angled triangle where  $\angle ABC$  is an obtuse angle.  $\overline{AD}$  is the altitude of  $\triangle ABC$  drawn from the vertex A to extended side  $\overline{BC}$ . Similarly,  $\overline{CE}$  is the altitude drawn from the vertex C to extended side  $\overline{AB}$ . And,  $\overline{BF}$  is the altitude drawn from B to  $\overline{CA}$ .

Now, observe the altitudes drawn in the triangles  $\triangle PQR$  and  $\triangle ABC$ .

It can be seen that altitudes in each triangle intersect each other at a common point.

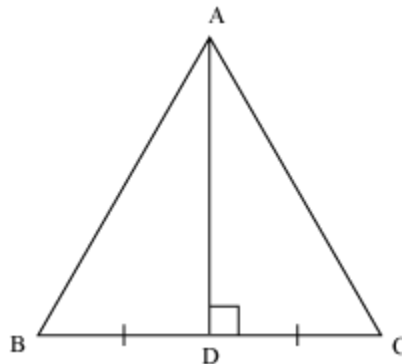
**Thus, altitudes of a triangle are concurrent.**

In  $\triangle PQR$  and  $\triangle ABC$ , G is the point of concurrence.

Let us look at another example now.

**Example:**

In triangle ABC,  $\overline{AD}$  is perpendicular to  $\overline{BC}$  such that  $\overline{BD} = \overline{CD}$ . Are the median and the altitude drawn from A to  $\overline{BC}$  same?



**Solution:**

Here,

$$\overline{AD} \perp \overline{BC}$$

Therefore,  $\overline{AD}$  is an altitude of  $\triangle ABC$  drawn from the vertex A to  $\overline{BC}$ .

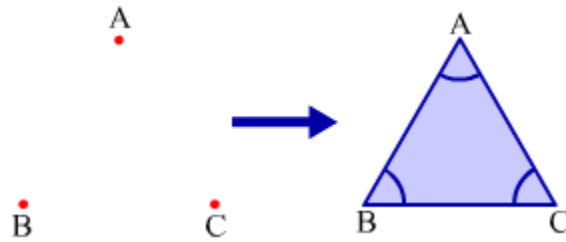
Also,  $\overline{BD} = \overline{CD}$

Therefore,  $\overline{AD}$  is a median of  $\triangle ABC$  drawn from the vertex A to  $\overline{BC}$ .

Thus, the altitude and the median drawn from A to  $\overline{BC}$  are the same.

## Angle Sum Property of Triangles

If we join any three non-collinear points in a plane, then we **get a triangle**. There are **three angles in a triangle**.



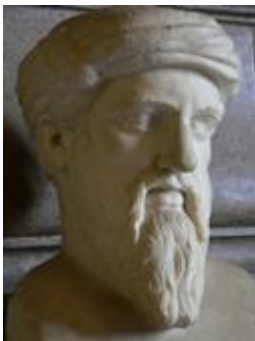
The sum of the three **interior angles** of a triangle is  $180^\circ$  and this property of a triangle is known as the angle sum property. This property holds true for all types of triangles, i.e., **acute-angled triangles**, **obtuse-angled triangles** and **right-angled triangles**. The angle sum property was identified by the Pythagorean school of Greek mathematicians (or the Pythagoreans) and proved by Euclid.

We will study the proof of the angle sum property of triangles and then solve some examples based on this property.

## Proving the Angle Sum Property of Triangles

### Know Your Scientist

#### Pythagoras



Pythagoras (570 BC–495 BC) was a great Greek mathematician and philosopher, often described as the first pure mathematician. He was born on the island of Samos and is best known for the Pythagoras theorem about right-angled triangles. He also made influential contributions to philosophy and religious teaching. He led a society that was part religious and part scientific. This society followed a code of secrecy, which is the reason why a sense of mystery surrounds the figure of Pythagoras.

## Euclid

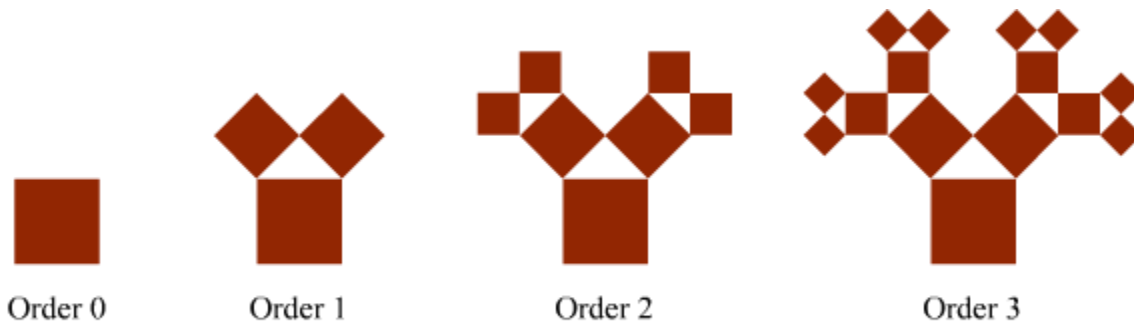
Euclid of Alexandria(325 BC–265 BC) was a great Greek mathematician. He is referred to as ‘the father of geometry’. Euclid taught at Alexandria during the reign of Ptolemy I, who ruled Egypt from 323 BC to 285 BC. Euclid wrote a series of books which are collectively known as the *Elements*. It is considered one of the most influential works in the history of mathematics.

The *Elements* served as the main textbook for teaching mathematics (especially geometry) from the time of its publication up until the early 20th century. In the *Elements*, Euclid defined most of the basic geometrical figures and deduced the principles of geometry through different sets of axioms.



### Did You Know?

In 1942, a Dutch mathematics teacher Albert E. Bosman invented a plane fractal constructed from a square. He named it the Pythagoras tree because of the presence of right-angled triangles in the figure.



Construction process of Pythagoras tree

### Facts about the Angle Sum Property

An important fact deduced through the angle sum property of triangles is that ***there can be no triangle with two right angles or two obtuse angles***. This fact can be proved as is shown.

Consider a  $\Delta ABC$  such that  $\angle A = 90^\circ$  and  $\angle B = 90^\circ$ .

According to the angle sum property, we have:

$$\angle A + \angle B + \angle C = 180^\circ$$

$$\Rightarrow 90^\circ + 90^\circ + \angle C = 180^\circ$$

$$\Rightarrow \angle C = 180^\circ - 180^\circ$$

$$\Rightarrow \angle C = 0^\circ$$

However, the above is not possible. So,  $\Delta ABC$  (or any other triangle) cannot have two right angles.

Similarly, we can prove that a triangle cannot have two obtuse angles.

**Whiz Kid**

**Relationship between the side lengths and the angle measurements of a triangle**

- The largest interior angle is **opposite** the largest side.
- The smallest interior angle is **opposite** the smallest side.
- The middle-sized interior angle is **opposite** the middle-sized side.

**Facts about the Angle Sum Property**

By the angle sum property, we can deduce the fact that ***there can be no triangle with all angles less than or greater than  $60^\circ$*** . This fact can be proved as is shown.

Consider a  $\Delta ABC$  with all angles equal to  $59^\circ$ .

According to the angle sum property, we should have  $\angle A + \angle B + \angle C = 180^\circ$ .

By adding the given angles, we obtain:

$$59^\circ + 59^\circ + 59^\circ = 177^\circ \neq 180^\circ$$

Since  $\Delta ABC$  does not satisfy the angle sum property, it cannot exist.

Now, consider a  $\Delta ABC$  with all angles equal to  $61^\circ$ .

According to the angle sum property, we should have  $\angle A + \angle B + \angle C = 180^\circ$ .

By adding the given angles, we obtain:

$$61^\circ + 61^\circ + 61^\circ = 183^\circ \neq 180^\circ$$

Since  $\Delta ABC$  does not satisfy the angle sum property, it cannot exist.

Thus, we have proved that a triangle cannot have all angles less than or greater than  $60^\circ$ .

### Whiz Kid

Sum of the interior angles of an  $n$ -sided polygon =  $(n - 2) \times 180^\circ$

For example:

Sum of the interior angles of a 6-sided polygon =  $(6 - 2) \times 180^\circ = 720^\circ$

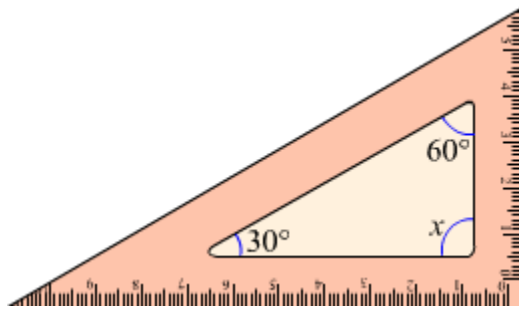
### Relation between the Vertex Angle and the Angles Made by the Bisectors of the Remaining Angles

#### Solved Examples

##### Easy

##### Example 1:

Find the measurement of  $x$  in the following figure.



##### Solution:

We know that the sum of the three angles of a triangle is  $180^\circ$ .

So, we have:

$$30^\circ + 60^\circ + x = 180^\circ$$

$$\Rightarrow 90^\circ + x = 180^\circ$$

$$\Rightarrow x = 90^\circ$$

##### Example 2:

**If the angles of a triangle are in the ratio 1 : 3 : 5, then what is the measure of each angle?**

**Solution:**

It is given that the angles of the triangle are in the ratio 1 : 3 : 5.

Let the angles be  $x$ ,  $3x$  and  $5x$ .

Now,  $x + 3x + 5x = 180^\circ$  (By the angle sum property of triangles)

$$\Rightarrow 9x = 180^\circ$$

$$\Rightarrow x = 20^\circ$$

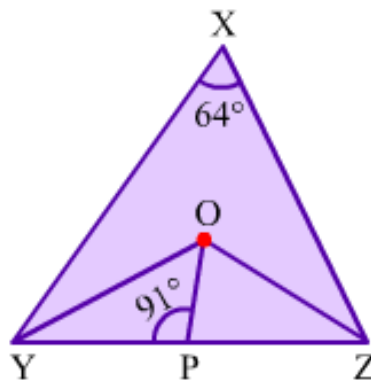
$$\text{So, } 3x = 3 \times 20^\circ = 60^\circ \text{ and } 5x = 5 \times 20^\circ = 100^\circ$$

Thus, the measures of the angles of the triangle are  $20^\circ$ ,  $60^\circ$  and  $100^\circ$ .

**Medium**

**Example 1:**

**In the given  $\Delta XYZ$ , YO, ZO and PO are the respective bisectors of  $\angle XYZ$ ,  $\angle XZY$  and  $\angle YOZ$ . Find the measure of  $\angle OYX$ .**



**Solution:**

As per the relation between the vertex angle and the angles made by the bisectors of the remaining two angles, we have:

$$\angle YOZ = 90^\circ + \frac{1}{2} \angle YXZ$$

$$\Rightarrow \angle YOZ = 90^\circ + \frac{1}{2} \times 64^\circ$$

$$\Rightarrow \angle YOZ = 90^\circ + 32^\circ$$

$$\Rightarrow \angle YOZ = 122^\circ$$

In  $\triangle OYP$ , we have:

$$\angle YOP = \frac{1}{2} \angle YOZ \text{ (because PO is the bisector of } \angle YOZ)$$

$$\Rightarrow \angle YOP = \frac{1}{2} \times 122^\circ$$

$$\Rightarrow \angle YOP = 61^\circ$$

Again in  $\triangle OYP$ , we have:

$$\angle YOP + \angle OPY + \angle PYO = 180^\circ \text{ (By the angle sum property)}$$

$$\Rightarrow 61^\circ + 91^\circ + \angle PYO = 180^\circ$$

$$\Rightarrow \angle PYO = 180^\circ - 152^\circ$$

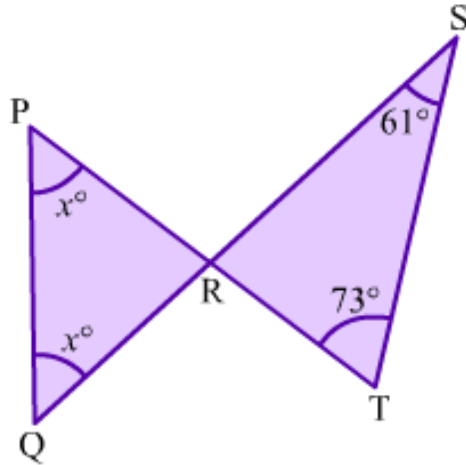
$$\Rightarrow \angle PYO = 28^\circ$$

We know that YO is the bisector of  $\angle XYZ$ .

$$\text{So, } \angle OYX = \angle PYO = 28^\circ$$

**Example 2:**

**Find the value of x in the given figure.**



**Solution:**

Using the angle sum property in  $\Delta RST$ , we obtain:

$$\angle RST + \angle RTS + \angle SRT = 180^\circ$$

$$\Rightarrow 61^\circ + 73^\circ + \angle SRT = 180^\circ$$

$$\Rightarrow 134^\circ + \angle SRT = 180^\circ$$

$$\Rightarrow \angle SRT = 180^\circ - 134^\circ$$

$$\Rightarrow \angle SRT = 46^\circ$$

In  $\Delta RST$  and  $\Delta RPQ$ , we have:

$$\angle PRQ = \angle SRT = 46^\circ \text{ (Vertically opposite angles)}$$

Using the angle sum property in  $\Delta RPQ$ , we obtain:

$$\angle RPQ + \angle RQP + \angle PRQ = 180^\circ$$

$$\Rightarrow x^\circ + x^\circ + 46^\circ = 180^\circ$$

$$\Rightarrow 2x^\circ + 46^\circ = 180^\circ$$

$$\Rightarrow 2x^\circ = 180^\circ - 46^\circ$$

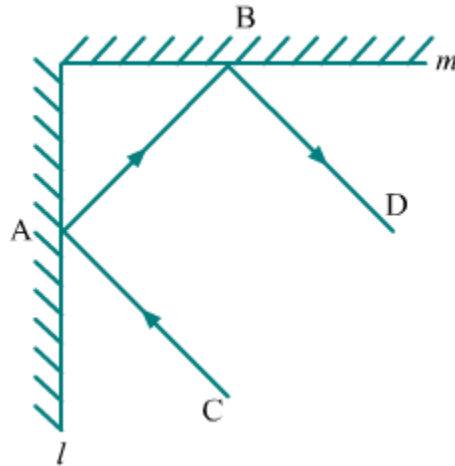
$$\Rightarrow 2x^\circ = 134^\circ$$

$$\Rightarrow x = 67$$

Hard

Example 1:

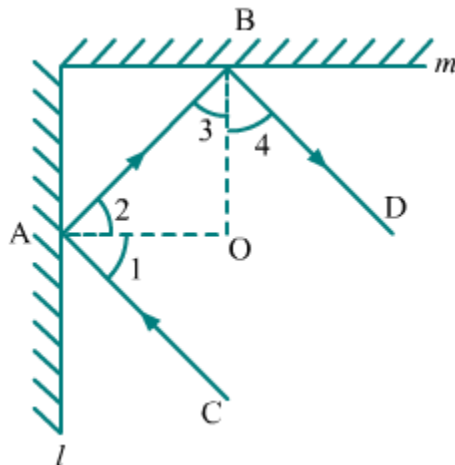
In the figure,  $l$  and  $m$  are two plane mirrors placed perpendicular to each other. Show that the incident ray  $CA$  is parallel to the reflected ray  $BD$ .



**Solution:**

It is given that mirrors  $l$  and  $m$  are perpendicular to each other.

*Construction:* Draw two perpendiculars  $OA$  and  $OB$  to  $l$  and  $m$  respectively. Mark the angles made by these perpendiculars with the incident and reflected rays as is shown.



$OA \perp OB$

$\therefore \angle BOA = 90^\circ$

In  $\triangle BOA$ , we have:

$$\angle 2 + \angle 3 + \angle BOA = 180^\circ$$

$$\Rightarrow \angle 2 + \angle 3 + 90^\circ = 180^\circ$$

$$\Rightarrow \angle 2 + \angle 3 = 90^\circ$$

$$\Rightarrow 2(\angle 2 + \angle 3) = 180^\circ$$

$$\Rightarrow 2\angle 2 + 2\angle 3 = 180^\circ \dots (1)$$

We know that:

Angle of incidence = Angle of reflection

For mirror  $l$ ,  $\angle 1$  is the angle of incidence and  $\angle 2$  is the angle of reflection.

$$\therefore \angle 1 = \angle 2$$

For mirror  $m$ ,  $\angle 3$  is the angle of incidence and  $\angle 4$  is the angle of reflection.

$$\therefore \angle 3 = \angle 4$$

So, by using equation 1, we get:

$$(\angle 2 + \angle 2) + (\angle 3 + \angle 3) = 180^\circ$$

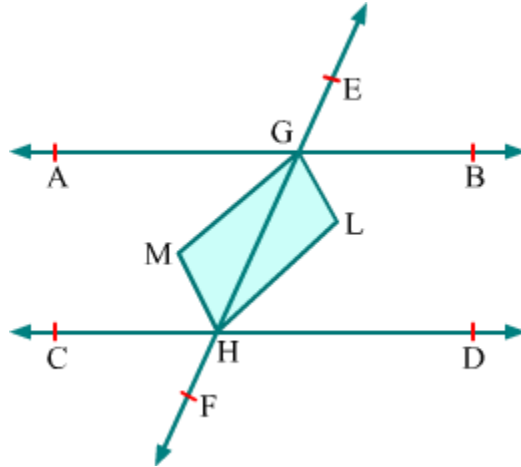
$$\Rightarrow (\angle 1 + \angle 2) + (\angle 3 + \angle 4) = 180^\circ \text{ (Since } \angle 1 = \angle 2 \text{ and } \angle 3 = \angle 4)$$

$$\Rightarrow \angle CAB + \angle ABD = 180^\circ$$

Since  $\angle CAB$  and  $\angle ABD$  are interior angles on the same side of transversal  $AB$  and their sum is  $180^\circ$ , **lines  $CA$  and  $BD$  must be parallel to each other. Therefore, the incident ray  $CA$  is parallel to the reflected ray  $BD$ .**

**Example 2:**

**In the given figure,  $AB$  is parallel to  $CD$ ;  $GM$ ,  $HM$ ,  $GL$  and  $HL$  are the bisectors of the two pairs of interior angles. Prove that  $\angle GLH = 90^\circ$ .**



**Solution:**

From the figure, we have:

$\angle AGH = \angle DHG$  (Alternate interior angles)

$$\Rightarrow \frac{1}{2} \angle AGH = \frac{1}{2} \angle DHG$$

$\Rightarrow \angle HGM = \angle GHL$  (&because GM bisects  $\angle AGH$  and HL bisects  $\angle DHG$ )

It can be said that lines GM and HL are intersected by the transversal GH at G and H respectively such that the alternate interior angles are equal, i.e.,  $\angle HGM = \angle GHL$ .

$\therefore GM \parallel HL$

Similarly, we can prove that  $GL \parallel HM$ . So, GMHL is a parallelogram.

We know that  $AB \parallel CD$  and EF is the transversal.

$\therefore \angle BGH + \angle DHG = 180^\circ$  (Interior angles on the same side of a transversal)

$$\Rightarrow \frac{1}{2} \angle BGH + \frac{1}{2} \angle DHG = 90^\circ$$

Since GL bisects  $\angle BGH$  and HL bisects  $\angle DHG$ , we obtain:

$$\angle LGH + \angle GHL = 90^\circ \dots (1)$$

Also,  $\angle LGH + \angle GHL + \angle GLH = 180^\circ$  (By the angle sum property)

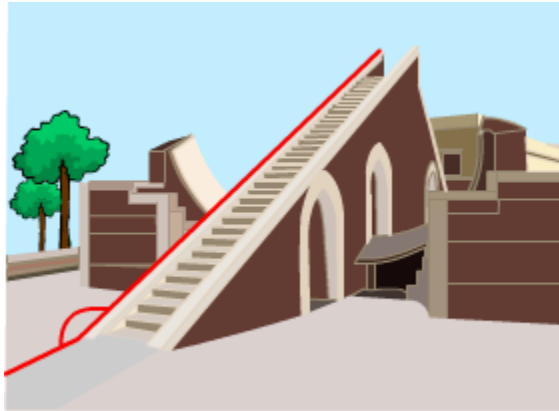
$$\therefore 90^\circ + \angle GLH = 180^\circ \text{ (Using equation 1)}$$

$$\Rightarrow \angle GLH = 90^\circ$$

Exterior Angle Property of Triangles

## Exterior Angles in Real Life

Look at the triangular structure in the figure.

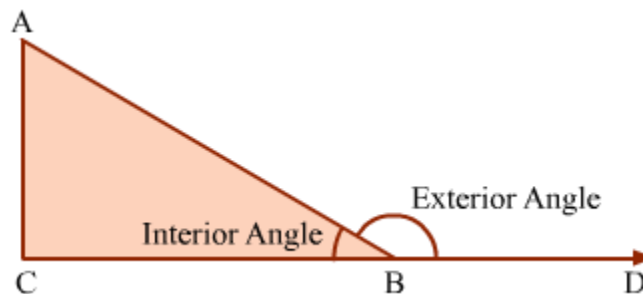


In the given figure, an open angle formed by the edge of the triangular structure with the horizontal plane is marked. This angle lies outside the triangle. Such angles are known as **exterior angles**.

In this lesson, we will study about exterior angles of triangles and the theorem based on them.

## Exterior Angles of Triangles

Look at the triangle shown.



It can be seen that in  $\triangle ABC$ , side CB is extended up to point D. This extended side forms an angle with side AB, i.e.,  $\angle ABD$ . This angle lies exterior to the triangle. Hence,  $\angle ABD$  is an exterior angle of  $\triangle ABC$ .

**An exterior angle of a triangle can be defined as follows:**

***The angle formed by a side of a triangle with an extended adjacent side is called an exterior angle of the triangle.***

It can be seen that exterior  $\angle ABD$  forms linear pair with interior  $\angle ABC$  of  $\triangle ABC$ . The other two interior angles of the triangle such as  $\angle ACB$  and  $\angle CAB$  do not form linear pair with  $\angle ABD$ .

Such angles are known as the **remote interior angles** of an exterior angle.

So,  $\angle ACB$  and  $\angle CAB$  are remote interior angles of exterior  $\angle ABD$ .

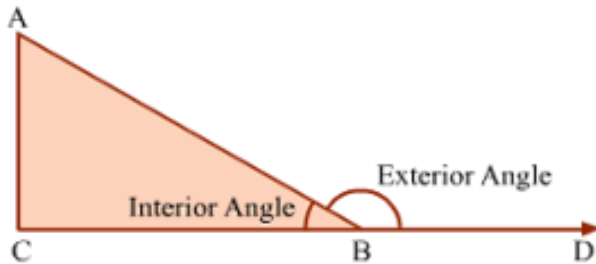
### **Exterior Angle Theorem or Remote Interior Angles Theorem and Its Proof**

#### **Corollary Related to Exterior Angle Theorem**

There is a corollary related to exterior angle theorem which states that:

**An exterior angle is always greater than each of its remote interior angles.**

Let us prove this corollary with the help of  $\triangle ABC$  shown in the figure.



Here,  $\angle ABD$  is an exterior angle of the triangle and its interior opposite or remote interior angles are  $\angle ACB$  and  $\angle CAB$ .

In a triangle, no interior angle can be zero angle or straight angle.

Thus,  $0^\circ < \angle ABC < 180^\circ$ ,  $0^\circ < \angle ACB < 180^\circ$  and  $0^\circ < \angle CAB < 180^\circ$

Now,  $\angle ABC < 180^\circ$

$\therefore 180^\circ - \angle ABC > 0^\circ$

$\Rightarrow \angle ABD > 0^\circ$

In triangle  $\triangle ABC$ , we have

$$\angle ABD = \angle ACB + \angle CAB \quad (\text{By exterior angle theorem})$$

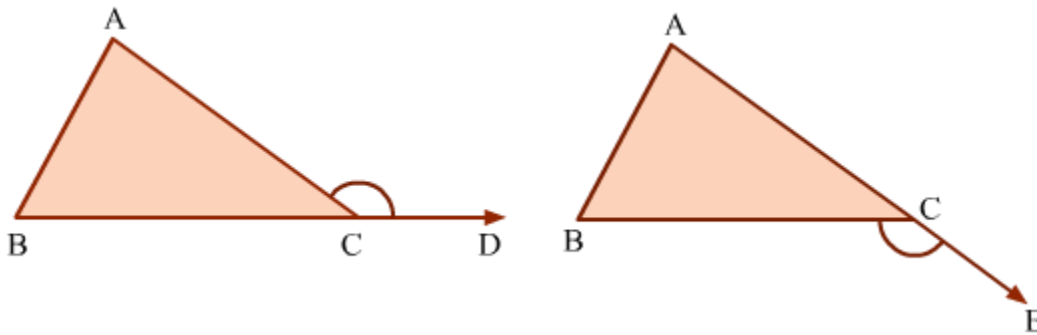
And,  $\angle ABD > 0$ ,  $\angle ACB > 0^\circ$  and  $\angle CAB > 0^\circ$  (Property of triangle)

**Therefore,  $\angle ABD > \angle ACB$  and  $\angle ABD > \angle CAB$**

Thus, an exterior angle is always greater than each of its remote interior angles.

### Two Exterior Angles at the Same Vertex are Equal

At any vertex, two exterior angles can be drawn by extending each of the two sides forming that vertex. These exterior angles are always of equal measure. Let us prove this using the  $\triangle ABC$  shown in the figure.



The figure clearly shows that two exterior angles can be drawn at vertex C—one by producing BC up to point D and the other by producing AC up to point E. The exterior angles thus obtained are  $\angle ACD$  and  $\angle BCE$ .

According to the exterior angle theorem, the measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles of the triangle.

$$\therefore \angle ACD = \angle ABC + \angle BAC \dots (1)$$

$$\text{And, } \angle BCE = \angle ABC + \angle BAC \dots (2)$$

Using equations 1 and 2, we get:

$$\angle ACD = \angle BCE$$

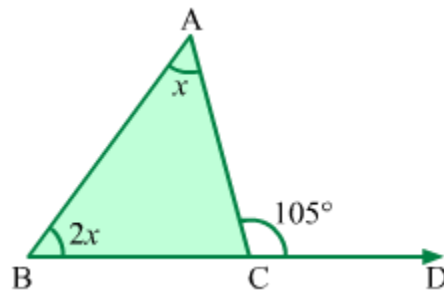
So, we can conclude that two exterior angles can be drawn at any vertex. The two angles thus drawn have an equal measure and are equal to the sum of the two opposite interior angles.

## Solved Examples

Easy

Example 1:

Find the value of  $x$  in the given figure.



**Solution:**

According to the exterior angle property of triangles, the measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles of the triangle.

So, we have:

$$x + 2x = 105^\circ$$

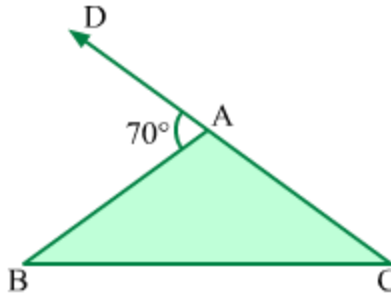
$$\Rightarrow 3x = 105^\circ$$

On dividing both sides of the equation by 3, we obtain:

$$\frac{3x}{3} = \frac{105^\circ}{3}$$
$$\Rightarrow x = 35^\circ$$

Example 2:

If  $\angle ABC = \angle ACB$  in  $\triangle ABC$ , then find the measure of  $\angle ABC$ .



**Solution:**

$\angle ABC$  and  $\angle ACB$  are interior angles opposite to the exterior angle at vertex A, i.e.,  $\angle BAD$ .

Therefore, by the exterior angle property of triangles, we obtain:

$$\angle ABC + \angle ACB = \angle BAD$$

$$\Rightarrow \angle ABC + \angle ACB = 70^\circ$$

It is given that  $\angle ABC = \angle ACB$

So, we obtain:

$$\angle ABC + \angle ABC = 70^\circ$$

$$\Rightarrow 2\angle ABC = 70^\circ$$

On dividing both sides of the equation by 2, we obtain:

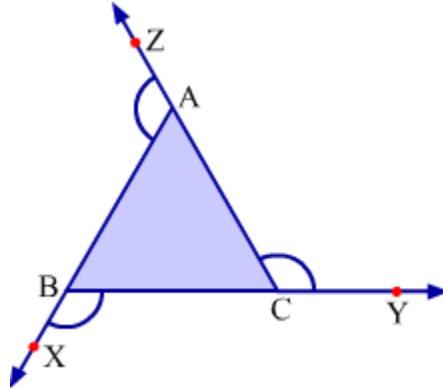
$$\frac{2\angle ABC}{2} = \frac{70^\circ}{2}$$

$$\Rightarrow \angle ABC = 35^\circ$$

**Medium**

**Example 1:**

The sides AB, BC and CA of  $\Delta ABC$  are produced up to points X, Y and Z respectively. Find the sum of the three exterior angles so formed.



**Solution:**

Using the exterior angle property, we obtain:

$$\angle BAZ = \angle ABC + \angle ACB \dots (1)$$

$$\angle CBX = \angle BAC + \angle ACB \dots (2)$$

$$\angle ACY = \angle BAC + \angle ABC \dots (3)$$

On adding equations 1, 2 and 3, we obtain:

$$\angle BAZ + \angle CBX + \angle ACY = \angle ABC + \angle ACB + \angle BAC + \angle ACB + \angle BAC + \angle ABC$$

$$\Rightarrow \angle BAZ + \angle CBX + \angle ACY = 2(\angle ABC + \angle ACB + \angle BAC)$$

According to the angle sum property of triangles, we have:

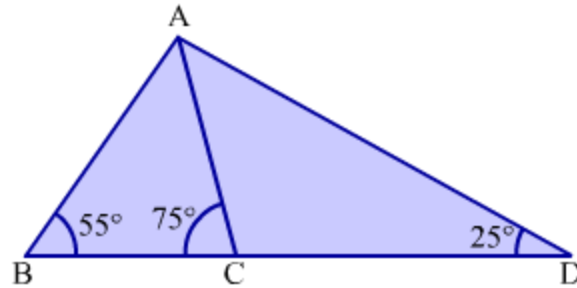
$$\angle ABC + \angle ACB + \angle BAC = 180^\circ$$

$$\therefore \angle BAZ + \angle CBX + \angle ACY = 2 \times 180^\circ = 360^\circ$$

Thus, the sum of the three exterior angles is  $360^\circ$ .

**Example 2:**

**Show that AC is the bisector of  $\angle BAD$  in the given figure.**



**Solution:**

On applying the angle sum property in  $\Delta ABC$ , we get:

$$\angle BAC + \angle ABC + \angle ACB = 180^\circ$$

$$\Rightarrow \angle BAC + 55^\circ + 75^\circ = 180^\circ$$

$$\Rightarrow \angle BAC = 180^\circ - 130^\circ$$

$$\Rightarrow \angle BAC = 50^\circ \dots (1)$$

Now, by using the exterior angle property, we get:

$$\angle ACB = \angle ADC + \angle CAD$$

$$\Rightarrow 75^\circ = 25^\circ + \angle CAD$$

$$\Rightarrow \angle CAD = 75^\circ - 25^\circ$$

$$\Rightarrow \angle CAD = 50^\circ$$

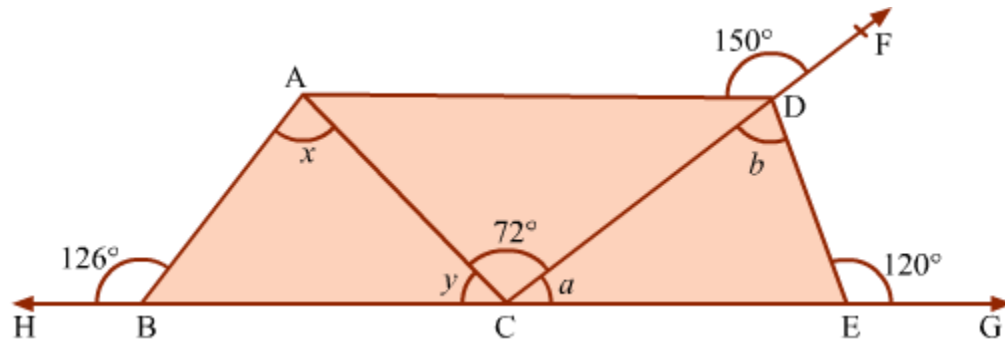
We know that  $\angle BAC + \angle CAD = \angle BAD$ . We have found  $\angle BAC = \angle CAD = 50^\circ$ .

Thus, AC is the bisector of  $\angle BAD$ .

**Hard**

**Example 1:**

If  $AD \parallel BE$  in the given figure, then find the values of  $a$ ,  $b$ ,  $x$  and  $y$ .



**Solution:**

From the figure, we have:

$$\angle ADC + \angle ADF = 180^\circ \text{ (Linear pair of angles)}$$

$$\Rightarrow \angle ADC + 150^\circ = 180^\circ$$

$$\Rightarrow \angle ADC = 30^\circ$$

Consider the parallel lines AD and BE and the transversal CF.

$$\angle ADF = \angle DCB \text{ (Corresponding angles)}$$

$$\Rightarrow 150^\circ = 72^\circ + y$$

$$\Rightarrow y = 78^\circ \dots (1)$$

Now,  $y + 72^\circ + a = 180^\circ$  (As they form line BCE)

$$\Rightarrow 78^\circ + 72^\circ + a = 180^\circ \text{ (Using equation 1)}$$

$$\Rightarrow a = 180^\circ - 150^\circ$$

$$\Rightarrow a = 30^\circ \dots (2)$$

Consider  $\triangle CDE$ .

$$\angle DEG = a + b \text{ (Exterior angle property)}$$

$$\Rightarrow 120^\circ = 30^\circ + b \text{ (Using equation 2)}$$

$$\Rightarrow b = 90^\circ$$

Now, consider  $\triangle ABC$ .

$\angle ABH = x + y$  (Exterior angle property)

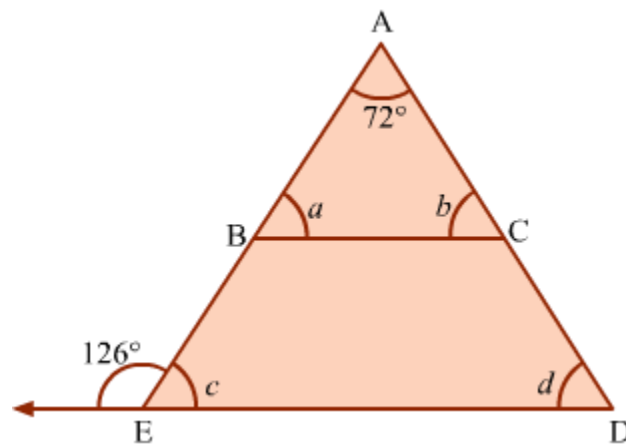
$$\Rightarrow 126^\circ = x + 78^\circ$$

$$\Rightarrow x = 48^\circ$$

Hence,  $a = 30^\circ$ ,  $b = 90^\circ$ ,  $x = 48^\circ$  and  $y = 78^\circ$ .

**Example 2:**

$\triangle ABC$  is placed atop trapezium  $EBCD$  in the given figure. Find the values of  $a$ ,  $b$ ,  $c$  and  $d$ .



**Solution:**

The exterior angle at E forms a linear pair with  $c$ .

$$\therefore 126^\circ + c = 180^\circ$$

$$\Rightarrow c = 180^\circ - 126^\circ$$

$$\Rightarrow c = 54^\circ$$

On using the exterior angle property in  $\triangle AED$ , we get:

$$126^\circ = 72^\circ + d$$

$$\Rightarrow d = 126^\circ - 72^\circ$$

$$\Rightarrow d = 54^\circ$$

Since EBCD is a trapezium, BC is parallel to ED. Also, BE and CD are transversals lying on the two parallel lines.

So, we have:

$$a = c = 54^\circ \text{ (Pair of corresponding angles)}$$

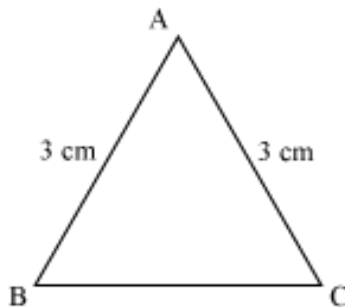
$$b = d = 54^\circ \text{ (Pair of corresponding angles)}$$

$$\text{Thus, } a = b = c = d = 54^\circ.$$

## Isosceles Triangles

We can classify triangles into different categories according to its sides (or angles). Now, let us study about a special type of triangle known as isosceles triangle.

“A triangle in which two sides are of equal length is called isosceles triangle.”



In  $\Delta ABC$ , two sides AB and AC are of equal length. Therefore,  $\Delta ABC$  is an isosceles triangle.

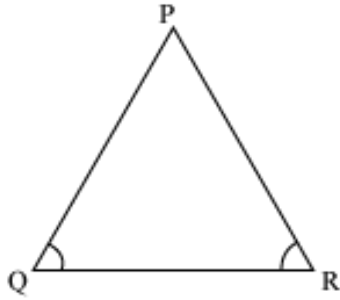
“In an isosceles triangle, angles opposite to equal sides are also equal.”

In the above figure,  $\angle B$  and  $\angle C$  are the angles opposite to equal sides AC and AB respectively.

$$\therefore \angle B = \angle C$$

Therefore, we can also define isosceles triangle in terms of angles.

“If two angles of a triangle are equal in measure, then the triangle is called isosceles triangle.”



In the given figure,

$$\angle Q = \angle R$$

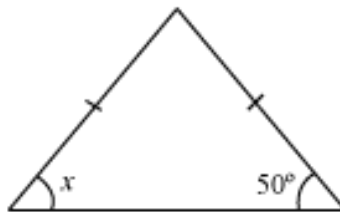
$\therefore \Delta PQR$  is an isosceles triangle.

Now, let us solve some examples involving isosceles triangles.

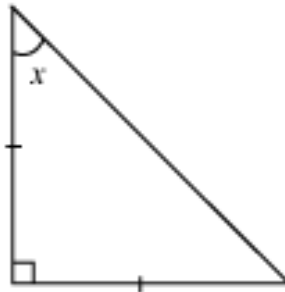
**Example 1:**

**Find the value of  $x$  for the following figures.**

(i)



(ii)



**Solution:**

(i) The given triangle is an isosceles triangle.

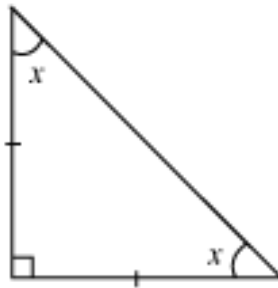
The angles opposite to equal sides are equal.

$$\therefore x = 50^\circ$$

(ii) The given triangle is an isosceles triangle.

The angles opposite to equal sides are equal.

Thus, we have



Now, using angle sum property of triangles, we obtain

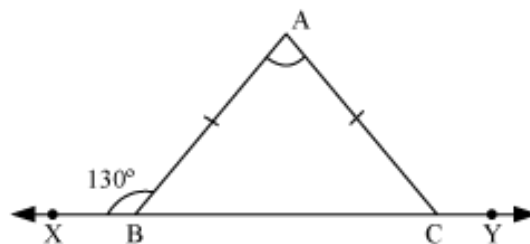
$$x + x + 90^\circ = 180^\circ$$

$$2x = 90^\circ$$

$$x = 45^\circ$$

**Example 2:**

**In the given figure, find  $\angle BAC$ .**



**Solution:**

It is given that  $\triangle ABC$  is an isosceles triangle. We know that in an isosceles triangle, angles opposite to equal sides are equal.

$$\therefore \angle ABC = \angle ACB \text{ (1)}$$

Now,  $\angle ABC = 180^\circ - \angle ABX$  (By linear pair axiom)

$$= 180^\circ - 130^\circ$$

$$= 50^\circ$$

From equation (1), we obtain

$$\angle ACB = 50^\circ$$

Using angle sum property in  $\triangle ABC$ , we obtain

$$\angle ABC + \angle ACB + \angle BAC = 180^\circ$$

$$50^\circ + 50^\circ + \angle BAC = 180^\circ$$

$$\angle BAC = 180^\circ - 100^\circ$$

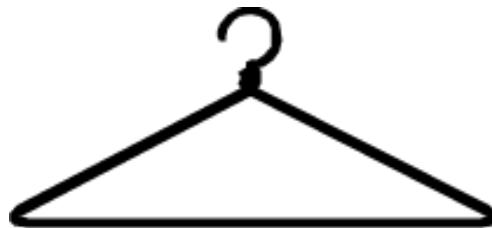
$$\angle BAC = 80^\circ$$

Thus, the measure of  $\angle BAC$  is  $80^\circ$ .

### **Angles Opposite to Equal Sides of an Isosceles Triangle are Equal**

#### **Isosceles Triangles around Us**

You must have seen triangular clothes hangers or coat hangers such as the one shown.



What do you observe about the sides of the hanger? The base is clearly the longest side, while the other two sides are equal. We know that a triangle with any two sides equal is called an isosceles triangle. So, the hanger is a real-life example of an isosceles triangle. Another example of an isosceles triangle is the roof of a hut. Yet another example is a diagonally cut slice of bread.

You can see that the angles opposite the equal sides of the clothes hanger are equal. In this lesson, we will discuss about the same, i.e., angles opposite equal sides of an isosceles triangle. We will also solve some problems based on this concept.

## Angles Opposite Equal Sides of an Isosceles Triangle Are Equal

### Know Your Scientist



Thales (624 BC–546 BC) is believed to be the first philosopher in the Greek tradition. A great mathematician and scientist, he is regarded as ‘the father of science’. Thales is credited for giving the first known mathematical proof, i.e., ‘a circle is bisected by its diameter’. He was also the first to prove the theorem: ‘any angle inscribed in a semicircle is a right angle’. For this reason, this theorem is called Thales Theorem.

Thales also generalized the concept of equality of the base angles of an isosceles triangle. He also propounded a few general notions such as ‘all straight angles are equal’, ‘equals subtracted from equals are equal’ and ‘equals added to equals are equal’. These theorems have helped in the evolution of modern mathematics.

### Did You Know?

The word ‘isosceles’ is a combination of the Greek words ‘isos’ meaning ‘equal’ and ‘skelos’ meaning ‘leg’. So, a triangle having two equal legs is called an isosceles triangle.

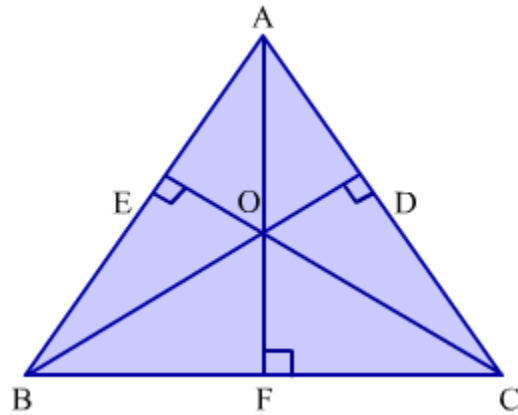
### Know More

- The altitude to the base of an isosceles triangle bisects the vertex angle.
- The altitude to the base of an isosceles triangle bisects the base.
- When the equal sides are at right angle, the triangle is called a ‘right isosceles triangle’.

### Whiz Kid

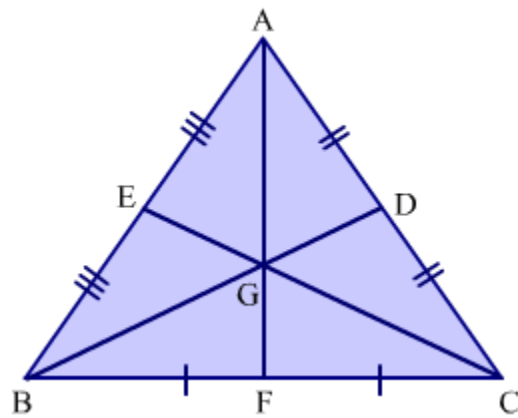
In an isosceles triangle, the orthocentre, the centroid, the incentre and the circumcentre, all lie on the median to the base.

**Orthocentre:** It is the point where the altitudes of the three sides of a triangle intersect.



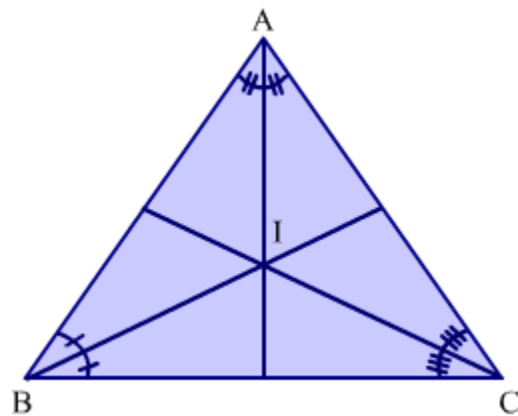
In  $\triangle ABC$ , O is the orthocentre.

**Centroid:** It is the point where the medians of the three sides of a triangle intersect.



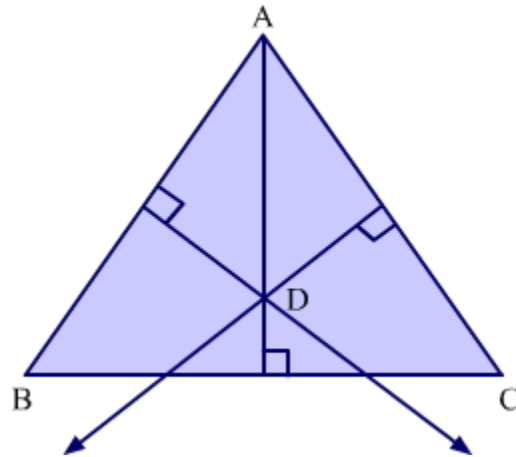
In  $\triangle ABC$ , G is the centroid.

**Incentre:** It is the point where the interior angle bisectors of a triangle intersect.



In  $\triangle ABC$ , I is the incentre.

**Circumcentre:** It is the point where the perpendicular bisectors of the three sides of a triangle intersect.



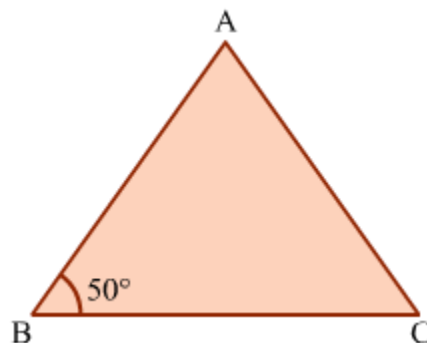
In  $\triangle ABC$ , D is the circumcentre.

### Solved Examples

Easy

**Example 1:**

In the given  $\triangle ABC$ ,  $AB = AC$ . What is the measure of  $\angle BAC$ ?



**Solution:**

$\triangle ABC$  is an isosceles triangle with  $AB = AC$ .

By the property of isosceles triangles, we obtain:

$$\angle ABC = \angle ACB$$

$$\Rightarrow \angle ACB = 50^\circ$$

By the angle sum property, we have:

$$\angle ABC + \angle ACB + \angle BAC = 180^\circ$$

$$\Rightarrow 50^\circ + 50^\circ + \angle BAC = 180^\circ$$

$$\Rightarrow \angle BAC = 180^\circ - 100^\circ$$

$$\Rightarrow \angle BAC = 80^\circ$$

### **Example 2:**

Ram lives in a triangular tree house. The measure of each base angle is the same and the length of the base is 4 m. He wants to cover the slant sides with metal sheets. If the boundary of the triangular structure is 10 m, then what length of sheet does Ram need to cover the slant sides (irrespective of the width of the sheet).

### **Solution:**

It is given that the base angles are equal; so, the triangular structure is isosceles. Hence, the slant sides are equal.

Now, we know that:

$$\text{Base} = 4 \text{ m}$$

$$\text{Boundary of the triangular structure} = \text{Perimeter of the isosceles triangle} = 10 \text{ m}$$

$$\Rightarrow 10 \text{ m} = 4 \text{ m} + 2 \times (\text{Measure of each slant side})$$

$$\Rightarrow 2 \times (\text{Measure of each slant side}) = 10 \text{ m} - 4 \text{ m}$$

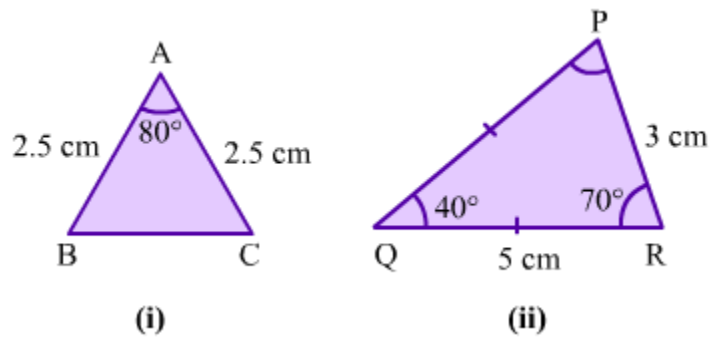
$$\Rightarrow 2 \times (\text{Measure of each slant side}) = 6 \text{ m}$$

Hence, Ram needs 6 m of metal sheet to cover the slant sides of the triangular structure.

### **Medium**

### **Example 1:**

Find the missing angles in the following triangles.



**Solution:**

1. In  $\triangle ABC$ ,  $AB = AC = 2.5$  cm

Since the angles opposite equal sides of a triangle are equal, we obtain:

$$\angle ABC = \angle ACB$$

$$\text{Let } \angle ABC = \angle ACB = x$$

By the angle sum property of triangles, we have:

$$\angle ABC + \angle ACB + \angle BAC = 180^\circ$$

$$\Rightarrow x + x + 80^\circ = 180^\circ$$

$$\Rightarrow 2x = 100^\circ$$

$$\Rightarrow x = 50^\circ$$

$$\text{Thus, } \angle ABC = \angle ACB = 50^\circ$$

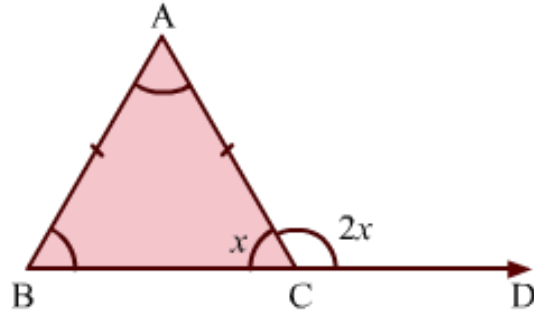
2. In  $\triangle PQR$ ,  $PQ = QR = 5$  cm and  $PR = 3$  cm

Since  $PQ = QR$ , we obtain

$$\angle QPR = \angle PRQ = 70^\circ$$

**Example 2:**

The shown  $\triangle ABC$  is isosceles with  $AB = AC$ . Find the measures of  $\angle BAC$ ,  $\angle ABC$  and  $\angle ACB$ .



**Solution:**

$\angle ACB$  and exterior angle  $\angle ACD$  form a linear pair.

$$\therefore \angle ACB + \angle ACD = 180^\circ$$

$$\Rightarrow x + 2x = 180^\circ$$

$$\Rightarrow 3x = 180^\circ$$

$$\Rightarrow \therefore x = 60^\circ$$

So,  $\angle ACB = 60^\circ$  and  $\angle ACD = 120^\circ$

$\triangle ABC$  is isosceles.

$\therefore \angle ABC = \angle ACB = 60^\circ$  ( $\because$  Angles opposite equal sides are equal)

Now, by the exterior angle property, we have:

$$\angle ACD = \angle BAC + \angle ABC$$

$$\Rightarrow 120^\circ = \angle BAC + 60^\circ$$

$$\Rightarrow \angle BAC = 60^\circ$$

Thus,  $\angle BAC = \angle ABC = \angle ACB = 60^\circ$ .

**Example 3:**

Prove that in an isosceles triangle, the angle bisector of the apex angle is the perpendicular bisector of the base.

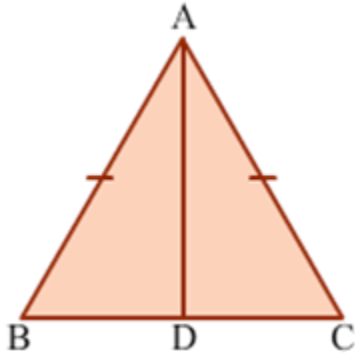
**Solution:**

**Given:** ABC is a triangle in which  $AB = AC$  and apex angle  $\angle A$ .

**To Prove:** AD is perpendicular bisector of BC and  $BD = DC$ .

**Construction:** Draw an angle bisector AD from A on BC.

**Proof:**



In  $\triangle ABD$  and  $\triangle ACD$ ,  
 $AB = AC$  (Given)

$AD = AD$  (Common side)  
 $\angle BAD = \angle CAD$  (AD is a bisector of  $\angle A$ )

So, by SAS congruence criterion,  
 $\triangle ADB \cong \triangle ADC$   
 $\Rightarrow BD = DC$  and  $\angle ADB = \angle ADC$  (CPCT)

Now,  
 $\angle ADB + \angle ADC = 180^\circ$  (Linear Pair)  
 $\Rightarrow \angle ADB + \angle ADB = 180^\circ$   
 $\Rightarrow 2\angle ADB = 180^\circ$

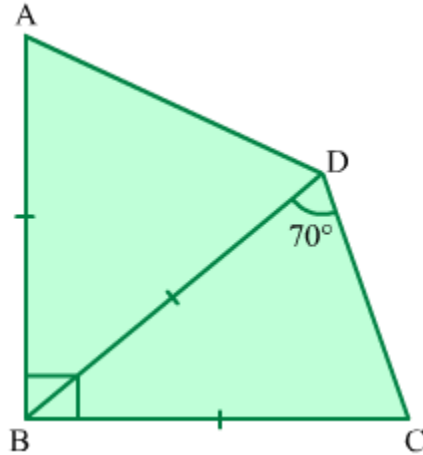
$$\Rightarrow \angle ADB = \frac{180^\circ}{2} = 90^\circ$$

Thus, AD is the perpendicular bisector of BC.

**Hard**

**Example 1:**

$\triangle ABD$  is isosceles (figure not drawn to scale) with  $AB = BD$ , while  $\triangle BCD$  is isosceles with  $BC = BD$ . Also,  $\angle BDC$  measures  $70^\circ$  and  $\angle ABC$  measures  $90^\circ$ . What is the measure of  $\angle ADC$ ?



**Solution:**

In  $\triangle BCD$ ,  $BC = BD$ .

$\therefore \angle BCD = \angle BDC = 70^\circ$  ( $\because$  Angles opposite equal sides are equal)

In  $\triangle BCD$ , by the angle sum property, we obtain:

$$\angle BCD + \angle BDC + \angle CBD = 180^\circ$$

$$\Rightarrow 70^\circ + 70^\circ + \angle CBD = 180^\circ$$

$$\Rightarrow 140^\circ + \angle CBD = 180^\circ$$

$$\Rightarrow \angle CBD = 40^\circ$$

It is given that  $\angle ABC = 90^\circ$ .

$$\Rightarrow \angle ABD + \angle CBD = 90^\circ$$

$$\Rightarrow \angle ABD + 40^\circ = 90^\circ$$

$$\Rightarrow \angle ABD = 50^\circ$$

In  $\triangle ABD$ , we have:

$$AB = BD$$

$\therefore \angle BAD = \angle BDA = x$  ( $\because$  Angles opposite equal sides are equal)

In  $\triangle ABD$ , by the angle sum property, we obtain:

$$\angle ABD + \angle BAD + \angle BDA = 180^\circ$$

$$\Rightarrow 50^\circ + x + x = 180^\circ$$

$$\Rightarrow 2x = 130^\circ$$

$$\Rightarrow x = 65^\circ$$

$$\text{So, } \angle BAD = \angle BDA = 65^\circ$$

Now,

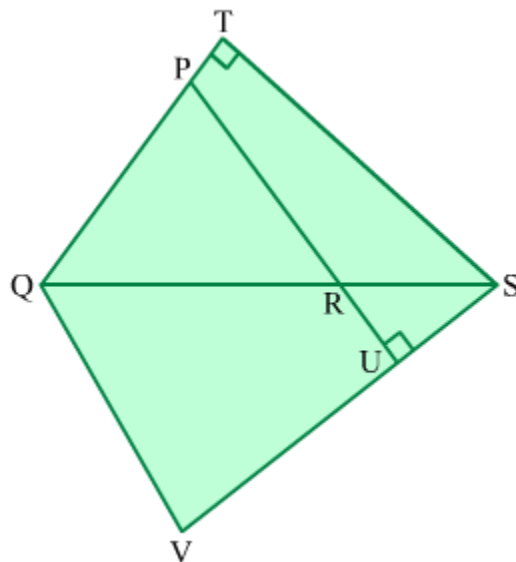
$$\angle ADC = \angle BDA + \angle CDB$$

$$\Rightarrow \angle ADC = 65^\circ + 70^\circ$$

$$\Rightarrow \angle ADC = 135^\circ$$

### Example 2:

In the given figure,  $\Delta PQR$  is isosceles with  $PQ = PR$ . Side  $SU$  is extended up to point  $V$  such that  $ST = SV$ . If  $\angle QTS = \angle RUS = 90^\circ$ , prove that  $QS$  bisects  $\angle TSU$  and hence show that  $\Delta QTS \cong \Delta QVS$ .



### Solution:

In  $\Delta PQR$ , we have:

$PQ = PR$  (Given)

$\therefore \angle PQR = \angle PRQ$  ( $\because$  Angles opposite equal sides are equal)

It is given that  $\angle QTS = 90^\circ$ .

Using the angle sum property in  $\Delta QTS$ , we obtain:

$$\angle TQS + \angle TSQ + \angle QTS = 180^\circ$$

$$\Rightarrow \angle TQS + \angle TSQ + 90^\circ = 180^\circ$$

$$\Rightarrow \angle TSQ = 90^\circ - \angle TQS \dots (1)$$

It is given that  $\angle RUS = 90^\circ$ .

Using the angle sum property in  $\Delta RUS$ , we obtain:

$$\angle SRU + \angle RUS + \angle RSU = 180^\circ$$

$$\Rightarrow \angle SRU + 90^\circ + \angle RSU = 180^\circ$$

$$\Rightarrow \angle RSU = 90^\circ - \angle SRU \dots (2)$$

$\angle SRU = \angle PRQ$  (Vertically opposite angles)

Also,  $\angle PQR = \angle PRQ$

$$\therefore \angle SRU = \angle PQR \dots (3)$$

Now, from equations (2) and (3), we get:

$$\angle RSU = 90^\circ - \angle PQR$$

$$\Rightarrow \angle RSU = 90^\circ - \angle TQS \dots (4) \quad [\because \angle PQR = \angle TQS]$$

From equations (1) and (4), we get:

$$\angle TSQ = \angle RSU$$

Hence, QS bisects  $\angle TSU$ .

Now, consider  $\Delta QTS$  and  $\Delta QVS$ .

$QS = QS$  (Common side)

$\angle TSQ = \angle QSV$  ( $\because \angle TSQ = \angle RSU$  and  $\angle RSU = \angle QSV$ )

$ST = SV$  (Given)

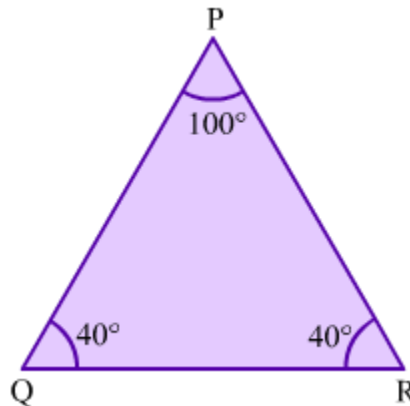
Thus, by the SAS congruency rule, we get:

$\triangle QTS \cong \triangle QVS$

### **Sides Opposite to Equal Angles of a Triangle are Equal**

#### **Observing the Equal Angles and the Sides Opposite to Them in an Isosceles Triangle**

Consider the following  $\triangle PQR$ .



Is  $\triangle PQR$  isosceles? We know that if two sides of a triangle are equal (or congruent), then the triangle is isosceles. However, in  $\triangle PQR$ , two angles are equal (or congruent). We have studied that the angles opposite to equal (or congruent) sides of an isosceles triangle are equal (or congruent). Is the converse of this property also true?

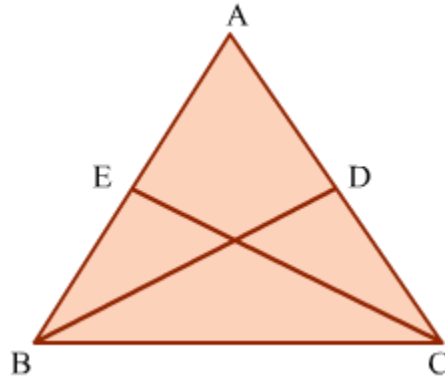
In this lesson, we will study about the equality of sides opposite equal angles in an isosceles triangle. We will also solve some examples related to this concept.

### **Sides Opposite To Equal Angles of a Triangle Are Equal**

#### **Whiz Kid**

In an isosceles triangle, the medians drawn from the base vertices to the opposite sides are of equal length.

For example:



The shown  $\triangle ABC$  is isosceles such that  $AB = AC$ .  $BD$  and  $CE$  are the respective medians from vertices  $B$  and  $C$  to sides  $AC$  and  $AB$ . Therefore,  $BD = CE$ .

### Solved Examples

#### Easy

#### Example 1:

In a  $\triangle ABC$ ,  $\angle BAC = 2x$  and  $\angle ABC = \angle ACB = x$ . Find the value of  $x$  and hence show that  $AB = AC$ .

#### Solution:

It is given that  $\angle BAC = 2x$  and  $\angle ABC = \angle ACB = x$ .

By applying the angle sum property, we obtain:

$$\angle BAC + \angle ABC + \angle ACB = 180^\circ$$

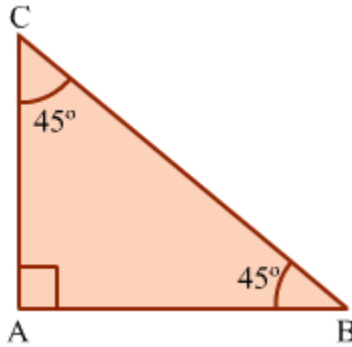
$$\Rightarrow 2x + x + x = 180^\circ$$

$$\Rightarrow 4x = 180^\circ$$

$$\Rightarrow x = 45^\circ$$

So,  $\angle BAC = 2x = 2 \times 45^\circ = 90^\circ$  and  $\angle ABC = \angle ACB = x = 45^\circ$

The given triangle can be drawn as is shown.



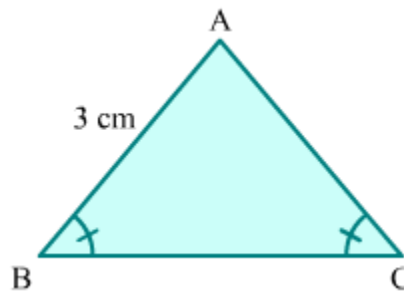
We know that the sides opposite equal angles of a triangle are equal.

$$\therefore AB = AC$$

Hence,  $\triangle ABC$  is isosceles with  $AB = AC$ .

**Example 2:**

In the given  $\triangle ABC$ ,  $\angle ABC = \angle ACB$  and the perimeter is 11 cm. Find the length of the base of the triangle.



**Solution:**

It is given that  $\angle ABC = \angle ACB$  and  $AB = 3$  cm.

We know that the sides opposite equal angles of a triangle are equal.

$$\therefore AC = AB = 3 \text{ cm}$$

$$\text{Perimeter of } \triangle ABC = AB + BC + AC$$

$$\Rightarrow 11 \text{ cm} = 3 \text{ cm} + BC + 3 \text{ cm}$$

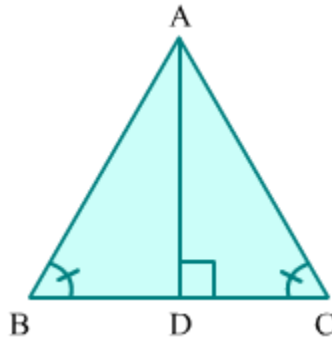
$$\Rightarrow BC = 5 \text{ cm}$$

Thus, the length of the base of  $\triangle ABC$  is 5 cm.

### Medium

#### Example 1:

In the given  $\triangle ABC$ ,  $\angle ABD = \angle ACD$  and  $AD$  is perpendicular to  $BC$ . Prove that  $AD$  bisects  $\angle BAC$ .



#### Solution:

In  $\triangle ABD$  and  $\triangle ACD$ , we have:

$$\angle ABD = \angle ACD \text{ (Given)}$$

$$\angle ADB = \angle ADC = 90^\circ$$

$$AB = AC \text{ } (\because \text{Sides opposite equal angles are equal})$$

Thus, by the AAS congruence rule, we have:

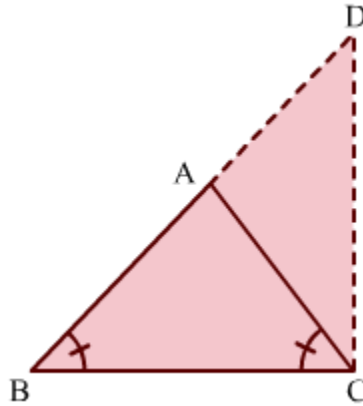
$$\triangle ABD \cong \triangle ACD$$

$$\therefore \angle BAD = \angle CAD \text{ (By CPCT)}$$

Thus,  $AD$  bisects  $\angle BAC$ .

#### Example 2:

In the given  $\triangle ABC$ ,  $\angle ABC = \angle ACB$ . Side  $BA$  is produced up to point  $D$  such that  $AB = AD$ . Prove that  $\angle BCD$  is a right angle.



**Solution:**

In  $\triangle ABC$ , we have:

$$\angle ABC = \angle ACB \dots (1) \text{ [Given]}$$

$$\therefore AB = AC \quad (\because \text{Sides opposite equal angles are equal})$$

Now,

$$AB = AD \quad (\text{Given})$$

$$AD = AC \quad (\because AB = AC)$$

Thus, in  $\triangle ADC$ , we have:

$$\angle ACD = \angle ADC \dots (2) \quad (\because \text{Angles opposite equal sides are equal})$$

On adding equations (1) and (2), we get:

$$\angle ACB + \angle ACD = \angle ABC + \angle ADC$$

$$\Rightarrow \angle BCD = \angle ABC + \angle BDC \quad (\because \angle ADC = \angle BDC)$$

On adding  $\angle BCD$  to both sides of the equation, we get:

$$\angle BCD + \angle BCD = \angle ABC + \angle BDC + \angle BCD$$

$$\Rightarrow 2\angle BCD = 180^\circ \quad (\text{By the angle sum property})$$

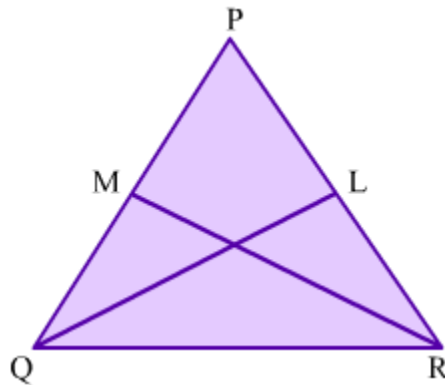
$$\Rightarrow \angle BCD = 90^\circ$$

Hence,  $\angle BCD$  is a right angle.

### Hard

#### Example 1:

The shown  $\Delta PQR$  is isosceles with  $PQ = PR$ .  $QL$  and  $RM$  are the respective medians from vertices  $Q$  and  $R$  to sides  $PR$  and  $PQ$ . Prove that the medians have the same length.



#### Solution:

It is given that  $PQ = PR$ .

Also,  $QL$  and  $RM$  are medians.

$\therefore PL = LR$  and  $PM = MQ$

So,

$$PL + LR = PR$$

$$LR + LR = PR$$

$$2LR = PR$$

$$LR = \frac{PR}{2} \dots (1)$$

$$\text{Similarly, } MQ = \frac{PQ}{2} \dots (2)$$

Since  $PQ = PR$ , using equations (1) and (2), we obtain:

$$LR = MQ \dots (3)$$

In  $\triangle QRL$  and  $\triangle RQM$ , we have:

$$LR = MQ \text{ (From (3))}$$

$$\angle LRQ = \angle MQR \text{ } (\because \text{Angles opposite equal sides are equal})$$

$$QR = RQ \text{ (Common side)}$$

$$\therefore \triangle QRL \cong \triangle RQM \text{ (By the SAS congruence criterion)}$$

$$\Rightarrow QL = RM \text{ (By CPCT)}$$

Thus, the medians  $QL$  and  $RM$  have the same length.

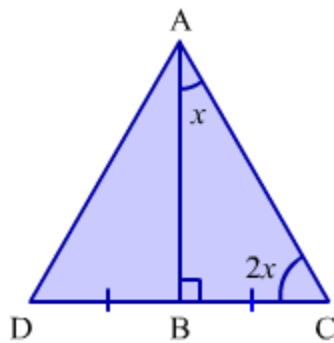
### Example 2:

In a right-angled triangle  $ABC$ ,  $\angle ACB = 2\angle BAC$ . Prove that  $AC = 2BC$ .

### Solution:

The given  $\triangle ABC$  can be drawn as is shown.

**Construction:** Produce  $CB$  up to point  $D$  such that  $BD = BC$ . Join point  $A$  to point  $D$ .



In  $\triangle ABD$  and  $\triangle ABC$ , we have:

$$BD = BC \text{ (By construction)}$$

$$\angle ABD = \angle ABC = 90^\circ$$

$AB = AB$  (Common side)

So, by the SAS congruence rule, we obtain:

$$\triangle ABD \cong \triangle ABC$$

$\Rightarrow AD = AC$  and  $\angle DAB = \angle BAC$  (By CPCT)

Let  $\angle BAC$  be  $x$ . Then,  $\angle DAB$  will also be  $x$ .

Now,  $\angle DAC = \angle DAB + \angle BAC$

$$\Rightarrow \angle DAC = x + x$$

$$\Rightarrow \angle DAC = 2x$$

$\Rightarrow \angle DAC = \angle ACB$  ( $\because \angle ACB = 2\angle BAC = 2x$ )

$\Rightarrow DC = AD$  ( $\because$  Sides opposite equal angles are equal)

Since  $BC = DB$ , we have:

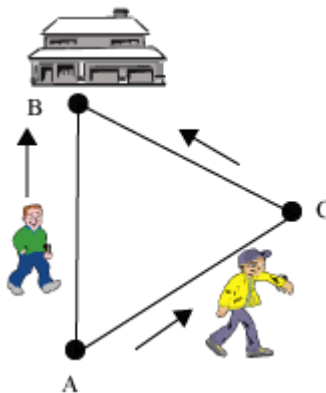
$$DC = 2BC$$

$$\Rightarrow 2BC = AD$$

$\Rightarrow 2BC = AC$  ( $\because$  We have proved  $AD = AC$ )

## Triangle Inequalities

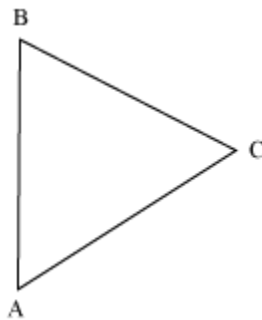
Rohit and Mohit are brothers. Their house is located at position B, as shown in the following figure.



They are standing at a position A and want to reach their house. Rohit chooses the path AB to reach his house while Mohit first goes to position C starting from A and then travels distance CB to reach his house. Both of them walk at the same speed. **Who will take less time to reach the house?**

Since their speeds are the same, more time will be taken by the person who has to cover the larger distance. Therefore, we can say that Rohit will take less time in reaching the house as the path followed by Rohit is shorter than the path followed by Mohit.

We can see that the paths followed by Rohit and Mohit form a triangle.



i.e. ABC is a triangle. From the above example, we can clearly see that length AB is shorter than length BC + CA. Or, we can say that length BC + CA is longer than length AB.

Therefore, we can conclude that

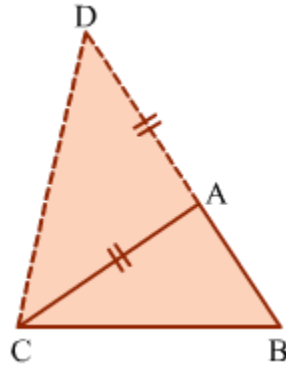
**“The sum of the lengths of any two sides of a triangle is always greater than the length of the third side of the triangle”.**

Now, from the above discussion, we can say that in  $\triangle ABC$ ,  $AB + BC > CA$ ,  $AB + CA > BC$  and  $CA + BC > AB$ .

Let us go through the proof of the above property.

**Proof:**

Let us take any  $\triangle ABC$  and extend the side AB to D such that  $AD = CA$ .



In  $\triangle ACD$ , we have

$$AD = CA$$

$$\Rightarrow \angle ACD = \angle ADC$$

$$\Rightarrow \angle ACD + \angle ACB > \angle ADC$$

$$\Rightarrow \angle BCD > \angle BDC \quad (\angle BDC \text{ is same as } \angle ADC)$$

$$\Rightarrow BD > BC \quad (\text{Side opposite to greater angle is longer})$$

$$\Rightarrow AB + AD > BC$$

$$\Rightarrow AB + CA > BC \quad (AD = CA)$$

Similarly, we can prove that  $AB + BC > CA$  and  $CA + BC > AB$ .

From the result  $CA + BC > AB$ , we can deduce that  $CA - AB < BC$ . Similarly, we can deduce results from the remaining two inequalities.

Thus, from this, we can conclude that

**“The difference between the lengths of any two sides of a triangle is always smaller than the length of the third side of the triangle”.**

The inequalities  $AB + CA > BC$ ,  $AB + BC > CA$  and  $CA + BC > AB$  are called triangle inequalities. They are necessary condition for the existence of a triangle with sides  $AB$ ,  $BC$  and  $CA$ . This tells that the straight line is the shortest path between any two points. Given three numbers  $a$ ,  $b$ ,  $c$ , then the necessary condition for the existence of a triangle with sides  $a$ ,  $b$ ,  $c$  are that  $b + c > a$ ,  $c + a > b$  and  $a + b > c$ .

Let us now look at some examples to understand this property of triangles better.

**Example 1:**

**Check whether it is possible to have a triangle with the following sides or not.**

1. **11.5 cm, 7.8 cm, 14.7 cm**
2. **7.4 cm, 18.5 cm, 10.9 cm**

**Solution:**

1. 11.5 cm, 7.8 cm, 14.7 cm

For a triangle, the sum of the lengths of any two sides must be greater than the third side.

Now,

$$11.5 \text{ cm} + 7.8 \text{ cm} = 19.3 \text{ cm} > 14.7 \text{ cm}$$

$$11.5 \text{ cm} + 14.7 \text{ cm} = 26.2 \text{ cm} > 7.8 \text{ cm}$$

$$7.8 \text{ cm} + 14.7 \text{ cm} = 22.5 \text{ cm} > 11.5 \text{ cm}$$

We can see that the sum of the lengths of any two sides is greater than the length of the third side.

Therefore, a triangle with sides of given lengths is possible.

2. 7.4 cm, 18.5 cm, 10.9 cm

For a triangle, the sum of the lengths of any two sides must be greater than the third side.

But in this case,

$$7.4 \text{ cm} + 10.9 \text{ cm} = 18.3 \text{ cm} < 18.5 \text{ cm}$$

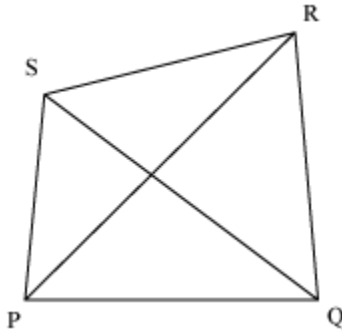
Therefore, a triangle with sides of given lengths is not possible.

**Example 2:**

**Can the sum of the four sides of a quadrilateral be equal to the sum of the lengths of its diagonals?**

**Solution:**

Consider a quadrilateral PQRS in which PR and QS are its two diagonals.



Now, we know that the sum of any two sides of a triangle is always greater than the third side. Therefore,

In  $\Delta PQR$ ,

$$PQ + QR > PR \dots (1)$$

In  $\Delta PRS$ ,

$$PS + RS > PR \dots (2)$$

In  $\Delta PQS$ ,

$$PQ + PS > QS \dots (3)$$

In  $\Delta SRQ$ ,

$$QR + RS > QS \dots (4)$$

On adding (1), (2), (3), and (4), we obtain

$$PQ + QR + PS + RS + PQ + PS + QR + RS > PR + PR + QS + QS$$

$$\Rightarrow (PQ + PQ) + (QR + QR) + (RS + RS) + (PS + PS) > (PR + PR) + (QS + QS)$$

$$\Rightarrow 2PQ + 2QR + 2RS + 2PS > 2PR + 2QS$$

$$\Rightarrow 2(PQ + QR + RS + PS) > 2(PR + QS)$$

On dividing both sides by 2, we obtain

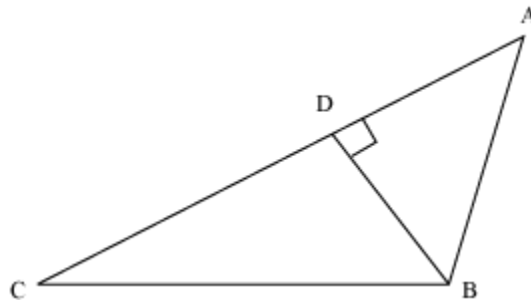
$$PQ + QR + RS + PS > PR + QS$$

Therefore, the sum of the lengths of the four sides of a quadrilateral is always greater than the sum of the lengths of its diagonals.

Hence, the sum of the lengths of the four sides cannot be equal to the sum of the lengths of its two diagonals.

**Example 3:**

**In the given figure,  $BD \perp AC$ .**



**Which of the following statements is correct?**

**(i)  $2 BD > AB + BC + CA$**

**(ii)  $2 BD < AB + BC + CA$**

**Solution:**

We know that the sum of any two sides of a triangle is always greater than the third side.

Therefore, in  $\triangle ABD$   $DA + AB > BD$  ... (i)

In  $\triangle BCD$ ,

$BC + CD > BD$  ... (ii)

On adding (i) and (ii), we obtain

$$AB + DA + BC + CD > BD + BD$$

$$\Rightarrow AB + BC + (CD + DA) > 2 BD$$

Now, from the figure,

$$CD + DA = CA$$

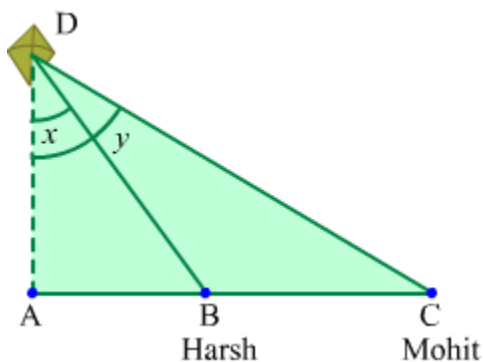
Therefore,

$$AB + BC + CA > 2 BD$$

$$\text{i.e. } 2 BD < AB + BC + CA$$

Therefore, statement (ii) is correct.

## Observing the Relation between the Angles and Sides of a Triangle



Harsh is flying a kite. The thread of the kite makes angle  $x$  with the vertical, as is shown in the figure. Sometime later, he gives the thread to his friend Mohit who is standing some distance away from him. At that position, the thread makes angle  $y$  with the vertical.

The figure clearly shows that angle  $y$  of  $\triangle ACD$  is greater than angle  $x$  of  $\triangle ABD$ . Also, side  $AC$  is greater than side  $AB$ . This shows us that if we increase the length of any side of a triangle, then the angle facing that side also increases.

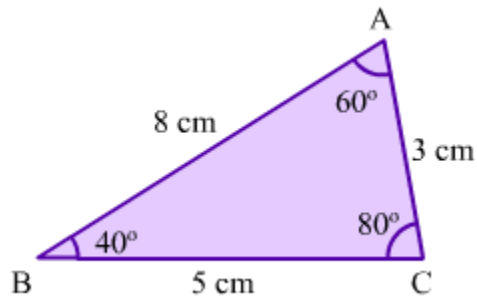
In this lesson, we will discuss the relation between the angles and sides of a triangle. We will then solve some examples relating to the same.

### Triangle Inequality Theorem

We know that if two sides of a triangle are equal, then the angles opposite these sides are also equal. Now, what **if all the sides of a triangle are unequal? What can be said about its angles?** The **triangle inequality theorem** describes such a triangle. It states that:

***If two sides of a triangle are unequal, then the longer side has the greater angle opposite it.***

Consider the given  $\triangle ABC$ .



Let us apply the stated theorem in this triangle.

$AB = 8$  cm is the longest side in  $\triangle ABC$ . Therefore, the angle opposite  $AB$ , i.e.,  $\angle BCA$  is the greatest angle of the triangle. Also,  $AC = 3$  cm is the shortest side in  $\triangle ABC$ . Therefore, the angle opposite  $AC$ , i.e.,  $\angle ABC$  is the smallest angle of the triangle.

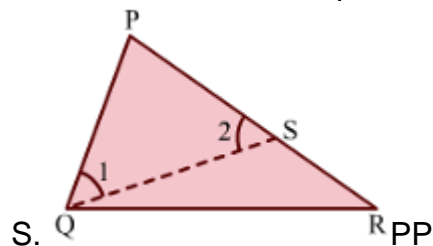
Thus, we can conclude that the angle opposite the longer side is greater. Therefore, the theorem holds true.

### Proving the Triangle Inequality Theorem

**Given:**  $\triangle PQR$  in which  $PR > PQ$

**To prove:**  $\angle Q > \angle R$

**Construction:** Mark a point  $S$  on  $PR$  such that  $PQ = PS$ . Join  $Q$  to



**Proof:** In  $\triangle PQS$ , we have:

$PQ = PS$  (By construction)

$\Rightarrow \angle 1 = \angle 2 \dots (1)$  [ $\because$  Angles opposite equal sides are equal]

In  $\triangle QRS$ ,  $\angle 2$  is the exterior angle; so, it is greater than the interior opposite angles of  $\triangle QRS$ .

$\therefore \angle 2 > \angle SRQ$

$$\Rightarrow \angle 2 > \angle PRQ \dots (2) [\because \angle PRQ = \angle SRQ]$$

From (1) and (2), we have:

$$\angle 1 > \angle PRQ \dots (3)$$

Now,  $\angle 1$  is a part of  $\angle PQR$ .

$$\text{So, } \angle PQR > \angle 1 \dots (4)$$

Thus, from (3) and (4), we can conclude that:

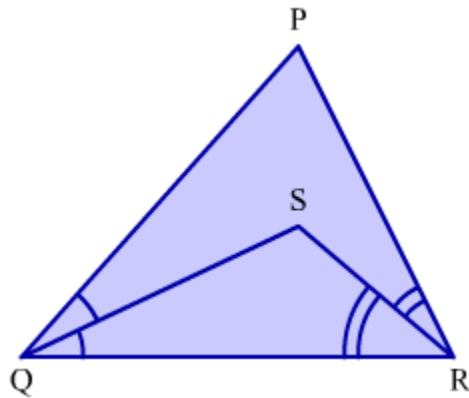
$$\angle PQR > \angle PRQ$$

### Solved Examples

#### Easy

#### Example 1:

In the given  $\triangle PQR$ ,  $PQ$  is greater than  $PR$ . Also,  $QS$  and  $RS$  are the respective bisectors of  $\angle PQR$  and  $\angle PRQ$ . Prove that  $\angle SRQ > \angle SQR$ .



#### Solution:

In  $\triangle PQR$ , we have:

$$PQ > PR \text{ (Given)}$$

$$\Rightarrow \angle PRQ > \angle PQR (\because \text{Angle opposite longer side is greater})$$

$$\Rightarrow \frac{\angle PRQ}{2} > \frac{\angle PQR}{2}$$

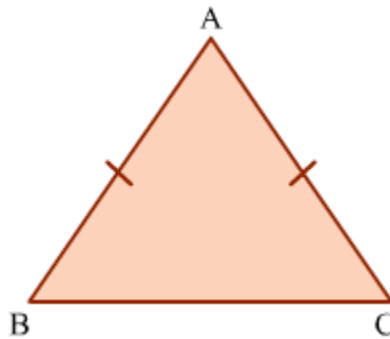
$\Rightarrow \angle SRQ > \angle SQR$  ( $\because$  RS bisects  $\angle PRQ$  and QS bisects  $\angle PQR$ )

**Example 2:**

ABC is an isosceles triangle with  $AB = AC$  and  $AB < BC$ . Prove that  $\angle BAC > \angle ABC$ .

**Solution:**

Consider the following  $\triangle ABC$  in which  $AB = AC$  and  $AB < BC$ .



In  $\triangle ABC$ , we have:

$\angle ABC = \angle ACB$  ... (1) [ $\because$  Angles opposite equal sides AB and AC are equal]

By the triangle inequality theorem, we have:

$\angle ACB < \angle BAC$  ... (2) [ $\because$  Angle opposite longer side BC is greater]

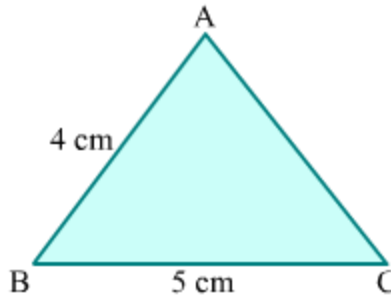
Thus, from (1) and (2), we obtain:

$\angle BAC > \angle ABC$

**Medium**

**Example 1:**

In the shown  $\triangle ABC$ ,  $AB = 4$  cm,  $BC = 5$  cm and the perimeter is 16 cm. Determine the smallest and greatest angles of the triangle.



**Solution:**

In  $\triangle ABC$ , we have:

$AB = 4 \text{ cm}$  and  $BC = 5 \text{ cm}$  (Given)

Perimeter =  $16 \text{ cm}$  (Also given)

$\Rightarrow AB + BC + CA = 16 \text{ cm}$  ( $\because$  Perimeter is the sum of all sides)

$\Rightarrow 4 \text{ cm} + 5 \text{ cm} + CA = 16 \text{ cm}$

$\Rightarrow CA = 7 \text{ cm}$

Now,  $CA = 7 \text{ cm}$  is the longest side of  $\triangle ABC$ . Thus, the angle opposite it, i.e.,  $\angle ABC$  is the greatest angle of the triangle.

$\therefore \angle ABC > \angle BAC$  and  $\angle ABC > \angle ACB \dots (1)$

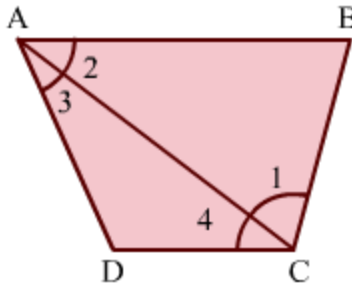
Also,  $BC > AB$

$\therefore \angle BAC > \angle ACB \dots (2)$

Thus, from (1) and (2), we can conclude that  $\angle ABC$  is the greatest angle and  $\angle ACB$  is the smallest angle in  $\triangle ABC$ .

**Example 2:**

Suppose  $AB$  is the longest side and  $CD$  the shortest side of the given quadrilateral  $ABCD$ . Then, prove that  $\angle BCD > \angle BAD$ .



**Solution:**

In  $\triangle ABC$ ,  $AB$  is the longest side.

So,  $AB > BC$

$\Rightarrow \angle 1 > \angle 2 \dots (1)$  [ $\because$  Angle opposite longer side is greater]

In  $\triangle ADC$ ,  $CD$  is the shortest side.

So,  $AD > CD$

$\Rightarrow \angle 4 > \angle 3 \dots (2)$  [ $\because$  Angle opposite longer side is greater]

On adding (1) and (2), we get:

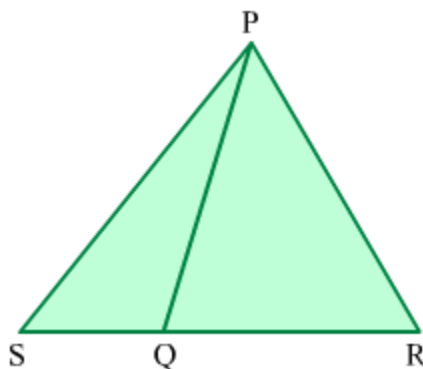
$$\angle 1 + \angle 4 > \angle 2 + \angle 3$$

$$\Rightarrow \angle BCD > \angle BAD$$

**Hard**

**Example 1:**

In the given figure,  $PQ = PR$ . Show that  $\angle PRS > \angle PSR$ .



**Solution:**

In  $\Delta PQR$ , we have:

$$PQ = PR$$

$$\Rightarrow \angle PRQ = \angle PQR \dots (1) \quad [\because \text{Angles opposite equal sides are equal}]$$

In  $\Delta PSQ$ ,  $SQ$  is produced to  $R$ .

$$\text{So, } \angle PQR = \angle PSQ + \angle SPQ \quad (\text{By the exterior angle property})$$

$$\Rightarrow \angle PQR > \angle PSQ \dots (2)$$

From (1) and (2), we obtain:

$$\angle PRQ > \angle PSQ$$

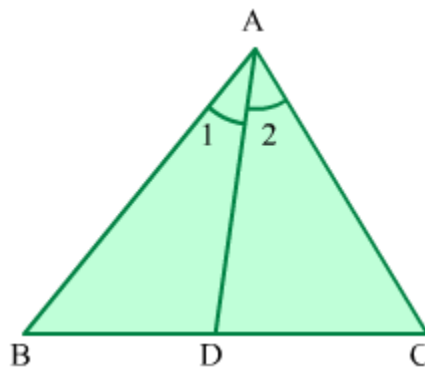
$$\Rightarrow \angle PRS > \angle PSR \quad (\because \angle PRQ = \angle PRS \text{ and } \angle PSQ = \angle PSR)$$

**Example 2:**

In a  $\Delta ABC$ ,  $AB > AC$  and  $AD$  is the bisector of  $\angle BAC$ . Show that  $\angle ADB > \angle ADC$ .

**Solution:**

Let the following  $\Delta ABC$  be the given triangle such that  $AB > AC$ . Also,  $AD$  is the bisector of  $\angle BAC$ .



In  $\Delta ABC$ , we have:

$$AB > AC \quad (\text{Given})$$

$\Rightarrow \angle ACB > \angle ABC$  ( $\because$  Angle opposite longer side is greater)

On adding  $\angle 1$  to both sides, we get:

$$\angle ACB + \angle 1 > \angle ABC + \angle 1$$

$\Rightarrow \angle ACB + \angle 2 > \angle ABC + \angle 1 \dots (1)$  [ $\because$  AD bisects  $\angle BAC$ ;  $\angle 1 = \angle 2$ ]

By the exterior angle property, we have:

$$\angle ADB = \angle ACB + \angle 2 \dots (2)$$

Similarly,  $\angle ADC = \angle ABC + \angle 1 \dots (3)$

Thus, by using (1), (2) and (3), we can conclude that:

$$\angle ADB > \angle ADC$$

### **Observation of the Sides of a Triangle by Seeing the Angles**

Consider the following triangular racetrack where two cars start from different points A and B to reach the finish point C.

The red car has to travel at an angle of  $65^\circ$ , while the blue car has to travel at an angle of  $50^\circ$  with respect to line AB to reach the finish point C. Do you think any one car has an advantage over the other?

On observing the triangular track, it seems that path BC is longer than path AC. So, clearly, the red car has an advantage over the blue car. Also, the angle opposite BC is greater than the angle opposite AC. So, what does this tell us about the relation between the sides and angles of the triangular racetrack?

Let us go through this lesson to learn about the relation between the sides and angles of a triangle. We will also solve some problems based on this relation.

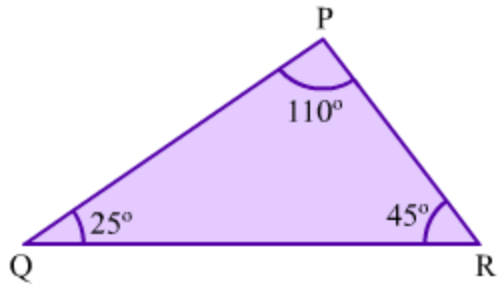
### **In a Triangle, the Side Opposite the Greater Angle is**

#### **Longer**

We have studied that in a triangle having two unequal sides, the angle opposite the longer side is greater. The converse of this property is also true. It states that:

***If two angles of a triangle are unequal, then the greater angle has the longer side opposite it. In other words, the smaller angle has the shorter side opposite it.***

Consider the following  $\triangle PQR$ .



Let us apply the stated property in this triangle.

$\angle QPR = 110^\circ$  is the greatest angle in  $\Delta PQR$ .

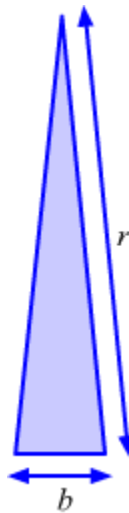
Therefore, the side opposite  $\angle QPR$ , i.e., QR is the longest side of the triangle. Also,  $\angle PQR = 25^\circ$  is the smallest angle in  $\Delta PQR$ .

Therefore, the side opposite  $\angle PQR$ , i.e., PR is the shortest side of the triangle.

We can apply this property to other triangles as well. Using this property, we can say that in a right triangle, the hypotenuse is the longest side.

### Whiz Kid

In trigonometry, a skinny triangle is one whose height is much greater than its base.



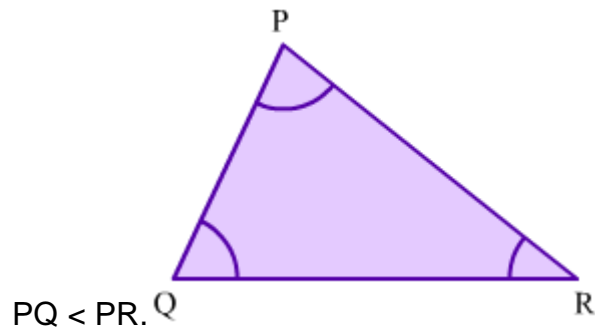
The given triangle is an example of a skinny triangle.

### Proving the Property

**Given:**  $\Delta PQR$  in which  $\angle Q > \angle R$

**To prove:**  $PR > PQ$ .

**Proof:** In  $\triangle PQR$ , we can have three possible cases: (1)  $PQ > PR$ , (2)  $PQ = PR$  and (3)



**CASE 1:**

When  $PQ > PR$ , we have:

$\angle R > \angle Q$  ( $\because$  Angle opposite longer side is greater)

But this contradicts the given hypothesis that  $\angle R < \angle Q$ . Thus,  $PQ > PR$  is not true.

**CASE 2:**

When  $PQ = PR$ , we have:

$\angle R = \angle Q$  ( $\because$  Angles opposite equal sides are equal)

But this too contradicts the given hypothesis that  $\angle R < \angle Q$ . Thus,  $PQ = PR$  is also not true.

So, we are left with the third possibility, i.e.,  $PQ < PR$  (or  $PR > PQ$ ), which must be true.

Thus, we have proven that if two angles of a triangle are unequal, then the greater angle has the longer side opposite it.

## Solved Examples

### Easy

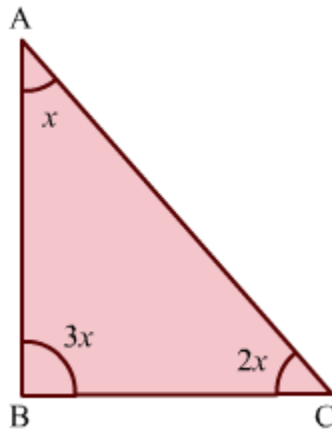
#### Example 1:

The angles of a triangle are in the ratio 1 : 2 : 3. Find the greatest angle and identify the longest side of the triangle.

**Solution:**

It is given that the angles of the triangle, say  $\triangle ABC$ , are in ratio 1 : 2 : 3.

Let the angles be  $x$ ,  $2x$  and  $3x$ , as is shown in the figure.



By the angle sum property of triangles, we have:

$$\angle BAC + \angle ABC + \angle ACB = 180^\circ$$

$$\Rightarrow x + 3x + 2x = 180^\circ$$

$$\Rightarrow 6x = 180^\circ$$

$$\Rightarrow x = 30^\circ$$

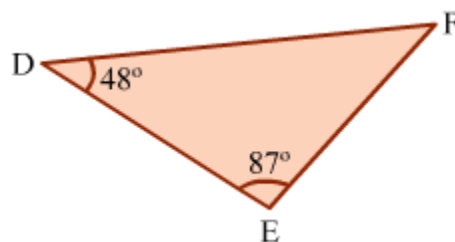
$$\text{So, } 2x = 2 \times 30^\circ = 60^\circ \text{ and } 3x = 3 \times 30^\circ = 90^\circ$$

Thus, the greatest angle in the given triangle is  $\angle ABC$ , i.e.,  $90^\circ$ .

Now, we know that the side opposite the greater angle is longer. In  $\triangle ABC$ , side AC is opposite the greatest angle; hence, it is the longest.

### Example 2:

Which side of the given triangle is the shortest?



**Solution:**

In order to find the shortest side of  $\triangle DEF$ , we need to figure out the smallest angle of the triangle. This is because the smallest side is opposite the smallest angle.

By the angle sum property of triangles, we have:

$$\angle EDF + \angle DEF + \angle EFD = 180^\circ$$

$$\Rightarrow 48^\circ + 87^\circ + \angle EFD = 180^\circ$$

$$\Rightarrow 135^\circ + \angle EFD = 180^\circ$$

$$\Rightarrow \angle EFD = 45^\circ$$

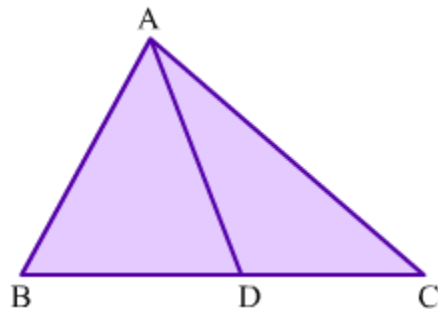
Clearly,  $\angle EFD$  has the smallest measure in the given triangle. So, the side opposite it, i.e.,  $DE$  is the shortest side of the triangle.

**Medium****Example 1:**

In the given  $\triangle ABC$ ,  $\angle ABD > \angle ACD$  and  $AD$  is the bisector of  $\angle BAC$ .

Prove that:

1.  $AB < AC$
2.  $\angle ADB < \angle ADC$

**Solution:**

1. It is given that  $\angle ABD > \angle ACD$ .  
 $\therefore AC > AB$  ( $\because$  Side opposite greater angle is longer)

Or,  $AB < AC$

2. In  $\triangle ABD$ , we have:

$$\angle ABD + \angle BAD + \angle ADB = 180^\circ \text{ (By the angle sum property)}$$

$$\Rightarrow \angle ACD + \angle BAD + \angle ADB < 180^\circ \text{ (}\because \angle ABD > \angle ACD\text{)}$$

$$\Rightarrow \angle ACD + \angle CAD + \angle ADB < 180^\circ \text{ (}\because \text{AD bisects } \angle BAC; \angle BAD = \angle CAD\text{)}$$

$$\Rightarrow \angle ADB < 180^\circ - (\angle ACD + \angle CAD)$$

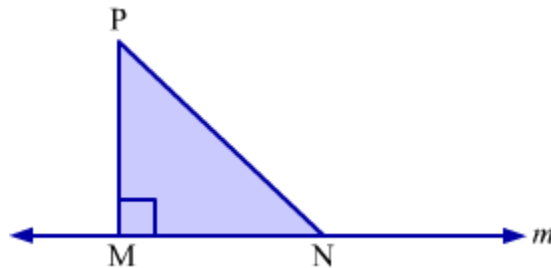
$$\Rightarrow \angle ADB < \angle ADC \text{ (By the angle sum property in } \triangle ADC\text{)}$$

**Example 2:**

Prove that of all the line segments that can be drawn to a given line from a point lying outside the line, the perpendicular line segment is the shortest.

**Solution:**

Let there be a straight line  $m$ , a point  $P$  lying outside the line and a point  $M$  lying on the line. Also,  $PM \perp m$  and  $N$  is any point other than  $M$  on  $m$ .



Since  $PM$  is perpendicular to  $m$ ,  $\angle PMN = 90^\circ$ .

In  $\triangle PMN$ , we have:

$$\Rightarrow \angle PMN + \angle MPN + \angle PNM = 180^\circ \text{ (By the angle sum property)}$$

$$\Rightarrow 90^\circ + \angle MPN + \angle PNM = 180^\circ$$

$$\Rightarrow \angle MPN + \angle PNM = 90^\circ$$

$$\Rightarrow \angle PNM < 90^\circ$$

$$\Rightarrow \angle PNM < \angle PMN$$

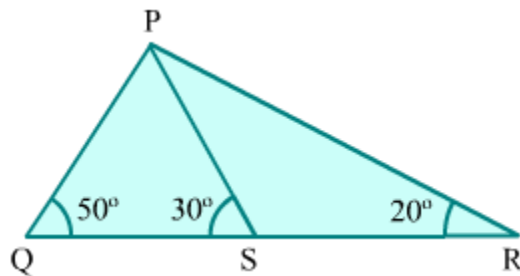
$$\Rightarrow PM < PN (\because \text{Side opposite greater angle is longer})$$

Thus, PM is the shortest of all line segments from point P to line  $m$ .

**Hard**

**Example 1:**

In the given  $\Delta PQR$ ,  $\angle PQR = 50^\circ$ ,  $\angle PRQ = 20^\circ$  and  $\angle PSQ = 30^\circ$ . Prove that  $QS > SR$ .



**Solution:**

In  $\Delta PQS$ , we have:

$$\angle PQR = 50^\circ \text{ and } \angle PSQ = 30^\circ \text{ (Given)}$$

By using the angle sum property in  $\Delta PQS$ , we obtain:

$$\angle PQR + \angle PSQ + \angle QPS = 180^\circ$$

$$\Rightarrow 50^\circ + 30^\circ + \angle QPS = 180^\circ$$

$$\Rightarrow \angle QPS = 180^\circ - (50^\circ + 30^\circ)$$

$$\Rightarrow \angle QPS = 100^\circ$$

So,  $\angle QPS > \angle PQR > \angle PSQ$

Since the side opposite the greater angle is longer, we get:

$$QS > PS > PQ \dots (1)$$

$\angle PSQ$  and  $\angle PSR$  form a linear pair.

$$\text{So, } \angle PSQ + \angle PSR = 180^\circ$$

$$\Rightarrow 30^\circ + \angle PSR = 180^\circ$$

$$\Rightarrow \angle PSR = 180^\circ - 30^\circ$$

$$\Rightarrow \angle PSR = 150^\circ$$

In  $\triangle PSR$ , we have:

$$\angle PSR + \angle SPR + \angle PRQ = 180^\circ \text{ (By the angle sum property)}$$

$$\Rightarrow 150^\circ + \angle SPR + 20^\circ = 180^\circ$$

$$\Rightarrow \angle SPR = 180^\circ - (150^\circ + 20^\circ)$$

$$\Rightarrow \angle SPR = 10^\circ$$

$$\text{So, } \angle PSR > \angle PRQ > \angle SPR$$

Since the side opposite the greater angle is longer, we get:

$$PR > PS > SR \dots (2)$$

By using (1) and (2), we can conclude that:

$$QS > SR$$

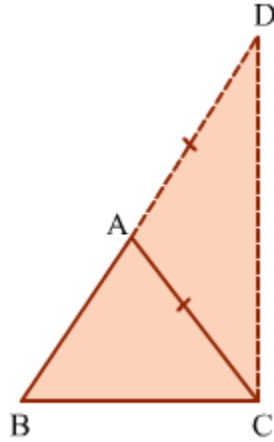
### **Example 2:**

Prove that the difference between any two sides of a triangle is less than the third side.

### **Solution:**

We have to prove that the difference between any two sides of a triangle (let us say  $\triangle ABC$ ) is less than the third side of the triangle. We will do so by showing that in  $\triangle ABC$ ,  $BC - AC < AB$ .

**Construction:** Extend side BA up to point D such that  $AD = AC$ . Join C to D.



In  $\triangle ACD$ , we have:

$$AD = AC \dots (1)$$

We know that in an isosceles triangle, the angles opposite equal sides are equal.

$$\therefore \angle ACD = \angle ADC$$

$$\Rightarrow \angle ACD + \angle ACB > \angle ADC$$

$$\Rightarrow \angle BCD > \angle ADC$$

We know that the side opposite the greater angle is longer. So, we obtain,  $BD > BC$

$$\Rightarrow AB + AD > BC \quad (\because BD = AB + AD)$$

$$\Rightarrow AB + AC > BC \quad (\text{Using equation 1})$$

$$\Rightarrow BC - AC < AB$$

Similarly, we can prove that  $AB - BC < AC$  and  $AC - AB < BC$ .

Thus, we have proved that the difference between any two sides of a triangle is less than the third side.