

29. Limits

Exercise 29.1

1. Question

Show that $\lim_{x \rightarrow 0} \frac{x}{|x|}$ does not exist.

Answer

Given

$$f(x) = \begin{cases} \frac{x}{|x|}, & x > 0 \\ \frac{x}{|x|}, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

To find $\lim_{x \rightarrow 0} f(x)$

To limit to exist, we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x)$(2)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} 1 = 1 \text{.....(3)}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} -1 = -1 \text{.....(4)}$$

From above equations

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x) \text{ (from 2)}$$

Thus, limit does not exist.

2. Question

Find k so that $\lim_{x \rightarrow 2} f(x)$ may exist, where $f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ x + k, & x > 2 \end{cases}$.

Answer

$$\text{Given } f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ x + k, & x > 2 \end{cases}$$

To find $\lim_{x \rightarrow 2} f(x)$

To limit to exist, we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

$$\text{thus } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} f(x)$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2 + h) = \lim_{h \rightarrow 0} 2(2 + h) + 3$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2 - h) = \lim_{h \rightarrow 0} (2 - h) + k$$

$$\lim_{x \rightarrow 2} f(x) = f(2) = 2(2) + 3 = 7$$

From (1)

$$\lim_{h \rightarrow 0} 2(2 + h) + 3 = \lim_{h \rightarrow 0} (2 - h) + k$$

$$2(2 + 0) + 3 = (2 - 0) + k$$

$$4 + 3 = 2 + k$$

$$5 = k$$

3. Question

Show that $\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.

Answer

$$f(x) = \frac{1}{x}$$

To find $\lim_{x \rightarrow 0} f(x)$

To limit to exist, we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus, to find the limit using the concept $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x)$(2)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} \frac{1}{0 + h} = \lim_{h \rightarrow 0} \frac{1}{h} = \infty \dots\dots(3)$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} \frac{1}{0 - h} = \lim_{h \rightarrow 0} \frac{-1}{h} = -\infty \dots\dots(4)$$

$$\lim_{x \rightarrow 0} f(x) = f(0) = \frac{1}{0} = \infty$$

From above equations

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x)$$

Thus, limit does not exist.

4. Question

Let $f(x)$ be a function defined by $f(x) = \begin{cases} \frac{3x}{|x| + 2x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Show that $f(x)$ does not exist.

Answer

$$\text{Given } f(x) = \begin{cases} \frac{3x}{|x| + 2x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{3x}{x + 2x}, & x > 0 \\ 0, & x = 0 \\ \frac{3x}{-x + 2x}, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ 3, & x < 0 \end{cases}$$

To find $\lim_{x \rightarrow 0} f(x)$

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x)$(2)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} 1 = 1 \text{.....(3)}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} 3 = 3 \text{.....(4)}$$

$$\lim_{x \rightarrow 0} f(x) = f(0) = 0$$

From above equations

$$\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0} f(x)$$

Thus, limit does not exist.

5. Question

Let $f(x) = \begin{cases} x + 1, & \text{if } x > 0 \\ x - 1, & \text{if } x < 0 \end{cases}$. Prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

Answer

$$\text{Given } f(x) = \begin{cases} x + 1, & x > 0 \\ x - 1, & x < 0 \end{cases}$$

To find whether $\lim_{x \rightarrow 0} f(x)$ exists?

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to limit to exist $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x)$(2)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} (0 + h) + 1 = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} (0 - h) - 1 = -1$$

From above equations

$$\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

Thus, the limit $\lim_{x \rightarrow 0} f(x)$ does not exists.

6. Question

Let $f(x) = \begin{cases} x + 5, & \text{if } x > 0 \\ x - 4, & \text{if } x < 0 \end{cases}$. Prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

Answer

$$\text{Given } f(x) = \begin{cases} x + 5, & x > 0 \\ x - 4, & x < 0 \end{cases}$$

To find whether $\lim_{x \rightarrow 0} f(x)$ exists?

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to limit to exist $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x)$(2)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} (0 + h) + 5 = 5$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} (0 - h) - 4 = -4$$

From above equations

$$\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

Thus, the limit $\lim_{x \rightarrow 0} f(x)$ does not exist.

7. Question

Find $\lim_{x \rightarrow 3} f(x)$, where

$$f(x) = \begin{cases} 4, & \text{if } x > 3 \\ x + 1, & \text{if } x < 3 \end{cases}$$

Answer

$$\text{Given } f(x) = \begin{cases} 4, & x > 3 \\ x + 1, & x < 3 \end{cases}$$

To find $\lim_{x \rightarrow 3} f(x)$

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3} f(x)$(2)

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{h \rightarrow 0} f(3 + h) = \lim_{h \rightarrow 0} 4 = 4 \text{.....(3)}$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{h \rightarrow 0} f(3 - h) = \lim_{h \rightarrow 0} (3 - h) + 1 = 4 \text{.....(4)}$$

From above equations

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x)$$

Thus from (2), (3) and (4)

$$\lim_{x \rightarrow 3} f(x) = 4$$

8. Question

If $f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x + 1), & x > 0 \end{cases}$. Find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$.

Answer

$$\text{Given } f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x + 1), & x > 0 \end{cases}$$

(i) To find $\lim_{x \rightarrow 0} f(x)$

To limit to exist, we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x)$(2)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} 3(0 + h + 1) = 3 \text{.....(3)}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} 2(0 - h) + 3 = 3 \dots (4)$$

$$\lim_{x \rightarrow 0} f(x) = f(0) = 2(0) + 3 = 3 \dots (5)$$

From above equations

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x) \text{ thus the limit exists}$$

Thus from (5)

$$\lim_{x \rightarrow 0} f(x) = 3$$

(ii) To find $\lim_{x \rightarrow 1} f(x)$

$$\text{To limit to exist, we know } \lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x) \dots (1)$$

$$\text{Thus to find the limit using the concept } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(x) \dots (2)$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1 + h) = \lim_{h \rightarrow 0} 2(1 + h) + 3 = 5 \dots (3)$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1 - h) = \lim_{h \rightarrow 0} 2(1 - h) + 3 = 5 \dots (4)$$

From above equations

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x)$$

Thus from (2), (3) and (4)

$$\lim_{x \rightarrow 1} f(x) = 5$$

9. Question

$$\text{Find } \lim_{x \rightarrow 1} f(x), \text{ if } f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$$

Answer

$$\text{Given } f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$$

To find $\lim_{x \rightarrow 1} f(x)$

$$\text{To limit to exist we know } \lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x) \dots (1)$$

$$\text{Thus to find the limit using the concept } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(x) \dots (2)$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1 + h) = \lim_{h \rightarrow 0} -(1 + h)^2 - 1 = \lim_{h \rightarrow 0} -1^2 - h^2 - 2h - 1 = -2 \dots (3)$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1 - h) = \lim_{h \rightarrow 0} (1 - h)^2 - 1 = \lim_{h \rightarrow 0} 1^2 - h^2 + 2h - 1 = 0 \dots (4)$$

From above equations

$$\lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x) \text{ thus the limit } \lim_{x \rightarrow 1} f(x) \text{ does not exist}$$

10. Question

Evaluate $\lim_{x \rightarrow 0} f(x)$, where

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Answer

$$\text{Given } f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{x}{x}, & x > 0 \\ 0, & x = 0 \\ -\frac{x}{x}, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

To find $\lim_{x \rightarrow 0} f(x)$

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x)$(2)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} 1 = 1 \text{.....(3)}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} -1 = -1 \text{.....(4)}$$

$$\lim_{x \rightarrow 0} f(x) = f(0) = 0$$

From above equations

$$\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0} f(x)$$

Thus limit does not exists

11. Question

Let $a_1, a_2, \dots, a_n$ be fixed real numbers such that $f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$

What is $\lim_{x \rightarrow a_1} f(x)$? For $a \neq a_1, a_2, \dots, a_n$ compute $\lim_{x \rightarrow a} f(x)$

Answer

Given: $f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$

$$\lim_{x \rightarrow a_1} f(x) = (a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n)$$

$$\lim_{x \rightarrow a_1} f(x) = 0$$

Now,

$$\lim_{x \rightarrow a} f(x) = (a - a_1)(a - a_2)(a - a_3) \dots (a - a_n)$$

12. Question

Find $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$

Answer

Given $f(x) = \frac{1}{x-1}$

To find $\lim_{x \rightarrow 1^+} f(x)$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1 + h) = \lim_{h \rightarrow 0} \frac{1}{(1 + h) - 1} = \lim_{h \rightarrow 0} \frac{1}{h} = \frac{1}{0} = \infty$$

13 A. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4}$$

Answer

Given $f(x) = \frac{x-3}{x^2-4}$

To find $\lim_{x \rightarrow 2^+} f(x)$

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{h \rightarrow 0} f(2 + h) = \lim_{h \rightarrow 0} \frac{(2 + h) - 3}{(2 + h)^2 - 4} = \lim_{h \rightarrow 0} \frac{h - 1}{2^2 + h^2 + 2h - 4} \\ &= \frac{0 - 1}{4 + 0^2 + 0 - 4} = -\frac{1}{0} = -\infty \end{aligned}$$

13 B. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4}$$

Answer

Given $f(x) = \frac{x-3}{x^2-4}$

To find $\lim_{x \rightarrow 2^-} f(x)$

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{h \rightarrow 0} f(2 - h) = \lim_{h \rightarrow 0} \frac{(2 - h) - 3}{(2 - h)^2 - 4} = \lim_{h \rightarrow 0} \frac{-h - 1}{2^2 + h^2 - 2h - 4} \\ &= \frac{0 - 1}{4 + 0^2 - 0 - 4} = -\frac{1}{0} = -\infty \end{aligned}$$

13 C. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow 0^+} \frac{1}{3x}$$

Answer

Given $f(x) = \frac{1}{3x}$

To find $\lim_{x \rightarrow 0^+} f(x)$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} \frac{1}{3(0 + h)} = \lim_{h \rightarrow 0} \frac{1}{3h} = \frac{1}{0} = \infty$$

13 D. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow -8^+} \frac{2x}{x+8}$$

Answer

$$\text{Given } f(x) = \frac{2x}{x+8}$$

Factorizing f(x)

$$f(x) = \frac{2x+16-16}{x+8}$$

$$f(x) = \frac{2(x+8)}{x+8} - \frac{16}{x+8}$$

$$f(x) = 2 - \frac{16}{x+8}$$

To find $\lim_{x \rightarrow -8^+} f(x)$

$$\begin{aligned} \lim_{x \rightarrow -8^+} f(x) &= \lim_{h \rightarrow 0} f(-8 + h) = \lim_{h \rightarrow 0} 2 - \frac{16}{(-8 + h) + 8} = \lim_{h \rightarrow 0} 2 - \frac{16}{h} = 2 - \infty \\ &= -\infty \end{aligned}$$

13 E. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow 0^+} \frac{2}{x^{1/5}}$$

Answer

$$\text{Given } f(x) = \frac{2}{x^{1/5}}$$

To find $\lim_{x \rightarrow 0^+} f(x)$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} \frac{2}{(0 + h)^{1/5}} = \lim_{h \rightarrow 0} \frac{2}{h^{1/5}} = \frac{2}{0} = \infty$$

13 F. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x$$

Answer

Some standard limit are:

$$\lim_{x \rightarrow 0} (\tan x) \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 0} (\sin x) \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 0} (\cos x) = 1$$

Thus to find:

$$\lim_{x \rightarrow \frac{\pi}{2}} \tan x = \lim_{x \rightarrow \frac{\pi}{2}} f(x)$$

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{h \rightarrow 0} f\left(\frac{\pi}{2} - h\right) = \lim_{h \rightarrow 0} \tan\left(\frac{\pi}{2} - h\right)$$

$$h) = \lim_{h \rightarrow 0} \coth = \lim_{h \rightarrow 0} \frac{1}{\tanh} = \lim_{h \rightarrow 0} \frac{h}{h \tanh} = \lim_{h \rightarrow 0} \frac{1}{h}$$

$= \infty$

13 G. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \sec x$$

Answer

Some standard limit are:

$$\lim_{x \rightarrow 0} (\tan x) \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 0} (\sin x) \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 0} (\cos x) = 1$$

Thus to find:

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \sec x = \lim_{x \rightarrow \frac{\pi}{2}^+} f(x)$$

$$\lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = \lim_{h \rightarrow 0} f\left(-\frac{\pi}{2} + h\right) = \lim_{h \rightarrow 0} \sec\left(-\frac{\pi}{2} + h\right)$$

$= -\infty$

$$h) = \lim_{h \rightarrow 0} -\operatorname{cosech} = \lim_{h \rightarrow 0} \frac{-1}{\sinh} = \lim_{h \rightarrow 0} \frac{-h}{h \tanh} = \lim_{h \rightarrow 0} -\frac{1}{h}$$

13 H. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow 0^-} \frac{x^2 - 3x + 2}{x^3 - 2x^2}$$

Answer

$$\text{Given } f(x) = \frac{x^2 - 3x + 2}{x^3 - 2x^2}$$

Factorizing $f(x)$

$$f(x) = \frac{x^2 - 2x - x + 2}{x^2(x-2)}$$

$$f(x) = \frac{x(x-2) - 1(x-2)}{x^2(x-2)}$$

$$f(x) = \frac{(x-1)(x-2)}{x^2(x-2)}$$

$$f(x) = \frac{(x-1)}{x^2}$$

To find $\lim_{x \rightarrow 0^-} f(x)$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} \frac{(0 - h) - 1}{(0 - h)^2} = \lim_{h \rightarrow 0} \frac{-h - 1}{h^2} = \frac{-1}{0} = -\infty$$

13 I. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow -2^+} \frac{x^2 - 1}{2x + 4}$$

Answer

$$\lim_{x \rightarrow -2^+} \frac{x^2 - 1}{2x + 4} = \lim_{h \rightarrow 0} \frac{[(-2 + h)^2 - 1]}{[2(-2 + h) + 4]} = \frac{h^2 - 4h + 3}{-4 + 2h + 4} = \infty$$

13 J. Question

Evaluate the following one - sided limits:

$$\lim_{x \rightarrow 0^-} (2 - \cot x)$$

Answer

Some standard limit are:

$$\lim_{x \rightarrow 0} (\tan x) \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 0} (\sin x) \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 0} (\cos x) = 1$$

Thus to find:

$$\lim_{x \rightarrow 0^-} 2 - \cot x = \lim_{x \rightarrow 0^-} f(x)$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) =$$

$$\lim_{h \rightarrow 0} 2 - \cot(0 - h) = \lim_{h \rightarrow 0} 2 - \cot(-h) = \lim_{h \rightarrow 0} 2 + \coth = \lim_{h \rightarrow 0} 2 + \frac{1}{\tanh} = \lim_{h \rightarrow 0} = 2 + \infty = \infty$$

13 K. Question

Evaluate the following one - sided limits:

$$(xi) \lim_{x \rightarrow 0^-} 1 + \operatorname{cosec} x$$

Answer

Some standard limit are:

$$\lim_{x \rightarrow 0} (\tan x) \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 0} (\sin x) \frac{1}{x} = 1$$

$$\lim_{x \rightarrow 0} (\cos x) = 1$$

Thus to find:

$$\lim_{x \rightarrow 0^-} 1 + \operatorname{cosec} x = \lim_{x \rightarrow 0^-} f(x)$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) =$$

$$\lim_{h \rightarrow 0} 1 + \operatorname{cosec}(0 - h) = \lim_{h \rightarrow 0} 1 + \operatorname{cosec}(-h) = \lim_{h \rightarrow 0} 1 - \operatorname{cosec} h = \lim_{h \rightarrow 0} 1 + \frac{-1}{\sinh} = 1 - \infty = -\infty$$

14. Question

Show that $\lim_{x \rightarrow 0} e^{-1/x}$ does not exist.

Answer

$$\text{Given } f(x) = e^{-\frac{1}{x}}$$

$$\text{To find } \lim_{x \rightarrow 0} f(x)$$

$$\text{To limit to exist we know } \lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x) \dots\dots(1)$$

$$\text{Thus to find the limit using the concept } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} f(x) \dots\dots(2)$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} e^{-\frac{1}{0+h}} = \lim_{h \rightarrow 0} e^{-\frac{1}{h}} = \frac{1}{e^{\frac{1}{0}}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \dots\dots(3)$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} e^{-\frac{1}{0-h}} = \lim_{h \rightarrow 0} e^{-\frac{1}{-h}} = \lim_{h \rightarrow 0} e^{\frac{1}{h}} = e^{\frac{1}{0}} = e^{\infty} = \infty \dots\dots(4)$$

$$\lim_{x \rightarrow 0} f(x) = f(0) = e^{-\frac{1}{0}} = \frac{1}{e^{\frac{1}{0}}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

From above equations

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x)$$

Thus, limit does not exist.

15 A. Question

Find:

$$\lim_{x \rightarrow 2} [x]$$

Answer

We know greatest integer $[x]$ is the integer part.

$$\text{For } f(x) = [x]$$

To find:

$$\lim_{x \rightarrow 2} f(x)$$

$$\text{To limit to exist we know } \lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x) \dots\dots(1)$$

$$\text{Thus to find the limit using the concept } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} f(x) \dots\dots(2)$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2 + h) = \lim_{h \rightarrow 0} [2 + h] = 2 \dots\dots(3)$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2 - h) = \lim_{h \rightarrow 0} [2 - h] = 1 \dots\dots(4)$$

$$\lim_{x \rightarrow 2} f(x) = f(2) = [2] = 2$$

From above equations

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} f(x)$$

Thus, the limit does not exist.

15 B. Question

Find:

$$\lim_{x \rightarrow \frac{5}{2}} [x]$$

Answer

We know greatest integer $[x]$ is the integer part.

For $f(x) = [x]$

To find:

$$\lim_{x \rightarrow 2.5} f(x)$$

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 2.5^+} f(x) = \lim_{x \rightarrow 2.5^-} f(x) = \lim_{x \rightarrow 2.5} f(x)$(2)

$$\lim_{x \rightarrow 2.5^+} f(x) = \lim_{h \rightarrow 0} f(2.5 + h) = \lim_{h \rightarrow 0} [2.5 + h] = 2 \text{.....(3)}$$

$$\lim_{x \rightarrow 2.5^-} f(x) = \lim_{h \rightarrow 0} f(2.5 - h) = \lim_{h \rightarrow 0} [2.5 - h] = 2 \text{.....(4)}$$

$$\lim_{x \rightarrow 2.5} f(x) = f(2.5) = [2.5] = 2$$

From above equations

$$\lim_{x \rightarrow 2.5^-} f(x) = \lim_{x \rightarrow 2.5^+} f(x) = \lim_{x \rightarrow 2.5} f(x)$$

Thus, limit does exists.

15 C. Question

Find:

$$\lim_{x \rightarrow 1} [x]$$

Answer

We know greatest integer $[x]$ is the integer part.

For $f(x) = [x]$

To find:

$$\lim_{x \rightarrow 1} f(x)$$

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(x)$(2)

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1 + h) = \lim_{h \rightarrow 0} [1 + h] = 1 \text{.....(3)}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1 - h) = \lim_{h \rightarrow 0} [1 - h] = 0 \text{.....(4)}$$

$$\lim_{x \rightarrow 1} f(x) = f(1) = [1] = 1$$

From above equations

$$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x)$$

Thus limit does not exists.

16. Question

Prove that $\lim_{x \rightarrow a^+} [x] = [a]$ for all $a \in \mathbb{R}$. Also, prove that $\lim_{x \rightarrow 1^-} [x] = 0$.

Answer

To Prove: $\lim_{x \rightarrow a^+} [x] = [a]$

$$\text{L.H.S} = \lim_{x \rightarrow a^+} [x] = \lim_{h \rightarrow 0} [a + h] = [a] \text{ (Since, } [a + h] = [a])$$

Hence, Proved.

Also,

To prove: $\lim_{x \rightarrow 1^-} [x] = 0$

$$\text{L.H.S} = \lim_{x \rightarrow 1^-} [x] = \lim_{h \rightarrow 0} [1 - h] = 0 \text{ (Since, } [1 - h] = 0)$$

Hence, Proved.

17. Question

Show that $\lim_{x \rightarrow 2^-} \frac{x}{[x]} \neq \lim_{x \rightarrow 2^+} \frac{x}{[x]}$.

Answer

We know greatest integer $[x]$ is the integer part.

For $f(x) = x/[x]$

To show

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

Proof:

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} f(x)$(2)

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2 + h) = \lim_{h \rightarrow 0} \frac{2+h}{[2+h]} = \frac{2+0}{2} = 1 \text{.....(3)}$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2 - h) = \lim_{h \rightarrow 0} \frac{2-h}{[2-h]} = \frac{2}{1} = 2 \text{.....(4)}$$

From above equations

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

18. Question

Find $\lim_{x \rightarrow 3^+} \frac{x}{[x]}$. Is it equal to $\lim_{x \rightarrow 3^-} \frac{x}{[x]}$.

Answer

We know greatest integer $[x]$ is the integer part.

For $f(x) = x/[x]$

To show

$$\lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 3^+} f(x)$$

Proof:

$$\text{To limit to exist we know } \lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x) \dots\dots(1)$$

$$\text{Thus to find the limit using the concept } \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3} f(x) \dots\dots(2)$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{h \rightarrow 0} f(3 + h) = \lim_{h \rightarrow 0} \frac{3 + h}{[3 + h]} = \frac{3 + 0}{3} = 1 \dots\dots(3)$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{h \rightarrow 0} f(3 - h) = \lim_{h \rightarrow 0} \frac{3 - h}{[3 - h]} = \frac{3 - 0}{2} = \frac{3}{2} \dots\dots(4)$$

From above equations

$$\lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 3^+} f(x)$$

19. Question

$$\text{Find } \lim_{x \rightarrow -5/2} [x].$$

Answer

We know greatest integer $[x]$ is the smallest integer nearest to that number .

$$\text{For } f(x) = [x]$$

To find:

$$\lim_{x \rightarrow -2.5} f(x)$$

$$\text{To limit to exist we know } \lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x) \dots\dots(1)$$

$$\text{Thus to find the limit using the concept } \lim_{x \rightarrow -2.5^+} f(x) = \lim_{x \rightarrow -2.5^-} f(x) = \lim_{x \rightarrow -2.5} f(x) \dots\dots(2)$$

$$\lim_{x \rightarrow -2.5^+} f(x) = \lim_{h \rightarrow 0} f(-2.5 + h) = \lim_{h \rightarrow 0} [-2.5 + h] = -3 \dots\dots(3)$$

$$\lim_{x \rightarrow -2.5^-} f(x) = \lim_{h \rightarrow 0} f(-2.5 - h) = \lim_{h \rightarrow 0} [-2.5 - h] = -3 \dots\dots(4)$$

$$\lim_{x \rightarrow -2.5} f(x) = f(-2.5) = [-2.5] = -3$$

From above equations

$$\lim_{x \rightarrow -2.5^-} f(x) = \lim_{x \rightarrow -2.5^+} f(x) = \lim_{x \rightarrow -2.5} f(x)$$

Thus limit does exists

20. Question

$$\text{Evaluate } \lim_{x \rightarrow 2} f(x) \text{ (if it exists), where } f(x) = \begin{cases} x - [x], & x < 2 \\ 4, & x = 2 \\ 3x - 5, & x > 2 \end{cases}$$

Answer

$$\text{Given } f(x) = \begin{cases} x - [x], & x < 2 \\ 4, & x = 2 \\ 3x - 5, & x > 2 \end{cases}$$

$$\text{To find } \lim_{x \rightarrow 2} f(x)$$

To limit to exist we know $\lim_{x \rightarrow h^+} f(x) = \lim_{x \rightarrow h^-} f(x) = \lim_{x \rightarrow h} f(x)$ (1)

Thus to find the limit using the concept $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} f(x)$(2)

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2 + h) = \lim_{h \rightarrow 0} 3(2 + h) - 5 = 6 + 0 - 5 = 1 \dots\dots(3)$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2 - h) = \lim_{h \rightarrow 0} 2 - h + [2 - h] = 2 - h + 1 = 3 \dots\dots(4)$$

$$\lim_{x \rightarrow 2} f(x) = f(2) = 4 \dots\dots(5)$$

From above equations

$$\lim_{x \rightarrow 3^+} f(x) \neq \lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 2} f(x)$$

Thus the limit does not exist

21. Question

Show that $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist.

Answer

To Prove: $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist

Let us take the left-hand limit for the function:

$$\text{L.H.L} = \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} \sin \left(\frac{1}{0-h} \right) = - \lim_{h \rightarrow 0} \sin \left(\frac{1}{h} \right)$$

Now, multiplying and dividing by h, we get,

$$\text{L.H.L} = - \frac{\lim_{h \rightarrow 0} \sin \left(\frac{1}{h} \right)}{\frac{1}{h}} \times \frac{1}{h} = -1 \times \frac{1}{0} = -\infty$$

Now, taking the right-hand limit of the function, we get,

$$\text{R.H.L} = \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h) = \lim_{h \rightarrow 0} \sin \left(\frac{1}{0+h} \right) = \lim_{h \rightarrow 0} \sin \left(\frac{1}{h} \right)$$

Now, multiplying and dividing by h, we get,

$$\text{R.H.L} = \frac{\lim_{h \rightarrow 0} \sin \left(\frac{1}{h} \right)}{\frac{1}{h}} \times \frac{1}{h} = 1 \times \frac{1}{0} = \infty$$

Clearly, L.H.L $\neq$ R.H.L

Hence, limit does not exist.

22. Question

Let $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{where } x \neq \frac{\pi}{2} \\ 3, & \text{where } x = \frac{\pi}{2} \end{cases}$ and if $\lim_{x \rightarrow \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$, find the value of k.

Answer

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{where } x \neq \frac{\pi}{2} \\ 3, & \text{where } x = \frac{\pi}{2} \end{cases}$$

Let us find the limit of the function at $x = \frac{\pi}{2}$.

$$\text{Let } y = x - \frac{\pi}{2}, \pi - 2x = -2y$$

Therefore,

$$\text{L.H.L} = \lim_{y \rightarrow 0^-} \frac{k \cos x}{\pi - 2x} = \lim_{h \rightarrow 0} \frac{\left(k \cos \left(y + \frac{\pi}{2} \right) \right)}{-2y} = \lim_{y \rightarrow 0} \frac{-k \sin y}{-2y} = \frac{k}{2}$$

$$\text{Now, } \frac{k}{2} = 3$$

Hence, $k = 6$.

Exercise 29.2

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{x^2 + 1}{x + 1}$$

Answer

$$\text{Given limit} \Rightarrow \lim_{x \rightarrow 1} \frac{x^2 + 1}{x + 1}$$

Putting the value of limits directly, i.e., $x = 1$, we have

$$\Rightarrow \frac{1^2 + 1}{1 + 1}$$

$$\Rightarrow \frac{2}{2}$$

$$\Rightarrow 1$$

Hence the value of the given limit is 1.

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2}$$

Answer

$$\text{Given limit} \Rightarrow \lim_{x \rightarrow 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2}$$

Putting the value of limits directly, i.e. $x = 0$, we have

$$\Rightarrow \frac{2(0^2) + 3(0) + 4}{0^2 + 3(0) + 2}$$

$$\Rightarrow \frac{4}{2}$$

$$\Rightarrow 2$$

Hence the value of the given limit is 2.

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \frac{\sqrt{2x+3}}{x+3}$$

Answer

Given limit $\Rightarrow \lim_{x \rightarrow 3} \frac{\sqrt{2x+3}}{x+3}$

Putting the value of limits directly, i.e. $x = 0$, we have

$$\Rightarrow \frac{\sqrt{2(3)+3}}{3+3}$$

$$\Rightarrow \frac{\sqrt{3}}{3}$$

$$\Rightarrow \frac{3}{6}$$

$$\Rightarrow \frac{1}{2}$$

Hence the value of the given limit is 0.5

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{\sqrt{x+8}}{\sqrt{x}}$$

Answer

Given limit $\Rightarrow \lim_{x \rightarrow 1} \frac{\sqrt{x+8}}{x}$

Putting the values of limits directly, i.e. $x = 1$, we have

$$\Rightarrow \frac{\sqrt{1+8}}{1}$$

$$\Rightarrow \frac{\sqrt{9}}{1}$$

$$\Rightarrow 3$$

Hence the value of the given limit is 3.

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{\sqrt{x} + \sqrt{a}}{x + a}$$

Answer

Given limit $\Rightarrow \lim_{x \rightarrow a} \frac{\sqrt{x} + \sqrt{a}}{x + a}$

Putting the values of limit directly, i.e. $x = a$, we have

$$\Rightarrow \frac{\sqrt{a} + \sqrt{a}}{a + a}$$

$$\Rightarrow \frac{2\sqrt{a}}{2a}$$

$$\Rightarrow \frac{1}{\sqrt{a}}$$

Hence the value of the given limit is $\Rightarrow \frac{1}{\sqrt{a}}$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{1 + (x-1)^2}{1 + x^2}$$

Answer

$$\text{Given limit} \Rightarrow \lim_{x \rightarrow 1} \frac{1 + (x-1)^2}{1 + x^2}$$

Putting the values of limits directly, i.e. $x = 1$, we have

$$\Rightarrow \frac{1 + (1-1)^2}{1 + 1^2}$$

$$\Rightarrow \frac{1}{2}$$

Hence the value of the given limit is 0.5

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x^{2/3} - 9}{x - 27}$$

Answer

$$\text{Given limit} \Rightarrow \lim_{x \rightarrow 0} \frac{x^{2/3} - 9}{x - 27}$$

Putting the value of limit directly, i.e. $x = 0$, we have

$$\Rightarrow \frac{0^{2/3} - 9}{0 - 27}$$

$$\Rightarrow \frac{-9}{-27}$$

$$\Rightarrow \frac{1}{3}$$

Hence the value of the given limit is $\Rightarrow \frac{1}{3}$

8. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} 9$$

Answer

$$\text{Given the limit} \Rightarrow \lim_{x \rightarrow 0} 9$$

Always remember the limiting value of a constant (such as 4, 13, b, etc.) is the constant itself.

So, the limiting value of constant 9 is itself, i.e., 9.

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} (3 - x)$$

Answer

$$\text{Given the limit} \Rightarrow \lim_{x \rightarrow 2} (3 - x)$$

Putting the limiting value directly, i.e. $x = 2$, we have

$$\Rightarrow (3 - 2)$$

$$\Rightarrow 1$$

Hence the value of the given limit is 1.

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -1} (4x^2 + 2)$$

Answer

$$\text{Given limit} \Rightarrow \lim_{x \rightarrow -1} (4x^2 + 2)$$

Putting the value of limits directly, we have

$$\Rightarrow (4(-1)^2 + 2)$$

$$\Rightarrow (4(1) + 2)$$

$$\Rightarrow 6$$

Hence the value of the given limit is 6.

11. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -1} \frac{x^3 - 3x + 1}{x - 1}$$

Answer

$$\text{Given the limit} \Rightarrow \lim_{x \rightarrow -1} \frac{x^3 - 3x + 1}{x - 1}$$

Putting the value of limits directly, i.e. $x = -1$, we have

$$\Rightarrow \frac{(-1)^3 - 3(-1) + 1}{(-1) - 1}$$

$$\Rightarrow \frac{-1 + 3 + 1}{-2}$$

$$\Rightarrow \frac{-3}{2}$$

Hence the value of the given limit is $\Rightarrow \frac{-3}{2}$

12. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{3x + 1}{x + 3}$$

Answer

$$\text{Given limit} \Rightarrow \lim_{x \rightarrow 0} \frac{3x+1}{x+3}$$

Putting the value of limit directly, i.e. $x = 0$, we have

$$\Rightarrow \frac{3(0)+1}{0+3}$$

$$\Rightarrow \frac{1}{3}$$

Hence the value of the given limit is $\Rightarrow \frac{1}{3}$

13. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x + 2}$$

Answer

$$\text{Given limit} \Rightarrow \frac{1}{3}$$

Putting the value of limits directly, i.e. $x = 3$, we have

$$\Rightarrow \frac{3^2 - 9}{3 + 2}$$

$$\Rightarrow \frac{0}{5}$$

$$\Rightarrow 0$$

Hence the value of the given limit is 0.

14. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{ax + b}{cx + d}, d \neq 0$$

Answer

$$\text{Given limit} \Rightarrow \frac{1}{3}$$

Putting the value of limits directly, i.e. $x = 0$, we have

$$\Rightarrow \lim_{x \rightarrow 0} \frac{ax + b}{cx + d}$$

$$\Rightarrow \frac{b}{d}$$

The given condition $d \neq 0$ is reasonable because the denominator cannot be zero.

Hence the value of the given limit is $\frac{b}{d}$.

Exercise 29.3**1. Question**

Evaluate the following limits:

$$\lim_{x \rightarrow -5} \frac{2x^2 + 9x - 5}{x + 5}$$

Answer

$$= \frac{2(-5)^2 + 9(-5) - 5}{(-5) + 5}$$

$$= \frac{50 - 50}{(-5) + 5}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow -5} \frac{2x^2 + 9x - 5}{x + 5}$$

$$= \lim_{x \rightarrow -5} \frac{2x^2 + 10x - x - 5}{x + 5}$$

$$= \lim_{x \rightarrow -5} \frac{2x(x + 5) - (x + 5)}{x + 5}$$

$$= \lim_{x \rightarrow -5} \frac{(2x - 1)(x + 5)}{x + 5}$$

$$= \lim_{x \rightarrow -5} 2x - 1$$

$$= 2(-5) - 1$$

$$= -11$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow -5} \frac{d(2x^2 + 9x - 5)}{d(x + 5)}$$

$$= \lim_{x \rightarrow -5} \frac{4x + 9}{1}$$

$$= 4(-5) + 9$$

$$= -11$$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3}$$

Answer

$$= \frac{(3)^2 - 4(3) + 3}{(3)^2 - 2(3) - 3}$$

$$= \frac{12 - 12}{(-9) + 9}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$\begin{aligned}
 &= \lim_{x \rightarrow 3} \frac{(x^2 - 4x + 3)}{(x^2 - 2x - 3)} \\
 &= \lim_{x \rightarrow 3} \frac{(x^2 - 3x - x + 3)}{(x^2 - 3x + x - 3)} \\
 &= \lim_{x \rightarrow 3} \frac{x(x-3) - 1(x-3)}{x(x-3) + 1(x-3)} \\
 &= \lim_{x \rightarrow 3} \frac{(x-3)(x-1)}{(x-3)(x+1)} \\
 &= \lim_{x \rightarrow 3} \frac{(x-1)}{(x+1)} \\
 &= \frac{(3-1)}{(3+1)} \\
 &= \frac{2}{4} = \frac{1}{2}
 \end{aligned}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$\begin{aligned}
 &= \lim_{x \rightarrow 3} \frac{d(x^2 - 4x + 3)}{d(x^2 - 2x - 3)} \\
 &= \lim_{x \rightarrow 3} \frac{2x-4}{2x-2} \\
 &= \frac{2(3)-4}{2(3)-2} \\
 &= \frac{2}{4} = \frac{1}{2}
 \end{aligned}$$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{x^2 - 9}$$

Answer

$$\begin{aligned}
 &= \frac{(3)^4 - 81}{(3)^2 - 9} \\
 &= \frac{81 - 81}{(-9) + 9}
 \end{aligned}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 3} \frac{(x^4 - 81)}{(x^2 - 9)}$$

$$= \lim_{x \rightarrow 3} \frac{(x^4 - 3^4)}{(x^2 - 3^2)}$$

$$= \lim_{x \rightarrow 3} \frac{((x^2)^2 - (3^2)^2)}{(x^2 - 3^2)}$$

Since $a^2 - b^2 = (a + b)(a - b)$

Thus

$$= \lim_{x \rightarrow 3} \frac{(x^2 - 3^2)(x^2 + 3^2)}{(x^2 - 3^2)}$$

$$= \lim_{x \rightarrow 3} (x^2 + 3^2)$$

$$= 3^2 + 3^2$$

$$= 18$$

Method 2:

By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 3} \frac{d(x^4 - 81)}{d(x^2 - 9)}$$

$$= \lim_{x \rightarrow 3} \frac{4x^3}{2x}$$

$$= \frac{4(3)^3}{2}$$

$$= 54$$

4. Question

Evaluate the following limits: $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4}$

Answer

$$= \frac{(2)^3 - 8}{(2)^2 - 4}$$

$$= \frac{8 - 8}{(4) - 4}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 2} \frac{(x^3 - 8)}{(x^2 - 4)}$$

$$= \lim_{x \rightarrow 2} \frac{(x^3 - 2^3)}{(x^2 - 2^2)}$$

$$= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2^2 + 2x)}{(x + 2)(x - 2)}$$

Since $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$\begin{aligned}
&= \lim_{x \rightarrow 2} \frac{(x^2 + 2^2 + 2x)}{(x + 2)} \\
&= \frac{(2^2 + 2^2 + 2(2))}{(2 + 2)} \\
&= \frac{3.4}{(4)} \\
&= 3
\end{aligned}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$\begin{aligned}
&= \lim_{x \rightarrow 2} \frac{d(x^3 - 8)}{d(x^2 - 4)} \\
&= \lim_{x \rightarrow 2} \frac{3x^2}{2x} \\
&= \lim_{x \rightarrow 2} \frac{3x}{2} \\
&= \frac{3(2)}{2} \\
&= 3
\end{aligned}$$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -1/2} \frac{8x^3 + 1}{2x + 1}$$

Answer

$$\begin{aligned}
&= \frac{8\left(-\frac{1}{2}\right)^3 + 1}{2\left(-\frac{1}{2}\right) + 1} \\
&= \frac{-1 + 1}{-1 + 1}
\end{aligned}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$\begin{aligned}
&= \lim_{x \rightarrow -\frac{1}{2}} \frac{8x^3 + 1}{2x + 1} \\
&= \lim_{x \rightarrow -\frac{1}{2}} \frac{(2x)^3 + (1)^3}{2x + 1}
\end{aligned}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$$\begin{aligned}
&= \lim_{x \rightarrow -\frac{1}{2}} \frac{(2x + 1)((2x)^2 + (1)^2 - 2x)}{2x + 1} \\
&= \lim_{x \rightarrow -\frac{1}{2}} (2x)^2 + (1)^2 - 2x
\end{aligned}$$

$$= (2(\frac{-1}{2}))^2 + (1)^2 - 2(\frac{-1}{2})$$

$$= 1 + 1 + 1$$

$$= 3$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow -\frac{1}{2}} \frac{d(8x^3 + 1)}{d(2x + 1)}$$

$$= \lim_{x \rightarrow -\frac{1}{2}} \frac{24x^2}{2}$$

$$= \lim_{x \rightarrow -\frac{1}{2}} 12x^2$$

$$= 12(-1/2)^2$$

$$= 12/4$$

$$= 3$$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 4} \frac{x^2 - 7x + 12}{x^2 - 3x - 4}$$

Answer

$$= \frac{(4)^2 - 7(4) + 12}{(4)^2 - 3(4) - 4}$$

$$= \frac{28 - 28}{-16 + 16}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 4} \frac{(x^2 - 7x + 12)}{(x^2 - 3x - 4)}$$

$$= \lim_{x \rightarrow 4} \frac{(x^2 - 3x - 4x + 12)}{(x^2 - 4x + x - 4)}$$

$$= \lim_{x \rightarrow 4} \frac{x(x - 3) - 4(x - 3)}{x(x - 4) + 1(x - 4)}$$

$$= \lim_{x \rightarrow 4} \frac{(x - 3)(x - 4)}{(x - 4)(x + 1)}$$

$$= \lim_{x \rightarrow 4} \frac{(x - 3)}{(x + 1)}$$

$$= \frac{(4 - 3)}{(4 + 1)}$$

$$= \frac{1}{5}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 4} \frac{d(x^2 - 7x + 12)}{d(x^2 - 3x - 4)}$$

$$= \lim_{x \rightarrow 4} \frac{2x - 7}{2x - 3}$$

$$= \frac{2(4) - 7}{2(4) - 3}$$

$$= \frac{1}{5}$$

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2}$$

Answer

$$= \frac{(2)^4 - 16}{2 - 2}$$

$$= \frac{16 - 16}{2 - 2}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 2} \frac{(x^4 - 16)}{(x - 2)}$$

$$= \lim_{x \rightarrow 2} \frac{(x^4 - 2^4)}{(x - 2)}$$

$$= \lim_{x \rightarrow 2} \frac{(x^2)^2 - (2^2)^2}{(x - 2)}$$

Since $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 2} \frac{(x^2 - 2^2)(x^2 + 2^2)}{(x - 2)}$$

$$= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)(x^2 + 2^2)}{(x - 2)}$$

$$= \lim_{x \rightarrow 2} (x + 2)(x^2 + 2^2)$$

$$= (2 + 2)(2^2 + 2^2)$$

$$= 32$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 2} \frac{d(x^4 - 16)}{d(x - 2)}$$

$$= \lim_{x \rightarrow 2} \frac{4x^3}{1}$$

$$= \lim_{x \rightarrow 2} 4x^3$$

$$= 4(2)^3$$

$$= 32$$

8. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 5} \frac{x^2 - 9x + 20}{x^2 - 6x + 5}$$

Answer

$$= \frac{(5)^2 - 9(5) + 20}{(5)^2 - 6(5) + 5}$$

$$= \frac{45 - 45}{30 - 30}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 5} \frac{(x^2 - 9x + 20)}{(x^2 - 6x + 5)}$$

$$= \lim_{x \rightarrow 5} \frac{(x^2 - 5x - 4x + 20)}{(x^2 - 5x - x + 5)}$$

$$= \lim_{x \rightarrow 5} \frac{x(x - 5) - 4(x - 5)}{x(x - 5) - 1(x - 5)}$$

$$= \lim_{x \rightarrow 5} \frac{(x - 5)(x - 4)}{(x - 5)(x - 1)}$$

$$= \lim_{x \rightarrow 5} \frac{(x - 4)}{(x - 1)}$$

$$= \frac{(5 - 4)}{(5 - 1)}$$

$$= \frac{1}{4}$$

Method 2:

By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 5} \frac{d(x^2 - 9x + 20)}{d(x^2 - 6x + 5)}$$

$$= \lim_{x \rightarrow 5} \frac{2x - 9}{2x - 6}$$

$$= \frac{2(5)-9}{2(5)-6}$$

$$= \frac{1}{4}$$

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -1} \frac{x^3 + 1}{x + 1}$$

Answer

$$= \frac{(-1)^3 + 1}{-1 + 1}$$

$$= \frac{-1 + 1}{-1 + 1}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow -1} \frac{(x^3 + 1)}{(x + 1)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow -1} \frac{(x + 1)(x^2 + 1^2 - x)}{(x + 1)}$$

$$= \lim_{x \rightarrow -1} (x^2 + 1^2 - x)$$

$$= (-1)^2 + (1)^2 - (-1)$$

$$= 3$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow -1} \frac{d(x^3 + 1)}{d(x + 1)}$$

$$= \lim_{x \rightarrow -1} \frac{3x^2}{1}$$

$$= \lim_{x \rightarrow -1} 3x^2$$

$$= 3(-1)^2$$

$$= 3$$

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 5} \frac{x^3 - 125}{x^2 - 7x + 10}$$

Answer

$$= \frac{(5)^3 - 125}{(5)^2 - 7(5) + 10}$$

$$= \frac{125 - 125}{35 - 35}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 5} \frac{(x^3 - 125)}{(x^2 - 7x + 10)}$$

$$= \lim_{x \rightarrow 5} \frac{(x^3 - 5^3)}{(x^2 - 5x - 2x + 10)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 5} \frac{(x - 5)(x^2 + 5^2 + 5x)}{(x^2 - 5x - 2x + 10)}$$

$$= \lim_{x \rightarrow 5} \frac{(x - 5)(x^2 + 5^2 + 5x)}{x(x - 5) - 2(x - 5)}$$

$$= \lim_{x \rightarrow 5} \frac{(x - 5)(x^2 + 5^2 + 5x)}{(x - 5)(x - 2)}$$

$$= \lim_{x \rightarrow 5} \frac{(x^2 + 5^2 + 5x)}{(x - 2)}$$

$$= \frac{(5^2 + 5^2 + 5(5))}{(5 - 2)}$$

$$= \frac{3 \cdot 5^2}{3}$$

$$= 25$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 5} \frac{d(x^3 - 125)}{d(x^2 - 7x + 10)}$$

$$= \lim_{x \rightarrow 5} \frac{3x^2}{2x - 7}$$

$$= \frac{3(5^2)}{2(5) - 7}$$

$$= \frac{75}{3}$$

$$= 25$$

11. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \sqrt{2}} \frac{x^2 - 2}{x^2 + \sqrt{2}x - 4}$$

Answer

$$= \frac{(\sqrt{2})^2 - 2}{(\sqrt{2})^2 + \sqrt{2}(\sqrt{2}) - 4}$$

$$= \frac{2-2}{4-4}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow \sqrt{2}} \frac{(x^2 - 2)}{(x^2 + \sqrt{2}x - 4)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{(x^2 - (\sqrt{2})^2)}{(x^2 + 2\sqrt{2}x - \sqrt{2}x - 4)}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})}{x(x + 2\sqrt{2}) - \sqrt{2}(x + 2\sqrt{2})}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})}{(x + 2\sqrt{2})(x - \sqrt{2})}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{(x + \sqrt{2})}{(x + 2\sqrt{2})}$$

$$= \frac{(\sqrt{2} + \sqrt{2})}{(\sqrt{2} + 2\sqrt{2})}$$

$$= \frac{(2\sqrt{2})}{(3\sqrt{2})}$$

$$= \frac{(2)}{(3)}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow \sqrt{2}} \frac{d(x^2 - 2)}{d(x^2 + \sqrt{2}x - 4)}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{2x}{2x + \sqrt{2}}$$

$$= \frac{2(\sqrt{2})}{2(\sqrt{2}) + \sqrt{2}}$$

$$= \frac{2\sqrt{2}}{3\sqrt{2}}$$

$$= \frac{2}{3}$$

12. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \sqrt{3}} \frac{x^2 - 3}{x^2 + 3\sqrt{3}x - 12}$$

Answer

$$= \frac{(\sqrt{3})^2 - 3}{(\sqrt{3})^2 + 3\sqrt{3}(\sqrt{3}) - 12}$$

$$= \frac{3-3}{12-12}$$

Since the form is

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x^2 - 3)}{(x^2 + 3\sqrt{3}x - 12)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x^2 - (\sqrt{3})^2)}{(x^2 + 4\sqrt{3}x - \sqrt{3}x - 12)}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})}{x(x + 4\sqrt{3}) - \sqrt{3}(x + 4\sqrt{3})}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})}{(x + 4\sqrt{3})(x - \sqrt{3})}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x + \sqrt{3})}{(x + 4\sqrt{3})}$$

$$= \frac{(\sqrt{3} + \sqrt{3})}{(\sqrt{3} + 4\sqrt{3})}$$

$$= \frac{(2\sqrt{3})}{(5\sqrt{3})}$$

$$= \frac{(2)}{(5)}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow \sqrt{3}} \frac{d(x^2 - 3)}{d(x^2 + 3\sqrt{3}x - 12)}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{2x}{2x + 3\sqrt{3}}$$

$$= \frac{2(\sqrt{3})}{2(\sqrt{3}) + 3\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{5\sqrt{3}}$$

$$= \frac{2}{5}$$

13. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \sqrt{3}} \frac{x^4 - 9}{x^2 + 4\sqrt{3}x - 15}$$

Answer

$$= \frac{(\sqrt{3})^4 - 9}{(\sqrt{3})^2 + 4\sqrt{3}(\sqrt{3}) - 15}$$

$$= \frac{9-9}{15-15}$$

Since the form is

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x^4 - 9)}{(x^2 + 4\sqrt{3}x - 15)}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x^2)^2 - (\sqrt{3}^2)^2}{(x^2 + 4\sqrt{3}x - 15)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x^2 + (\sqrt{3})^2)(x^2 - (\sqrt{3})^2)}{(x^2 + 5\sqrt{3}x - \sqrt{3}x - 15)}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})(x^2 + (\sqrt{3})^2)}{x(x + 5\sqrt{3}) - \sqrt{3}(x + 5\sqrt{3})}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})(x^2 + (\sqrt{3})^2)}{(x + 5\sqrt{3})(x - \sqrt{3})}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{(x + \sqrt{3})(x^2 + (\sqrt{3})^2)}{(x + 5\sqrt{3})}$$

$$= \frac{(\sqrt{3} + \sqrt{3})(\sqrt{3}^2 + (\sqrt{3})^2)}{(\sqrt{3} + 5\sqrt{3})}$$

$$= \frac{(2\sqrt{3})(2.3)}{(6\sqrt{3})}$$

$$= 2$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow \sqrt{3}} \frac{d(x^4 - 9)}{d(x^2 + 4\sqrt{3}x - 15)}$$

$$= \lim_{x \rightarrow \sqrt{3}} \frac{4x^3}{2x + 4\sqrt{3}}$$

$$= \frac{4(\sqrt{3})^3}{2(\sqrt{3}) + 4\sqrt{3}}$$

$$= \frac{12\sqrt{3}}{6\sqrt{3}}$$

$$= 2$$

14. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \left(\frac{x}{x-2} - \frac{4}{x^2-2x} \right)$$

Answer

$$= \lim_{x \rightarrow 2} \left(\frac{x}{x-2} - \frac{4}{x^2-2x} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x}{x-2} - \frac{4}{x(x-2)} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x}{1} - \frac{4}{x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x^2-4}{x} \right) \left(\frac{1}{x-2} \right)$$

Since $a^2-b^2 = (a+b)(a-b)$

$$= \lim_{x \rightarrow 2} \left(\frac{x^2-2^2}{x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x-2}{x} \right) \left(\frac{x+2}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x+2}{x} \right)$$

$$= \frac{4}{2}$$

$$= 2$$

15. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \left(\frac{1}{x^2+x-2} - \frac{x}{x^3-1} \right)$$

Answer

$$\lim_{x \rightarrow 1} \left(\frac{1}{x^2+x-2} - \frac{x}{x^3-1} \right) = \lim_{x \rightarrow 1} \left(\frac{1}{x^2+2x-x-2} - \frac{x}{x^3-1} \right)$$

$$\Rightarrow \lim_{x \rightarrow 1} \left(\frac{1}{x^2+x-2} - \frac{x}{x^3-1} \right) = \lim_{x \rightarrow 1} \left(\frac{1}{x(x+2)-1(x+2)} - \frac{x}{x^3-1} \right)$$

$$\Rightarrow \lim_{x \rightarrow 1} \left(\frac{1}{x^2+x-2} - \frac{x}{x^3-1} \right) = \lim_{x \rightarrow 1} \left(\frac{1}{(x+2)(x-1)} - \frac{x}{(x-1)(x^2+x+1)} \right)$$

$$\Rightarrow \lim_{x \rightarrow 1} \left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1} \right) = \lim_{x \rightarrow 1} \frac{1}{x - 1} \left(\frac{1}{x + 2} - \frac{x}{x^2 + x + 1} \right)$$

$$\Rightarrow \lim_{x \rightarrow 1} \left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1} \right) = \lim_{x \rightarrow 1} \frac{1}{x - 1} \left(\frac{x^2 + x + 1 - x(x + 2)}{(x + 2)(x^2 + x + 1)} \right)$$

$$\lim_{x \rightarrow 1} \left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1} \right) = \frac{-1}{(x + 2)(x^2 + x + 1)}$$

$$\text{Hence, } \lim_{x \rightarrow 1} \left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1} \right) = \frac{-1}{9}$$

16. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \left(\frac{1}{x - 3} - \frac{2}{x^2 - 4x + 3} \right)$$

Answer

$$= \lim_{x \rightarrow 3} \left(\frac{1}{x - 3} - \frac{2}{x^2 - 3x - x + 3} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{1}{x - 3} - \frac{2}{x(x - 3) - 1(x - 3)} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{1}{x - 3} - \frac{2}{(x - 3)(x - 1)} \right)$$

$$= \lim_{x \rightarrow 3} \frac{1}{x - 3} \left(1 - \frac{2}{(x - 1)} \right)$$

$$= \lim_{x \rightarrow 3} \frac{1}{x - 3} \left(\frac{x - 1 - 2}{(x - 1)} \right)$$

$$= \lim_{x \rightarrow 3} \frac{1}{x - 3} \left(\frac{x - 3}{(x - 1)} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{1}{(x - 1)} \right)$$

$$= \frac{1}{(3 - 1)}$$

$$= \frac{1}{2}$$

17. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \left(\frac{1}{x - 2} - \frac{2}{x^2 - 2x} \right)$$

Answer

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x - 2} - \frac{2}{x^2 - 2x} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x - 2} - \frac{2}{x(x - 2)} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{1} - \frac{2}{x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x-2}{x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x} \right)$$

$$= \frac{1}{2}$$

18. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1/4} \frac{4x-1}{2\sqrt{x}-1}$$

Answer

$$= \frac{4\left(\frac{1}{4}\right)-1}{2\left(\sqrt{\frac{1}{4}}\right)-1}$$

$$= \frac{1-1}{1-1}$$

Since the form is indeterminate

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 1/4} \frac{(4x-1)}{(2\sqrt{x}-1)}$$

$$= \lim_{x \rightarrow 1/4} \frac{(2\sqrt{x})^2 - (1)^2}{(2\sqrt{x}-1)}$$

Since $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a+b)(a-b)$

$$= \lim_{x \rightarrow 1/4} \frac{(2\sqrt{x}-1)(2\sqrt{x}+1)}{(2\sqrt{x}-1)}$$

$$= \lim_{x \rightarrow 1/4} (2\sqrt{x}+1)$$

$$= \left(2\sqrt{\frac{1}{4}} + 1 \right)$$

$$= \left(\frac{2}{2} + 1 \right)$$

$$= 2$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 1/4} \frac{d(4x-1)}{d(2\sqrt{x}-1)}$$

$$= \lim_{x \rightarrow \frac{1}{4}} \frac{4}{2\left(\frac{1}{2}\right)x^{-\frac{1}{2}}}$$

$$= \frac{4}{\left(1/\sqrt{\frac{1}{4}}\right)}$$

$$= 2$$

19. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{\sqrt{x} - 2}$$

Answer

$$= \frac{4^2 - 16}{(\sqrt{4}) - 2}$$

$$= \frac{16 - 16}{2 - 2}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 4} \frac{(x^2 - 16)}{(\sqrt{x} - 2)}$$

$$= \lim_{x \rightarrow 4} \frac{(x)^2 - (4)^2}{(\sqrt{x} - 2)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 4} \frac{(x - 4)(x + 4)}{(\sqrt{x} - 2)}$$

$$= \lim_{x \rightarrow 4} \frac{((\sqrt{x})^2 - (2)^2)(x + 4)}{(\sqrt{x} - 2)}$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)(x + 4)}{(\sqrt{x} - 2)}$$

$$= \lim_{x \rightarrow 4} (\sqrt{x} + 2)(x + 4)$$

$$= (\sqrt{4} + 2)(4 + 4)$$

$$= (2 + 2)(4 + 4)$$

$$= 32$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 4} \frac{d(x^2 - 16)}{d(\sqrt{x} - 2)}$$

$$= \lim_{x \rightarrow 4} \frac{2x}{\left(\frac{1}{2}\right)^{x-\frac{1}{2}}}$$

$$= \lim_{x \rightarrow 4} 4x^{\frac{3}{2}}$$

$$= 4(4)^{3/2}$$

$$= 32$$

20. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{(a+x)^2 - a^2}{x}$$

Answer

$$= \frac{(a)^2 - a^2}{0}$$

Since the form is indeterminate

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 0} \frac{(a+x)^2 - a^2}{x}$$

Since $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a+b)(a-b)$

$$= \lim_{x \rightarrow 0} \frac{(a+x+a)(a+x-a)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{(2a+x)(x)}{x}$$

$$= \lim_{x \rightarrow 0} (2a+x)$$

$$= 2a + 0$$

$$= 2a$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 0} \frac{d((a+x)^2 - a^2)}{d(x)}$$

$$= \lim_{x \rightarrow 0} \frac{2(a+x)}{1}$$

$$= 2(a+0)$$

$$= 2a$$

21. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^3 - 2x^2} \right)$$

Answer

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^3 - 2x^2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2(x-2)} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{1} - \frac{4}{x^2} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{x^2} \right) \left(\frac{1}{x-2} \right)$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 2} \left(\frac{x + 2}{x^2} \right) \left(\frac{x-2}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x + 2}{x^2} \right)$$

$$= \frac{4}{4}$$

$$= 1$$

22. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{3}{x^2 - 3x} \right)$$

Answer

$$= \lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{3}{x^2 - 3x} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{3}{x(x-3)} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{1}{1} - \frac{3}{x} \right) \left(\frac{1}{x-3} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{x-3}{x} \right) \left(\frac{1}{x-3} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{1}{x} \right)$$

$$= \frac{1}{3}$$

23. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2}{x^2 - 1} \right)$$

Answer

$$= \lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right)$$

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{2}{(x+1)(x-1)} \right)$$

$$= \lim_{x \rightarrow 1} \left(\frac{1}{1} - \frac{2}{x+1} \right) \left(\frac{1}{x-1} \right)$$

$$= \lim_{x \rightarrow 1} \left(\frac{x-1}{x+1} \right) \left(\frac{1}{x-1} \right)$$

$$= \lim_{x \rightarrow 1} \left(\frac{1}{x+1} \right)$$

$$= \frac{1}{2}$$

24. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} (x^2 - 9) \left(\frac{1}{x+3} + \frac{1}{x-3} \right)$$

Answer

Since $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 3} (x+3)(x-3) \left(\frac{1}{x+3} + \frac{1}{x-3} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{(x+3)(x-3)}{x+3} + \frac{(x+3)(x-3)}{x-3} \right)$$

$$= \lim_{x \rightarrow 3} \left(\frac{(x-3)}{1} + \frac{(x+3)}{1} \right)$$

$$= \lim_{x \rightarrow 3} 2x$$

$$= 6$$

25. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1}$$

Answer

$$= \frac{(1)^4 - 3(1)^3 + 2}{(1)^3 - 5(1)^2 + 3(1) + 1}$$

$$= \frac{3-3}{5-5}$$

Since the form is indeterminate

$$= \frac{0}{0}$$

Method 1: factorization

$$\begin{aligned}
 &= \lim_{x \rightarrow 1} \frac{(x)^4 - 3(x)^3 + 2}{(x)^3 - 5(x)^2 + 3(x) + 1} \\
 &= \lim_{x \rightarrow 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1} \\
 &= \lim_{x \rightarrow 1} \frac{x^4 - 2x^3 - x^3 + 2}{x^3 - x^2 - 3x^2 - x^2 + 3x + 1} \\
 &= \lim_{x \rightarrow 1} \frac{x^3(x-1) - 2(x^3-1)}{x^2(x-1) - 1(x^2-1) - 3x(x-1)}
 \end{aligned}$$

Since $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a+b)(a-b)$

$$\begin{aligned}
 &= \lim_{x \rightarrow 1} \frac{x^3(x-1) - 2(x-1)(x^2 + 1^2 + x)}{x^2(x-1) - 1(x-1)(x+1) - 3x(x-1)} \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)(x^3 - 2(x^2 + 1^2 + x))}{(x-1)(x^2 - 1(x+1) - 3x)} \\
 &= \lim_{x \rightarrow 1} \frac{x^3 - 2(x^2 + 1^2 + x)}{x^2 - 1(x+1) - 3x} \\
 &= \frac{1^3 - 2(1^2 + 1^2 + 1)}{1^2 - 1(1+1) - 3(1)} \\
 &= \frac{1 - 2(3)}{1 - 1(2) - 3(1)} \\
 &= \frac{-5}{-4} \\
 &= \frac{5}{4}
 \end{aligned}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$\begin{aligned}
 &= \lim_{x \rightarrow 1} \frac{d((x)^4 - 3(x)^3 + 2)}{d((x)^3 - 5(x)^2 + 3(x) + 1)} \\
 &= \lim_{x \rightarrow 1} \frac{4x^3 - 9x^2}{3x^2 - 10x + 3} \\
 &= \frac{4(1)^3 - 9(1)^2}{3(1)^2 - 10(1) + 3} \\
 &= \frac{-5}{-4} \\
 &= \frac{5}{4}
 \end{aligned}$$

26. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6}$$

Answer

$$= \frac{(2)^3 + 3(2)^2 - 9(2) - 2}{(2)^3 - 2 - 6}$$

$$= \frac{20-20}{8-8}$$

Since the form is indeterminate

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 2} \frac{(x)^3 + 3(x)^2 - 9(x) - 2}{(x)^3 - x - 6}$$

By long division method

$$= \lim_{x \rightarrow 2} 1 + \frac{3x^2 - 8x + 4}{x^3 - x - 6}$$

$$= \lim_{x \rightarrow 2} 1 + \frac{3x^2 - 6x - 2x + 4}{x^3 - 4x + 3x - 6}$$

$$= \lim_{x \rightarrow 2} 1 + \frac{3x(x-2) - 2(x-2)}{x(x^2-4) + 3(x-2)}$$

Since $a^3-b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2-b^2 = (a+b)(a-b)$

$$= \lim_{x \rightarrow 2} 1 + \frac{(x-2)(3x-2)}{x(x^2-2^2) + 3(x-2)}$$

$$= \lim_{x \rightarrow 2} 1 + \frac{(x-2)(3x-2)}{x(x-2)(x+2) + 3(x-2)}$$

$$= \lim_{x \rightarrow 2} 1 + \frac{(x-2)(3x-2)}{(x-2)[x(x+2) + 3]}$$

$$= \lim_{x \rightarrow 2} 1 + \frac{(3x-2)}{[x(x+2) + 3]}$$

$$= 1 + \frac{(3 \cdot 2 - 2)}{[2(2+2) + 3]}$$

$$= 1 + \frac{4}{11}$$

$$= \frac{15}{11}$$

Method2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 2} \frac{d((x)^3 + 3(x)^2 - 9(x) - 2)}{d((x)^3 - x - 6)}$$

$$= \lim_{x \rightarrow 2} \frac{3x^2 + 6x - 9}{3x^2 - 1}$$

$$= \frac{3(2)^2 + 6(2) - 9}{3(2)^2 - 1}$$

$$= \frac{24-9}{12-1}$$

$$= \frac{15}{11}$$

27. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{1 - x^{-1/3}}{1 - x^{-2/3}}$$

Answer

$$= \frac{-(1)^{-\frac{1}{3}} + 1}{-(1)^{-\frac{2}{3}} + 1}$$

$$= \frac{-1 + 1}{-1 + 1}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 1} \frac{-(x)^{-\frac{1}{3}} + 1}{-\left((x)^{-\frac{1}{3}}\right)^2 + 1}$$

Since $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 1} \frac{-(x)^{-\frac{1}{3}} + 1}{\left[-(x)^{-\frac{1}{3}} + 1\right] \left[(x)^{-\frac{1}{3}} + 1\right]}$$

$$= \lim_{x \rightarrow 1} \frac{1}{\left[(x)^{-\frac{1}{3}} + 1\right]}$$

$$= \frac{1}{\left[(x)^{-\frac{1}{3}} + 1\right]}$$

$$= \frac{1}{[1 + 1]}$$

$$= \frac{1}{[2]}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 1} \frac{d(-(x)^{-\frac{1}{3}} + 1)}{d(-(x)^{-\frac{2}{3}} + 1)}$$

$$= \lim_{x \rightarrow 1} \frac{\frac{1}{3}x^{-\frac{4}{3}}}{\frac{2}{3}x^{-\frac{5}{3}}}$$

$$= \lim_{x \rightarrow 1} \frac{1}{2} x^{\frac{1}{3}}$$

$$= \frac{1}{2} (1)^{\frac{1}{3}}$$

$$= \frac{1}{2}$$

28. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^3 - 3x^2 + x - 3}$$

Answer

$$= \frac{(3)^2 - (3) - 6}{(3)^3 - 3(3)^2 + 3 - 3}$$

$$= \frac{9-9}{12-12}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 3} \frac{\{(x)^2 - (x) - 6\}}{\{(x)^3 - 3(x)^2 + x - 3\}}$$

$$= \lim_{x \rightarrow 3} \frac{\{x^2 - 3x + 2x - 6\}}{\{x^3 - 3x^2 + x - 3\}}$$

$$= \lim_{x \rightarrow 3} \frac{\{x(x-3) + 2(x-3)\}}{\{x^2(x-3) + 1(x-3)\}}$$

$$= \lim_{x \rightarrow 3} \frac{\{(x+2)(x-3)\}}{\{(x^2+1)(x-3)\}}$$

$$= \lim_{x \rightarrow 3} \frac{\{x+2\}}{\{x^2+1\}}$$

$$= \frac{\{3+2\}}{\{3^2+1\}}$$

$$= \frac{5}{10}$$

$$= \frac{1}{2}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 3} \frac{d\{(x)^2 - (x) - 6\}}{d\{(x)^3 - 3(x)^2 + x - 3\}}$$

$$= \lim_{x \rightarrow 3} \frac{2x-1}{3x^2-6x+1}$$

$$= \frac{2(3)-1}{3(3)^2-6(3)+1}$$

$$= \frac{5}{10}$$

$$= \frac{1}{2}$$

29. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -2} \frac{x^3 + x^2 + 4x + 12}{x^3 - 3x + 2}$$

Answer

$$= \frac{(-2)^3 + (-2)^2 + 4(-2) + 12}{(-2)^3 - 3(-2) + 2}$$

$$= \frac{16-16}{8-8}$$

Since the form is

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow -2} \frac{\{x^3 + x^2 + 4x + 12\}}{\{x^3 - 3x + 2\}}$$

By long division method

$$= \lim_{x \rightarrow -2} 1 + \frac{\{x^2 + 7x + 10\}}{\{x^3 - 3x + 2\}}$$

$$= \lim_{x \rightarrow -2} 1 + \frac{\{x^2 + 5x + 2x + 10\}}{\{x^3 - 4x + x + 2\}}$$

$$= \lim_{x \rightarrow -2} 1 + \frac{\{x(x + 5) + 2(x + 5)\}}{\{x(x^2 - 2^2) + 1(x + 2)\}}$$

Since $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow -2} 1 + \frac{\{(x + 5)(x + 2)\}}{\{x(x + 2)(x - 2) + 1(x + 2)\}}$$

$$= \lim_{x \rightarrow -2} 1 + \frac{\{(x + 5)(x + 2)\}}{(x + 2)\{x(x - 2) + 1\}}$$

$$= \lim_{x \rightarrow -2} 1 + \frac{\{(x + 5)\}}{\{x(x - 2) + 1\}}$$

$$= 1 + \frac{\{(-2 + 5)\}}{\{-2(-2 - 2) + 1\}}$$

$$= 1 + \frac{3}{\{8 + 1\}}$$

$$= 1 + \frac{3}{\{9\}}$$

$$= 1 + \frac{1}{3}$$

$$= \frac{4}{3}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow -2} \frac{d\{(x)^3 + (x)^2 + 4(x) + 12\}}{d\{(x)^3 - 3(x) + 2\}}$$

$$= \lim_{x \rightarrow -2} \frac{3x^2 + 2x + 4}{3x^2 - 3}$$

$$= \frac{3(-2)^2 + 2(-2) + 4}{3(-2)^2 - 3}$$

$$= \frac{16 - 4}{12 - 3}$$

$$= \frac{12}{9} = \frac{4}{3}$$

30. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1}$$

Answer

$$= \frac{(1)^3 + 3(1)^2 - 6(1) + 2}{(1)^3 + 3(1)^2 - 3(1) - 1}$$

$$= \frac{6 - 6}{3 - 3}$$

Since the form is

$$= \frac{0}{0}$$

Method 1: factorization

$$= \lim_{x \rightarrow 1} \frac{\{x^3 + 3x^2 - 6x + 2\}}{\{x^3 + 3x^2 - 3x - 1\}}$$

by dividing

$$= \lim_{x \rightarrow 1} 1 + \frac{-3x + 3}{\{x^3 - 1 + 3x^2 - 3x\}}$$

Since $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a + b)(a - b)$

$$= \lim_{x \rightarrow 1} 1 + \frac{-3x + 3}{(x - 1)(x^2 + 1 + x) + 3x(x - 1)}$$

$$= \lim_{x \rightarrow 1} 1 + \frac{-3(x - 1)}{(x - 1)[(x^2 + 1 + x) + 3x]}$$

$$= \lim_{x \rightarrow 1} 1 + \frac{-3}{[x^2 + 1 + 4x]}$$

$$= 1 + \frac{-3}{[1^2 + 1 + 4 \cdot 1]}$$

$$= 1 + \frac{-3}{[6]}$$

$$= 1 + \frac{-1}{[2]}$$

$$= \frac{1}{2}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$= \lim_{x \rightarrow 1} \frac{d\{(x)^3 + 3(x)^2 - 6(x) + 2\}}{d\{(x)^3 + 3(x)^2 - 3(x) - 1\}}$$

$$= \lim_{x \rightarrow 1} \frac{3x^2 + 6x - 6}{3x^2 + 6x - 3}$$

$$= \frac{3(1)^2 + 6(1) - 6}{3(1)^2 + 6(1) - 3}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

31. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \left\{ \frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right\}$$

Answer

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 2x^2 - x^2 + 2x} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^2(x-2) - x(x-2)} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{(x^2 - x)(x-2)} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{1}{1} - \frac{2(2x-3)}{(x^2 - x)} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x^2 - x - 4x + 6}{x^2 - x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x^2 - 5x + 6}{x^2 - x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x^2 - 2x - 3x + 6}{x^2 - x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x(x-2) - 3(x-2)}{x^2 - x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{(x-3)(x-2)}{x^2 - x} \right) \left(\frac{1}{x-2} \right)$$

$$= \lim_{x \rightarrow 2} \left(\frac{x-3}{x^2 - x} \right)$$

$$= \frac{2-3}{4-2}$$

$$= \frac{-1}{2}$$

32. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{\sqrt{x^2 - 1} + \sqrt{x - 1}}{\sqrt{x^2 - 1}}, x > 1$$

Answer

$$\begin{aligned} &= \lim_{x \rightarrow 1} \frac{\sqrt{x^2 - 1} + \sqrt{x - 1}}{\sqrt{x^2 - 1}} \\ &= \lim_{x \rightarrow 1} \frac{\sqrt{(x + 1)(x - 1)} + \sqrt{x - 1}}{\sqrt{(x - 1)(x + 1)}} \\ &= \lim_{x \rightarrow 1} \frac{(\sqrt{(x + 1)} + 1)\sqrt{x - 1}}{\sqrt{(x - 1)(x + 1)}} \\ &= \lim_{x \rightarrow 1} \frac{(\sqrt{(x + 1)} + 1)}{\sqrt{(x + 1)}} \\ &= \frac{(\sqrt{(1 + 1)} + 1)}{\sqrt{(1 + 1)}} \\ &= \frac{\sqrt{2} + 1}{\sqrt{2}} \end{aligned}$$

33. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \left\{ \frac{x - 2}{x^2 - x} - \frac{1}{x^3 - 3x^2 + 2x} \right\}$$

Answer

$$\begin{aligned} &= \lim_{x \rightarrow 1} \left(\frac{x - 2}{x^2 - x} - \frac{1}{x^3 - 3x^2 + 2x} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{x - 2}{x^2 - x} - \frac{1}{x^3 - 2x^2 - x^2 + 2x} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{x - 2}{x^2 - x} - \frac{1}{x^2(x - 2) - x(x - 2)} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{x - 2}{x^2 - x} - \frac{1}{(x^2 - x)(x - 2)} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{x - 2}{1} - \frac{1}{(x - 2)} \right) \left(\frac{1}{x^2 - x} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{(x - 2)^2 - 1}{x - 2} \right) \left(\frac{1}{x^2 - x} \right) \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \left(\frac{1}{x^2 - x} \right) \left(\frac{x^2 - 4x + 3}{x - 2} \right) \\
&= \lim_{x \rightarrow 1} \left(\frac{1}{x(x-1)} \right) \left(\frac{x^2 - 3x - x + 3}{x - 2} \right) \\
&= \lim_{x \rightarrow 1} \left(\frac{1}{x(x-1)} \right) \left(\frac{x(x-3) - 1(x-3)}{x - 2} \right) \\
&= \lim_{x \rightarrow 1} \left(\frac{1}{x(x-1)} \right) \left(\frac{(x-1)(x-3)}{x - 2} \right) \\
&= \lim_{x \rightarrow 1} \left(\frac{x-3}{x(x-2)} \right) \\
&= \frac{1-3}{1(1-2)} \\
&= 2
\end{aligned}$$

34. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$$

Answer

$$\begin{aligned}
&= \frac{(1)^7 - 2(1)^5 + 1}{(1)^3 - 3(1)^2 + 2} \\
&= \frac{2-2}{3-3}
\end{aligned}$$

Since the form is indeterminant

$$= \frac{0}{0}$$

Method 1: factorization

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{\{(x)^7 - 2(x)^5 + 1\}}{\{(x)^3 - 3(x)^2 + 2\}} \\
&= \lim_{x \rightarrow 1} \frac{\{(x)^7 - 1(x)^5 - x^5 + 1\}}{\{(x)^3 - (x)^2 - 2x^2 + 2\}} \\
&= \lim_{x \rightarrow 1} \frac{\{(x)^5(x^2 - 1) - (x^5 - 1)\}}{\{(x)^3 - (x)^2 - 2x^2 + 2\}} \\
&= \lim_{x \rightarrow 1} \frac{\{(x)^5(x^2 - 1) - (x-1)(x^4 + x^3 + x^2 + x + 1)\}}{\{(x)^3 - (x)^2 - 2x^2 + 2\}}
\end{aligned}$$

Since $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$ & $a^2 - b^2 = (a+b)(a-b)$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{\{(x)^5(x-1)(x+1) - (x-1)(x^4 + x^3 + x^2 + x + 1)\}}{x^2(x-1) - 2(x^2 - 1)} \\
&= \lim_{x \rightarrow 1} \frac{\{(x)^5(x-1)(x+1) - (x-1)(x^4 + x^3 + x^2 + x + 1)\}}{x^2(x-1) - 2(x-1)(x+1)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{(x-1)\{(x)^5(x+1) - (x^4 + x^3 + x^2 + x + 1)\}}{(x-1)[x^2 - 2(x+1)]} \\
&= \lim_{x \rightarrow 1} \frac{\{(x)^5(x+1) - (x^4 + x^3 + x^2 + x + 1)\}}{[x^2 - 2(x+1)]} \\
&= \frac{\{(1)^5(1+1) - (1^4 + 1^3 + 1^2 + 1 + 1)\}}{[1^2 - 2(1+1)]} \\
&= \frac{2-5}{1-4} \\
&= \frac{-3}{-3} \\
&= 1
\end{aligned}$$

Method 2: By L hospital rule:

Differentiating numerator and denominator separately:

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{d\{(x)^7 - 2(x)^5 + 1\}}{d\{(x)^3 - 3(x)^2 + 2\}} \\
&= \lim_{x \rightarrow 1} \frac{7x^6 - 10x^4}{3x^2 - 6x} \\
&= \frac{7(1)^6 - 10(1)^4}{3(1)^2 - 6(1)} \\
&= \frac{-3}{-3} \\
&= 1
\end{aligned}$$

Exercise 29.4

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x}$$

Answer

Given $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x+x^2} - 1)(\sqrt{1+x+x^2} + 1)}{x(\sqrt{1+x+x^2} + 1)}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{1 + x + x^2 - 1}{x(\sqrt{1+x+x^2} + 1)} \\
&= \lim_{x \rightarrow 0} \frac{x(1+x)}{x(\sqrt{1+x+x^2} + 1)}
\end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x} = \lim_{x \rightarrow 0} \frac{(1+x)}{(\sqrt{1+x+x^2} + 1)}$$

Now we can see that the indeterminate form is removed, so substituting x as 0

$$\text{We get, } \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - 1}{x} = \frac{1}{1+1} = \frac{1}{2}$$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}}$$

Answer

$$\text{Given } \lim_{x \rightarrow 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}}$$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}} = \lim_{x \rightarrow 0} \frac{2x}{(\sqrt{a+x} - \sqrt{a-x})} \frac{(\sqrt{a+x} + \sqrt{a-x})}{(\sqrt{a+x} + \sqrt{a-x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{2x(\sqrt{a+x} + \sqrt{a-x})}{a+x-a+x}$$

$$= \lim_{x \rightarrow 0} \frac{2x(\sqrt{a+x} + \sqrt{a-x})}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{a+x} + \sqrt{a-x})}{1}$$

Now we can see that the indeterminate form is removed, so substituting x as 0

$$\text{We get } \lim_{x \rightarrow 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}} = \sqrt{a} + \sqrt{a} = 2\sqrt{a}$$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{a^2 + x^2} - a}{x^2}$$

Answer

$$\text{Given } \lim_{x \rightarrow 0} \frac{\sqrt{a^2 + x^2} - a}{x^2}$$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{a^2 + x^2} - a}{x^2} = \lim_{x \rightarrow 0} \frac{(\sqrt{a^2 + x^2} - a)(\sqrt{a^2 + x^2} + a)}{x^2(\sqrt{a^2 + x^2} + a)}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{(a^2 + x^2 - a^2)}{x^2(\sqrt{a^2 + x^2} + a)}$$

$$= \lim_{x \rightarrow 0} \frac{x^2}{x^2(\sqrt{a^2 + x^2} + a)}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{a^2 + x^2} - a}{x^2} = \lim_{x \rightarrow 0} \frac{1}{(\sqrt{a^2 + x^2} + a)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

$$\text{We get, } \lim_{x \rightarrow 0} \frac{\sqrt{a^2 + x^2} - a}{x^2} = \frac{1}{a+a} = \frac{1}{2a}$$

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x}$$

Answer

$$\text{Given } \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x}$$

To find: the limit of the given equation when x tends to 0

Substituting x as 0 we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{2x(\sqrt{1+x} + \sqrt{1-x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{1+x - 1+x}{2x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \rightarrow 0} \frac{2x}{2x(\sqrt{1+x} + \sqrt{1-x})}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x} = \lim_{x \rightarrow 0} \frac{1}{(\sqrt{1+x} + \sqrt{1-x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

$$\text{We get } \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x} = \frac{1}{1+1} = \frac{1}{2}$$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{\sqrt{3-x} - 1}{2-x}$$

Answer

Given $\lim_{x \rightarrow 2} \frac{\sqrt{3-x}-1}{2-x}$

To find: the limit of the given equation when x tends to 2

Substituting x as 2, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\lim_{x \rightarrow 2} \frac{\sqrt{3-x}-1}{2-x} = \lim_{x \rightarrow 2} \frac{(\sqrt{3-x}-1)(\sqrt{3-x}+1)}{(2-x)(\sqrt{3-x}+1)}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 2} \frac{(3-x-1)}{(2-x)(\sqrt{3-x}+1)}$$

$$= \lim_{x \rightarrow 2} \frac{(2-x)}{(2-x)(\sqrt{3-x}+1)}$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{\sqrt{3-x}-1}{2-x} = \lim_{x \rightarrow 2} \frac{1}{(\sqrt{3-x}+1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 2

$$\text{We get } \lim_{x \rightarrow 2} \frac{\sqrt{3-x}-1}{2-x} = \frac{1}{1+1} = \frac{1}{2}$$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$$

Answer

Given $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$

To find: the limit of the given equation when x tends to 3

Substituting x as 3, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}} = \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{(\sqrt{x-2}-\sqrt{4-x})(\sqrt{x-2}+\sqrt{4-x})}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{(x-2-4+x)(1)}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{(2x-6)(1)}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2}+\sqrt{4-x})}{2(x-3)(1)}$$

$$\Rightarrow \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}} = \lim_{x \rightarrow 3} \frac{(1)(\sqrt{x-2}+\sqrt{4-x})}{2(1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 3

We get $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}} = \frac{1+1}{2} = 1$

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x-1}{\sqrt{x^2+3}-2}$$

Answer

Given $\lim_{x \rightarrow 0} \frac{x-1}{\sqrt{x^2+3}-2}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we find that it is in non-indeterminant form so by substituting x as 0 we will directly get the answer

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x-1}{\sqrt{x^2+3}-2} = \frac{0-1}{\sqrt{0+3}-2}$$

We get $\lim_{x \rightarrow 0} \frac{x-1}{\sqrt{x^2+3}-2} = \frac{-1}{\sqrt{3}-2}$ as the answer

8. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x-1}$$

Answer

Given $\lim_{x \rightarrow 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x-1}$

To find: the limit of the given equation when x tends to 1

Substituting x as 1, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x-1} = \lim_{x \rightarrow 1} \frac{(\sqrt{5x-4}-\sqrt{x})(\sqrt{5x-4}+\sqrt{x})}{(x-1)(\sqrt{5x-4}+\sqrt{x})}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 1} \frac{(5x-4-x)}{(x-1)} \frac{1}{(\sqrt{5x-4}+\sqrt{x})}$$

$$= \lim_{x \rightarrow 1} \frac{4(x-1)}{(x-1)} \frac{1}{(\sqrt{5x-4}+\sqrt{x})}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x-1} = \lim_{x \rightarrow 1} \frac{4}{1} \frac{1}{(\sqrt{5x-4}+\sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x \rightarrow 1} \frac{\sqrt{5x-4}-\sqrt{x}}{x-1} = \frac{4}{1+1} = 2$

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2}$$

Answer

Given $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2}$

To find: the limit of the given equation when x tends to 1

Substituting x as 1, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2} = \lim_{x \rightarrow 1} \frac{(x-1)}{(\sqrt{x^2+3}-2)} \frac{(\sqrt{x^2+3}+2)}{(\sqrt{x^2+3}+2)}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 1} \frac{(x-1)}{(x^2+3-4)} \frac{(\sqrt{x^2+3}+2)}{1}$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)}{(x-1)(x+1)} \frac{(\sqrt{x^2+3}+2)}{1}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2} = \lim_{x \rightarrow 1} \frac{1}{(x+1)} \frac{(\sqrt{x^2+3}+2)}{1}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

We get $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2} = \frac{4}{1+1} = 2$

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+3}-\sqrt{6}}{x^2-9}$$

Answer

Given $\lim_{x \rightarrow 3} \frac{\sqrt{x+3}-\sqrt{6}}{x^2-9}$

To find: the limit of the given equation when x tends to 3

Substituting x as 3, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 3} \frac{\sqrt{x+3}-\sqrt{6}}{x^2-9} = \lim_{x \rightarrow 3} \frac{(\sqrt{x+3}-\sqrt{6})}{(x^2-9)} \frac{(\sqrt{x+3}+\sqrt{6})}{(\sqrt{x+3}+\sqrt{6})}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 3} \frac{(x+3-6)}{(x^2-9)} \frac{1}{(\sqrt{x+3}+\sqrt{6})}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)}{(x-3)(x+3)} \frac{1}{(\sqrt{x+3}+\sqrt{6})}$$

$$\Rightarrow \lim_{x \rightarrow 3} \frac{\sqrt{x+3} - \sqrt{6}}{x^2 - 9} = \lim_{x \rightarrow 3} \frac{1}{(x+3)} \frac{1}{(\sqrt{x+3} + \sqrt{6})}$$

Now we can see that the indeterminate form is removed, so substituting x as 3

$$\text{We get } \lim_{x \rightarrow 3} \frac{\sqrt{x+3} - \sqrt{6}}{x^2 - 9} = \frac{1}{6(2\sqrt{6})} = \frac{1}{12\sqrt{6}}$$

11. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^2 - 1}$$

Answer

$$\text{Given } \lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^2 - 1}$$

To find: the limit of the given equation when x tends to 1

Substituting x as 1 we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(\sqrt{5x-4} - \sqrt{x})(\sqrt{5x-4} + \sqrt{x})}{(x^2 - 1)(\sqrt{5x-4} + \sqrt{x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 1} \frac{(5x - 4 - x)}{(x^2 - 1)} \frac{1}{(\sqrt{5x-4} + \sqrt{x})}$$

$$= \lim_{x \rightarrow 1} \frac{4(x-1)}{(x-1)(x+1)} \frac{1}{(\sqrt{5x-4} + \sqrt{x})}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{4}{(x+1)} \frac{1}{(\sqrt{5x-4} + \sqrt{x})}$$

Now we can see that the indeterminate form is removed, so substituting x as 1

$$\text{We get } \lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^2 - 1} = \frac{4}{2(2)} = 1$$

12. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$$

Answer

$$\text{Given } \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{(1+x-1)}{x} \frac{1}{(\sqrt{1+x}+1)}$$

$$= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{1+x}+1)}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \rightarrow 0} \frac{1}{(\sqrt{1+x}+1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

$$\text{We get } \lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{1+1} = \frac{1}{2}$$

13. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{\sqrt{x^2+1}-\sqrt{5}}{x-2}$$

Answer

$$\text{Given } \lim_{x \rightarrow 2} \frac{\sqrt{x^2+1}-\sqrt{5}}{x-2}$$

To find: the limit of the given equation when x tends to 2

Substituting x as 2, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 2} \frac{\sqrt{x^2+1}-\sqrt{5}}{x-2} = \lim_{x \rightarrow 2} \frac{(\sqrt{x^2+1}-\sqrt{5})(\sqrt{x^2+1}+\sqrt{5})}{(x-2)(\sqrt{x^2+1}+\sqrt{5})}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 2} \frac{(x^2+1-5)}{x-2} \frac{1}{(\sqrt{x^2+1}+\sqrt{5})}$$

$$= \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{x-2} \frac{1}{(\sqrt{x^2+1}+\sqrt{5})}$$

$$= \lim_{x \rightarrow 2} \frac{(x+2)}{1} \frac{1}{(\sqrt{x^2+1}+\sqrt{5})}$$

Now we can see that the indeterminant form is removed, so substituting x as 2

$$\text{We get } \lim_{x \rightarrow 2} \frac{\sqrt{x^2+1}-\sqrt{5}}{x-2} = \frac{2+2}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$$

14. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$$

Answer

Given $\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$

To find: the limit of the given equation when x tends to 2

Substituting x as 2, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x}-\sqrt{2}} = \lim_{x \rightarrow 2} \frac{(x-2)}{(\sqrt{x}-\sqrt{2})} \frac{(\sqrt{x}+\sqrt{2})}{(\sqrt{x}+\sqrt{2})}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x}+\sqrt{2})}{(x-2) \cdot 1}$$

$$= \lim_{x \rightarrow 2} \frac{(\sqrt{x}+\sqrt{2})}{1}$$

Now we can see that the indeterminate form is removed, so substituting x as 2

We get $\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x}-\sqrt{2}} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

15. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 7} \frac{4 - \sqrt{9+x}}{1 - \sqrt{8-x}}$$

Answer

Given $\lim_{x \rightarrow 7} \frac{4 - \sqrt{9+x}}{1 - \sqrt{8-x}}$

To find: the limit of the given equation when x tends to 7

Substituting x as 7, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 7} \frac{(4 - \sqrt{9+x})(1 + \sqrt{8-x})(4 + \sqrt{9+x})}{(1 - \sqrt{8-x})(1 + \sqrt{8-x})(4 + \sqrt{9+x})}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 7} \frac{(16 - 9 - x)(1 + \sqrt{8-x})}{(1 - 8 + x)(4 + \sqrt{9+x})}$$

$$= \lim_{x \rightarrow 7} \frac{(7-x)(1 + \sqrt{8-x})}{(-7+x)(4 + \sqrt{9+x})}$$

$$\Rightarrow \lim_{x \rightarrow 7} \frac{4 - \sqrt{9+x}}{1 - \sqrt{8-x}} = \lim_{x \rightarrow 7} \frac{-(1 + \sqrt{8-x})}{(4 + \sqrt{9+x})}$$

Now we can see that the indeterminate form is removed, so substituting x as 7

We get $\lim_{x \rightarrow 7} \frac{4 - \sqrt{9+x}}{1 - \sqrt{8-x}} = \frac{-2}{8} = -\frac{1}{4}$

16. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{a+x} - \sqrt{a}}{x\sqrt{a^2+ax}}$$

Answer

Given $\lim_{x \rightarrow 0} \frac{\sqrt{a+x} - \sqrt{a}}{x\sqrt{a^2+ax}}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation,

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{a+x} - \sqrt{a}}{x\sqrt{a^2+ax}} = \lim_{x \rightarrow 0} \frac{(\sqrt{a+x} - \sqrt{a})(\sqrt{a+x} + \sqrt{a})}{(x\sqrt{a^2+ax})(\sqrt{a+x} + \sqrt{a})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{(a+x-a)}{(x\sqrt{a^2+ax})} \frac{1}{(\sqrt{a+x} + \sqrt{a})}$$

$$= \lim_{x \rightarrow 0} \frac{(x)}{(x\sqrt{a^2+ax})} \frac{1}{(\sqrt{a+x} + \sqrt{a})}$$

$$= \lim_{x \rightarrow 0} \frac{1}{(\sqrt{a^2+ax})(\sqrt{a+x} + \sqrt{a})}$$

Now we can see that the indeterminate form is removed, so substituting x as 0

$$\text{We get } \lim_{x \rightarrow 0} \frac{\sqrt{a+x} - \sqrt{a}}{x\sqrt{a^2+ax}} = \frac{1}{a(2\sqrt{a})} = \frac{1}{2a\sqrt{a}}$$

17. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 5} \frac{x-5}{\sqrt{6x-5} - \sqrt{4x+5}}$$

Answer

Given $\lim_{x \rightarrow 5} \frac{x-5}{\sqrt{6x-5} - \sqrt{4x+5}}$

To find: the limit of the given equation when x tends to 5

Substituting x as 5, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 5} \frac{(x-5)}{(\sqrt{6x-5} - \sqrt{4x+5})} \frac{(\sqrt{6x-5} + \sqrt{4x+5})}{(\sqrt{6x-5} + \sqrt{4x+5})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 5} \frac{(x-5)}{(6x-5-4x-5)} \frac{(\sqrt{6x-5} + \sqrt{4x+5})}{1}$$

$$= \lim_{x \rightarrow 5} \frac{(x-5) (\sqrt{6x-5} + \sqrt{4x+5})}{2(x-5) \cdot 1}$$

$$= \lim_{x \rightarrow 5} \frac{1 (\sqrt{6x-5} + \sqrt{4x+5})}{2 \cdot 1}$$

Now we can see that the indeterminant form is removed, so substituting x as 5

$$\text{We get } \lim_{x \rightarrow 5} \frac{x-5}{\sqrt{6x-5} - \sqrt{4x+5}} = \frac{\sqrt{25} + \sqrt{25}}{2} = \frac{10}{2} = 5$$

18. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^3 - 1}$$

Answer

$$\text{Given } \lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^3 - 1}$$

To find: the limit of the given equation when x tends to 1

Substituting x as 1, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 1} \frac{(\sqrt{5x-4} - \sqrt{x})(\sqrt{5x-4} + \sqrt{x})}{(x^3 - 1)(\sqrt{5x-4} + \sqrt{x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 1} \frac{(5x - 4 - x)}{(x^3 - 1)(\sqrt{5x-4} + \sqrt{x})}$$

$$= \lim_{x \rightarrow 1} \frac{4(x - 1)}{(x - 1)(x^2 + x + 1)(\sqrt{5x-4} + \sqrt{x})}$$

$$= \lim_{x \rightarrow 1} \frac{4}{(x^2 + x + 1)(\sqrt{5x-4} + \sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

$$\text{We get } \lim_{x \rightarrow 1} \frac{\sqrt{5x-4} - \sqrt{x}}{x^3 - 1} = \frac{4}{(1+1+1)(1+1)} = \frac{2}{3}$$

19. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{\sqrt{1+4x} - \sqrt{5+2x}}{x - 2}$$

Answer

$$\text{Given } \lim_{x \rightarrow 2} \frac{\sqrt{1+4x} - \sqrt{5+2x}}{x - 2}$$

To find: the limit of the given equation when x tends to 2

Substituting x as 2, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 2} \frac{(\sqrt{1+4x} - \sqrt{5+2x})}{(x-2)} \frac{(\sqrt{1+4x} + \sqrt{5+2x})}{(\sqrt{1+4x} + \sqrt{5+2x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 2} \frac{(1 + 4x - 5 - 2x)}{(x-2)} \frac{1}{(\sqrt{1+4x} + \sqrt{5+2x})}$$

$$= \lim_{x \rightarrow 2} \frac{2(x-2)}{(x-2)} \frac{1}{(\sqrt{1+4x} + \sqrt{5+2x})}$$

$$= \lim_{x \rightarrow 2} \frac{2}{1} \frac{1}{(\sqrt{1+4x} + \sqrt{5+2x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 2

$$\text{We get } \lim_{x \rightarrow 2} \frac{\sqrt{1+4x} - \sqrt{5+2x}}{x-2} = \frac{2}{(3+3)} = \frac{1}{3}$$

20. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1}$$

Answer

$$\text{Given } \lim_{x \rightarrow 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1}$$

To find: the limit of the given equation when x tends to 1

Substituting x as 1, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(\sqrt{3+x} - \sqrt{5-x})}{(x^2 - 1)} \frac{(\sqrt{3+x} + \sqrt{5-x})}{(\sqrt{3+x} + \sqrt{5-x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 1} \frac{(3 + x - 5 + x)}{(x^2 - 1)} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x \rightarrow 1} \frac{2(x-1)}{(x-1)(x+1)} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x \rightarrow 1} \frac{2}{(x+1)} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

$$\text{We get } \lim_{x \rightarrow 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1} = \frac{2}{(2)(2+2)} = \frac{1}{4}$$

21. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x}$$

Answer

Given $\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation,

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x^2} - \sqrt{1-x^2})(\sqrt{1+x^2} + \sqrt{1-x^2})}{x(\sqrt{1+x^2} + \sqrt{1-x^2})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{(1+x^2 - 1+x^2)}{x} \frac{1}{(\sqrt{1+x^2} + \sqrt{1-x^2})}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(2x^2)}{x} \frac{1}{(\sqrt{1+x^2} + \sqrt{1-x^2})}$$

$$= \lim_{x \rightarrow 0} \frac{(2x)}{1} \frac{1}{(\sqrt{1+x^2} + \sqrt{1-x^2})}$$

Now we can see that the indeterminate form is removed, so substituting x as 0

We get $\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x} = \frac{0}{2} = 0$

22. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - \sqrt{x+1}}{2x^2}$$

Answer

Given $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - \sqrt{x+1}}{2x^2}$

To find: the limit of the given equation when x tends to 0

Substituting x as 0, we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x+x^2} - \sqrt{x+1})(\sqrt{1+x+x^2} + \sqrt{x+1})}{2x^2(\sqrt{1+x+x^2} + \sqrt{x+1})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{(1+x+x^2 - x - 1)}{2x^2} \frac{1}{(\sqrt{1+x+x^2} + \sqrt{x+1})}$$

$$= \lim_{x \rightarrow 0} \frac{(x^2)}{2x^2} \frac{1}{(\sqrt{1+x+x^2} + \sqrt{x+1})}$$

$$= \lim_{x \rightarrow 0} \frac{(1)}{2} \frac{1}{(\sqrt{1+x+x^2} + \sqrt{x+1})}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

$$\text{We get } \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2}-\sqrt{x+1}}{2x^2} = \frac{1}{2(1+1)} = \frac{1}{4}$$

23. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{4 - x}$$

Answer

$$\text{Given } \lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{4 - x}$$

To find: the limit of the given equation when x tends to 4

Substituting x as 4, we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 4} \frac{(2 - \sqrt{x})(2 + \sqrt{x})}{(4 - x)(2 + \sqrt{x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 4} \frac{(4 - x)(1)}{(4 - x)(2 + \sqrt{x})}$$

$$= \lim_{x \rightarrow 4} \frac{1(1)}{1(2 + \sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 4

$$\text{We get } \lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{4 - x} = \frac{1}{2(\sqrt{4})} = \frac{1}{4}$$

24. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{x - a}{\sqrt{x} - \sqrt{a}}$$

Answer

$$\text{Given } \lim_{x \rightarrow a} \frac{x - a}{\sqrt{x} - \sqrt{a}}$$

To find: the limit of the given equation when x tends to a

Substituting x as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$\Rightarrow \lim_{x \rightarrow a} \frac{x - a}{\sqrt{x} - \sqrt{a}} = \lim_{x \rightarrow a} \frac{(x - a)(\sqrt{x} + \sqrt{a})}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow a} \frac{(x - a)(\sqrt{x} + \sqrt{a})}{(x - a)(1)}$$

$$= \lim_{x \rightarrow a} \frac{1(\sqrt{x} + \sqrt{a})}{1(1)}$$

Now we can see that the indeterminate form is removed, so substituting x as a

$$\text{We get } \lim_{x \rightarrow a} \frac{x-a}{\sqrt{x}-\sqrt{a}} = \sqrt{a} + \sqrt{a} = 2\sqrt{a}$$

25. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x}$$

Answer

$$\text{Given } \lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x}$$

To find: the limit of the given equation when x tends to 0

Substituting 0 as we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{1+3x} - \sqrt{1-3x})}{x} \frac{(\sqrt{1+3x} + \sqrt{1-3x})}{(\sqrt{1+3x} + \sqrt{1-3x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{(1+3x - 1+3x)}{x} \frac{1}{(\sqrt{1+3x} + \sqrt{1-3x})}$$

$$= \lim_{x \rightarrow 0} \frac{(6x)}{x} \frac{1}{(\sqrt{1+3x} + \sqrt{1-3x})}$$

$$= \lim_{x \rightarrow 0} \frac{(6)}{1} \frac{1}{(\sqrt{1+3x} + \sqrt{1-3x})}$$

Now we can see that the indeterminate form is removed, so substituting x as 0

$$\text{We get } \lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x} = \frac{6}{1+1} = 3$$

26. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{2-x} - \sqrt{2+x}}{x}$$

Answer

$$\text{Given } \lim_{x \rightarrow 0} \frac{\sqrt{2-x} - \sqrt{2+x}}{x}$$

To find: the limit of the given equation when x tends to 0

Substituting 0 as we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{2-x} - \sqrt{2+x})}{x} \frac{(\sqrt{2-x} + \sqrt{2+x})}{(\sqrt{2-x} + \sqrt{2+x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{(2 - x - 2 - x)}{x} \frac{1}{(\sqrt{2 - x} + \sqrt{2 + x})}$$

$$= \lim_{x \rightarrow 0} \frac{(-2x)}{x} \frac{1}{(\sqrt{2 - x} + \sqrt{2 + x})}$$

$$= \lim_{x \rightarrow 0} \frac{(-2)}{1} \frac{1}{(\sqrt{2 - x} + \sqrt{2 + x})}$$

Now we can see that the indeterminate form is removed, so substituting x as 0

We get $\lim_{x \rightarrow 0} \frac{\sqrt{2-x}-\sqrt{2+x}}{x} = \frac{-2}{\sqrt{2}+\sqrt{2}} = -\frac{1}{\sqrt{2}}$

27. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1}$$

Answer

Given $\lim_{x \rightarrow 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2 - 1}$

To find: the limit of the given equation when x tends to 1

Substituting 1 as we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 1} \frac{(\sqrt{3+x} - \sqrt{5-x})(\sqrt{3+x} + \sqrt{5-x})}{x^2 - 1} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 1} \frac{(3 + x - 5 + x)}{x^2 - 1} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x \rightarrow 1} \frac{2(x-1)}{(x-1)(x+1)} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

$$= \lim_{x \rightarrow 1} \frac{2}{(x+1)} \frac{1}{(\sqrt{3+x} + \sqrt{5-x})}$$

Now we can see that the indeterminate form is removed, so substituting x as 1

We get $\lim_{x \rightarrow 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1} = \frac{2}{2(2+2)} = \frac{1}{4}$

28. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$$

Answer

Given $\lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$

To find: the limit of the given equation when x tends to 1

Substituting 1 as we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)(\sqrt{x}+1)}{(3x^2+3x-6)(\sqrt{x}+1)}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 1} \frac{(2x-3)(x-1)}{(3x^2+3x-6)(\sqrt{x}+1)}$$

$$= \lim_{x \rightarrow 1} \frac{(2x-3)(x-1)}{3(x^2+x-2)(\sqrt{x}+1)}$$

$$= \lim_{x \rightarrow 1} \frac{(2x-3)(x-1)}{3(x-1)(x+2)(\sqrt{x}+1)}$$

$$= \lim_{x \rightarrow 1} \frac{(2x-3)}{3(x+2)(\sqrt{x}+1)}$$

Now we can see that the indeterminate form is removed, so substituting x as 1

$$\text{We get } \lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6} = \frac{2-3}{(3)(3)(2)} = \frac{-1}{18}$$

29. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1+x}}{\sqrt{1+x^3} - \sqrt{1+x}}$$

Answer

$$\text{Given } \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1+x}}{\sqrt{1+x^3} - \sqrt{1+x}}$$

To find: the limit of the given equation when x tends to 0

Substituting 0 as we get an indeterminate form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x^2} - \sqrt{1+x})(\sqrt{1+x^2} + \sqrt{1+x})(\sqrt{1+x^3} + \sqrt{1+x})}{(\sqrt{1+x^3} - \sqrt{1+x})(\sqrt{1+x^2} + \sqrt{1+x})(\sqrt{1+x^3} + \sqrt{1+x})}$$

Formula: $(a+b)(a-b) = a^2 - b^2$

$$= \lim_{x \rightarrow 0} \frac{(1+x^2-1-x)(1)(\sqrt{1+x^3} + \sqrt{1+x})}{(1+x^3-1-x)(\sqrt{1+x^2} + \sqrt{1+x})(1)}$$

$$= \lim_{x \rightarrow 0} \frac{(x^2-x)(1)(\sqrt{1+x^3} + \sqrt{1+x})}{(x^3-x)(\sqrt{1+x^2} + \sqrt{1+x})(1)}$$

$$= \lim_{x \rightarrow 0} \frac{x(x-1)(1)(\sqrt{1+x^3} + \sqrt{1+x})}{x(x^2-1)(\sqrt{1+x^2} + \sqrt{1+x})(1)}$$

$$= \lim_{x \rightarrow 0} \frac{(x-1)}{(x^2-1)} \frac{(1)}{(\sqrt{1+x^2} + \sqrt{1+x})} \frac{(\sqrt{1+x^2} + \sqrt{1+x})}{(1)}$$

Now we can see that the indeterminant form is removed, so substituting x as 0

$$\text{We get } \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1+x}}{\sqrt{1+x^2} - \sqrt{1+x}} = \frac{1+1}{1+1} = \frac{2}{2} = 1$$

30. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$$

Answer

$$\text{Given } \lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$$

To find: the limit of the given equation when x tends to 1

Substituting 1 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{x \rightarrow 1} \frac{(x^2 - \sqrt{x}) (\sqrt{x} + 1) (x^2 + \sqrt{x})}{(\sqrt{x} - 1) (\sqrt{x} + 1) (x^2 + \sqrt{x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow 1} \frac{(x^4 - x) (\sqrt{x} + 1) (1)}{(x - 1) (1) (x^2 + \sqrt{x})}$$

$$= \lim_{x \rightarrow 1} \frac{x(x^3 - 1) (\sqrt{x} + 1) (1)}{(x - 1) (1) (x^2 + \sqrt{x})}$$

$$= \lim_{x \rightarrow 1} \frac{x(x - 1)(x^2 + x + 1) (\sqrt{x} + 1) (1)}{(x - 1) (1) (x^2 + \sqrt{x})}$$

$$= \lim_{x \rightarrow 1} \frac{x(x^2 + x + 1) (\sqrt{x} + 1) (1)}{1 (1) (x^2 + \sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting x as 1

$$\text{We get } \lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1} = \frac{(3)(2)}{2} = 3$$

31. Question

Evaluate the following limits:

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}, x \neq 0$$

Answer

$$\text{Given } \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

To find: the limit of the given equation when h tends to 0

Substituting 0 as we get an indeterminant form of $\frac{0}{0}$

Rationalizing the given equation

$$= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{h \rightarrow 0} \frac{(x+h-x)}{h} \frac{(1)}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{(h)}{h} \frac{(1)}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{(1)}{1} \frac{(1)}{(\sqrt{x+h} + \sqrt{x})}$$

Now we can see that the indeterminant form is removed, so substituting h as 0

$$\text{We get } \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

32. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - (\sqrt{5} + \sqrt{2})}{x^2 - 10}$$

Answer

$$\text{Given } \lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - (\sqrt{5} + \sqrt{2})}{x^2 - 10}$$

To find: the limit of the given equation when x tends to $\sqrt{10}$

Re-writing the equation as

$$= \lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - \sqrt{(\sqrt{5} + \sqrt{2})^2}}{x^2 - 10}$$

$$= \lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - \sqrt{5+2+2\sqrt{10}}}{x^2 - 10}$$

$$= \lim_{x \rightarrow \sqrt{10}} \frac{\sqrt{7+2x} - \sqrt{7+2\sqrt{10}}}{x^2 - 10}$$

Now rationalizing the above equation

$$= \lim_{x \rightarrow \sqrt{10}} \frac{(\sqrt{7+2x} - \sqrt{7+2\sqrt{10}})(\sqrt{7+2x} + \sqrt{7+2\sqrt{10}})}{x^2 - 10 (\sqrt{7+2x} + \sqrt{7+2\sqrt{10}})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow \sqrt{10}} \frac{(7+2x - (7+2\sqrt{10}))}{x^2 - 10} \frac{(1)}{(\sqrt{7+2x} + \sqrt{7+2\sqrt{10}})}$$

$$= \lim_{x \rightarrow \sqrt{10}} \frac{(2x - 2\sqrt{10})}{x^2 - 10} \frac{(1)}{(\sqrt{7+2x} + \sqrt{7+2\sqrt{10}})}$$

$$= \lim_{x \rightarrow \sqrt{10}} \frac{2(x - \sqrt{10})}{(x + \sqrt{10})(x - \sqrt{10})} \frac{(1)}{(\sqrt{7 + 2x} + \sqrt{7 + 2\sqrt{10}})}$$

$$= \lim_{x \rightarrow \sqrt{10}} \frac{2(1)}{(x + \sqrt{10})(1)} \frac{(1)}{(\sqrt{7 + 2x} + \sqrt{7 + 2\sqrt{10}})}$$

Now we can see that the indeterminant form is removed, so substituting x as $\sqrt{10}$

$$= \frac{2}{2\sqrt{10}} \frac{1}{(2\sqrt{7 + 2\sqrt{10}})}$$

$$= \frac{1}{2\sqrt{10}} \frac{1}{(\sqrt{7 + 2\sqrt{10}})}$$

33. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5 + 2x} - (\sqrt{3} + \sqrt{2})}{x^2 - 6}$$

Answer

Given $\lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5 + 2x} - (\sqrt{3} + \sqrt{2})}{x^2 - 6}$

To find: the limit of the given equation when x tends to $\sqrt{6}$

Re-writing the equation as

$$= \lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5 + 2x} - \sqrt{(\sqrt{3} + \sqrt{2})^2}}{x^2 - 6}$$

$$= \lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5 + 2x} - \sqrt{3 + 2 + 2\sqrt{6}}}{x^2 - 6}$$

$$= \lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5 + 2x} - \sqrt{5 + 2\sqrt{6}}}{x^2 - 6}$$

Now rationalizing the above equation

$$= \lim_{x \rightarrow \sqrt{6}} \frac{(\sqrt{5 + 2x} - \sqrt{5 + 2\sqrt{6}})(\sqrt{5 + 2x} + \sqrt{5 + 2\sqrt{6}})}{x^2 - 6(\sqrt{5 + 2x} + \sqrt{5 + 2\sqrt{6}})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow \sqrt{6}} \frac{(5 + 2x - (5 + 2\sqrt{6}))}{x^2 - 6} \frac{(1)}{(\sqrt{5 + 2x} + \sqrt{5 + 2\sqrt{6}})}$$

$$= \lim_{x \rightarrow \sqrt{6}} \frac{(2x - 2\sqrt{6})}{x^2 - 6} \frac{(1)}{(\sqrt{5 + 2x} + \sqrt{5 + 2\sqrt{6}})}$$

$$= \lim_{x \rightarrow \sqrt{6}} \frac{2(x - \sqrt{6})}{(x + \sqrt{6})(x - \sqrt{6})} \frac{(1)}{(\sqrt{5 + 2x} + \sqrt{5 + 2\sqrt{6}})}$$

$$= \lim_{x \rightarrow \sqrt{6}} \frac{2(1)}{(x + \sqrt{6})(1)} \frac{(1)}{(\sqrt{5 + 2x} + \sqrt{5 + 2\sqrt{6}})}$$

Now we can see that the indeterminant form is removed, so substituting x as $\sqrt{6}$

$$\lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{5 + 2x} - (\sqrt{3} + \sqrt{2})}{x^2 - 6} = \frac{2}{2\sqrt{6}} \frac{1}{(2\sqrt{5 + 2\sqrt{6}})} = \frac{1}{2\sqrt{6}} \frac{1}{(\sqrt{5 + 2\sqrt{6}})}$$

34. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \sqrt{2}} \frac{\sqrt{3 + 2x} - (\sqrt{2} + 1)}{x^2 - 2}$$

Answer

Given $\lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{3 + 2x} - (\sqrt{2} + \sqrt{1})}{x^2 - 2}$

To find: the limit of the given equation when x tends to $\sqrt{2}$

Re-writing the equation as

$$= \lim_{x \rightarrow \sqrt{2}} \frac{\sqrt{3 + 2x} - \sqrt{(\sqrt{2} + \sqrt{1})^2}}{x^2 - 2}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{\sqrt{3 + 2x} - \sqrt{2 + 1 + 2\sqrt{2}}}{x^2 - 2}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{\sqrt{3 + 2x} - \sqrt{3 + 2\sqrt{2}}}{x^2 - 2}$$

Now rationalizing the above equation

$$= \lim_{x \rightarrow \sqrt{2}} \frac{(\sqrt{3 + 2x} - \sqrt{3 + 2\sqrt{2}}) (\sqrt{3 + 2x} + \sqrt{3 + 2\sqrt{2}})}{x^2 - 2 (\sqrt{3 + 2x} + \sqrt{3 + 2\sqrt{2}})}$$

Formula: $(a + b)(a - b) = a^2 - b^2$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{(3 + 2x - (3 + 2\sqrt{2}))}{x^2 - 2} \frac{(1)}{(\sqrt{3 + 2x} + \sqrt{3 + 2\sqrt{2}})}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{(2x - 2\sqrt{2})}{x^2 - 2} \frac{(1)}{(\sqrt{3 + 2x} + \sqrt{3 + 2\sqrt{2}})}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{2(x - \sqrt{2})}{(x + \sqrt{2})(x - \sqrt{2})} \frac{(1)}{(\sqrt{3 + 2x} + \sqrt{3 + 2\sqrt{2}})}$$

$$= \lim_{x \rightarrow \sqrt{2}} \frac{2(1)}{(x + \sqrt{2})(1)} \frac{(1)}{(\sqrt{3 + 2x} + \sqrt{3 + 2\sqrt{2}})}$$

Now we can see that the indeterminant form is removed, so substituting x as $\sqrt{2}$

$$\Rightarrow \lim_{x \rightarrow \sqrt{6}} \frac{\sqrt{3 + 2x} - (\sqrt{2} + \sqrt{1})}{x^2 - 2} = \frac{2}{2\sqrt{2}} \frac{1}{(2\sqrt{3 + 2\sqrt{2}})} = \frac{1}{2\sqrt{2}} \frac{1}{(\sqrt{3 + 2\sqrt{2}})}$$

Exercise 29.5

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$$

Answer

We need to find the limit for: $\lim_{x \rightarrow a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$$

As Z does not match exactly with the form as described above so we need to do some manipulations-

$$Z = \lim_{x \rightarrow a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a}$$

$$\Rightarrow Z = \lim_{x \rightarrow a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x+2 - (a+2)}$$

Let $x + 2 = y$ and $a+2 = k$

As $x \rightarrow a$; $y \rightarrow k$

$$\therefore Z = \lim_{y \rightarrow k} \frac{(y)^{5/2} - (k)^{5/2}}{y-k}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$$

$$\therefore Z = \frac{5}{2} k^{\frac{5}{2}-1} = \frac{5}{2} k^{\frac{3}{2}} = \frac{5}{2} (a+2)^{\frac{3}{2}}$$

$$\text{Hence, } \lim_{x \rightarrow a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x-a} = \frac{5}{2} (a+2)^{\frac{3}{2}}$$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$$

Answer

We need to find the limit for: $\lim_{x \rightarrow a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z does not match exactly with the form as described above so we need to do some manipulations-

$$Z = \lim_{x \rightarrow a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x - a}$$

$$\Rightarrow Z = \lim_{x \rightarrow a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x+2 - (a+2)}$$

Let $x + 2 = y$ and $a+2 = k$

As $x \rightarrow a$; $y \rightarrow k$

$$\therefore Z = \lim_{y \rightarrow k} \frac{(y)^{3/2} - (k)^{3/2}}{y - k}$$

Use the formula: $\lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$$\therefore Z = \frac{3}{2} k^{\frac{3}{2}-1} = \frac{3}{2} k^{\frac{1}{2}} = \frac{3}{2} (a+2)^{\frac{1}{2}}$$

$$\text{Hence, } \lim_{x \rightarrow a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x - a} = \frac{3}{2} \sqrt{a+2}$$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$$

Answer

We need to find the limit for: $\lim_{x \rightarrow a} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

As limit can be find out simply by putting $x = a$ because it is not taking indeterminate form(0/0) form, so we will be putting $x = a$

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$$

$$\Rightarrow Z = \frac{(1+a)^6 - 1}{(1+a)^2 - 1} = \frac{\{(1+a)^2\}^3 - 1}{(1+a)^2 - 1}$$

This can be further simplified using $a^3 - 1 = (a-1)(a^2 + a + 1)$

$$\Rightarrow Z = \frac{\{(1+a)^2 - 1\} \{(1+a)^4 + (1+a)^2 + 1\}}{(1+a)^2 - 1}$$

$$\Rightarrow Z = (1+a)^4 + (1+a)^2 + 1$$

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{x^{2/7} - a^{2/7}}{x - a}$$

Answer

We need to find the limit for: $\lim_{x \rightarrow a} \frac{x^{2/7} - a^{2/7}}{x - a}$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{x^{2/7} - a^{2/7}}{x - a}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z matches exactly with the form as described above so we don't need to do any manipulations-

$$Z = \lim_{x \rightarrow a} \frac{x^{2/7} - a^{2/7}}{x - a}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

$$\therefore Z = \frac{2}{7} a^{\frac{2}{7}-1} = \frac{2}{7} a^{-\frac{5}{7}}$$

$$\text{Hence, } \lim_{x \rightarrow a} \frac{x^{2/7} - a^{2/7}}{x - a} = \frac{2}{7} a^{-\frac{5}{7}}$$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{x^{5/7} - a^{5/7}}{x^{2/7} - a^{2/7}}$$

Answer

$$\text{We need to find the limit for: } \lim_{x \rightarrow a} \frac{x^{5/7} - a^{5/7}}{x^{2/7} - a^{2/7}}$$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{x^{5/7} - a^{5/7}}{x^{2/7} - a^{2/7}}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z does not match exactly with the form as described above so we need to do some manipulations-

$$Z = \lim_{x \rightarrow a} \frac{x^{5/7} - a^{5/7}}{x^{2/7} - a^{2/7}}$$

Dividing numerator and denominator by $(x-a)$, we get

$$Z = \lim_{x \rightarrow a} \frac{\frac{x^{5/7} - a^{5/7}}{x - a}}{\frac{x^{2/7} - a^{2/7}}{x - a}}$$

Using algebra of limits, we have -

$$Z = \frac{\lim_{x \rightarrow a} \frac{x^{5/7} - a^{5/7}}{x - a}}{\lim_{x \rightarrow a} \frac{x^{2/7} - a^{2/7}}{x - a}}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

$$\therefore Z = \frac{\frac{5}{2}a^{\frac{5}{7}-1}}{\frac{2}{7}a^{\frac{12}{7}-1}} = \frac{5a^{\frac{-2}{7}}}{2a^{\frac{-5}{7}}} = \frac{5}{2}a^{\frac{3}{7}}$$

$$\text{Hence, } \lim_{x \rightarrow a} \frac{\frac{5}{2}x^{\frac{5}{7}-1}}{\frac{2}{7}x^{\frac{12}{7}-1}} = \frac{5}{2}a^{\frac{3}{7}}$$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -1/2} \frac{8x^3 + 1}{2x + 1}$$

Answer

We need to find the limit for: $\lim_{x \rightarrow -1/2} \frac{8x^3 + 1}{2x + 1}$

As limit can't be find out simply by putting $x = (-1/2)$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow -\frac{1}{2}} \frac{8x^3 + 1}{2x + 1}$$

$$\Rightarrow Z = \lim_{x \rightarrow -\frac{1}{2}} \frac{(2x)^3 - (-1)^3}{2x - (-1)}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z matches exactly with the form as described above so we don't need to do any manipulations-

$$Z = \lim_{x \rightarrow -\frac{1}{2}} \frac{(2x)^3 - (-1)^3}{2x - (-1)}$$

$$\text{Let } y = 2x$$

$$\text{As } x \rightarrow -1/2 \Rightarrow 2x = y \rightarrow -1$$

$$\therefore Z = \lim_{y \rightarrow -1} \frac{y^3 - (-1)^3}{y - (-1)}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

$$\therefore Z = 3(-1)^{3-1} = 3(-1)^2 = 3$$

$$\text{Hence, } \lim_{x \rightarrow -\frac{1}{2}} \frac{8x^3 + 1}{2x + 1} = 3$$

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 27} \frac{(x^{1/3} + 3)(x^{1/3} - 3)}{x - 27}$$

Answer

$$\text{We need to find the limit for: } \lim_{x \rightarrow 27} \frac{(x^{1/3} + 3)(x^{1/3} - 3)}{x - 27}$$

As limit can't be find out simply by putting $x = 27$ because it is taking indeterminate form(0/0) form, so we

need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow 27} \frac{(x^{1/3}+3)(x^{1/3}-3)}{x-27}$$

Using algebra of limits, we have-

$$Z = \lim_{x \rightarrow 27} \left(x^{\frac{1}{3}} + 3 \right) \times \lim_{x \rightarrow 27} \frac{(x^{1/3}-3)}{x-27}$$

$$\Rightarrow Z = (27^{1/3} + 3) \times \lim_{x \rightarrow 27} \frac{(x^{1/3}-3)}{x-27}$$

$$\Rightarrow Z = 6 \lim_{x \rightarrow 27} \frac{(x^{1/3}-3)}{x-27}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$$

As Z matches exactly with the form as described above so we don't need to do any manipulations-

$$Z = 6 \lim_{x \rightarrow 27} \frac{x^{\frac{1}{3}} - (27)^{\frac{1}{3}}}{x-27}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$$

$$\therefore Z = 6 \times \frac{1}{3} (27)^{\frac{1}{3}-1} = 2 \times (27)^{-\frac{2}{3}} = 2 \times 3^{-2} = \frac{2}{9}$$

$$\text{Hence, } \lim_{x \rightarrow 27} \frac{(x^{1/3}+3)(x^{1/3}-3)}{x-27} = \frac{2}{9}$$

8. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 4} \frac{x^3 - 64}{x^2 - 16}$$

Answer

$$\text{We need to find the limit for: } \lim_{x \rightarrow 4} \frac{x^3 - 64}{x^2 - 16}$$

As limit can't be find out simply by putting $x = 4$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow 4} \frac{x^3 - 64}{x^2 - 16}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x-a} = na^{n-1}$$

As Z does not match exactly with the form as described above so we need to do some manipulations-

$$Z = \lim_{x \rightarrow 4} \frac{x^3 - 64}{x^2 - 16} = \lim_{x \rightarrow 4} \frac{x^3 - 4^3}{x^2 - 4^2}$$

Dividing numerator and denominator by $(x-4)$, we get

$$Z = \lim_{x \rightarrow 4} \frac{\frac{x^3 - 4^3}{x-4}}{\frac{x^2 - 4^2}{x-4}}$$

Using algebra of limits, we have -

$$Z = \frac{\lim_{x \rightarrow 4} \frac{x^3 - 4^3}{x - 4}}{\lim_{x \rightarrow 4} \frac{x^2 - 4^2}{x - 4}}$$

Use the formula: $\lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$$\therefore Z = \frac{3 \times (4)^{3-1}}{2 \times (4)^{2-1}} = \frac{3 \times 16}{2 \times 4} = 6$$

Hence, $\lim_{x \rightarrow 4} \frac{x^3 - 64}{x^2 - 16} = 6$

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1}$$

Answer

We need to find the limit for: $\lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1}$

As limit can't be find out simply by putting $x = 1$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

Let, $Z = \lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1}$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

Formula to be used: $\lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

As Z does not match exactly with the form as described above so we need to do some manipulations-

$$Z = \lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1} = \lim_{x \rightarrow 1} \frac{x^{15} - 1^{15}}{x^{10} - 1^{10}}$$

Dividing numerator and denominator by (x-1),we get

$$Z = \lim_{x \rightarrow 1} \frac{\frac{x^{15} - 1^{15}}{x - 1}}{\frac{x^{10} - 1^{10}}{x - 1}}$$

Using algebra of limits, we have -

$$Z = \frac{\lim_{x \rightarrow 1} \frac{x^{15} - 1^{15}}{x - 1}}{\lim_{x \rightarrow 1} \frac{x^{10} - 1^{10}}{x - 1}}$$

Use the formula: $\lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$$\therefore Z = \frac{15 \times (1)^{15-1}}{10 \times (1)^{10-1}} = \frac{15}{10} = \frac{3}{2}$$

Hence, $\lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1} = \frac{3}{2}$

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -1} \frac{x^3 + 1}{x + 1}$$

Answer

We need to find the limit for: $\lim_{x \rightarrow -1} \frac{x^3+1}{x+1}$

As limit can't be find out simply by putting $x = -1$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow -1} \frac{x^3+1}{x+1}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z does matches exactly with the form as described above so we don't need to do any manipulations-

$$Z = \lim_{x \rightarrow -1} \frac{x^3+1}{x+1} = \lim_{x \rightarrow -1} \frac{x^3 - (-1)^3}{x - (-1)}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

$$\therefore Z = 3(-1)^{3-1} = 3$$

$$\text{Hence, } \lim_{x \rightarrow -1} \frac{x^3+1}{x+1} = 3$$

11. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{x^{2/3} - a^{2/3}}{x^{3/4} - a^{3/4}}$$

Answer

$$\text{We need to find the limit for: } \lim_{x \rightarrow a} \frac{x^{2/3} - a^{2/3}}{x^{3/4} - a^{3/4}}$$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{x^{2/3} - a^{2/3}}{x^{3/4} - a^{3/4}}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z does not match exactly with the form as described above so we need to do some manipulations-

$$Z = \lim_{x \rightarrow a} \frac{x^{2/3} - a^{2/3}}{x^{3/4} - a^{3/4}}$$

Dividing numerator and denominator by $(x-a)$, we get

$$Z = \lim_{x \rightarrow a} \frac{\frac{x^{2/3} - a^{2/3}}{x - a}}{\frac{x^{3/4} - a^{3/4}}{x - a}}$$

Using algebra of limits, we have -

$$Z = \frac{\lim_{x \rightarrow a} \frac{x^{2/3} - a^{2/3}}{x - a}}{\lim_{x \rightarrow a} \frac{x^{3/4} - a^{3/4}}{x - a}}$$

Use the formula: $\lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$$\therefore Z = \frac{\frac{2}{3} \times (a)^{\frac{2}{3}-1}}{\frac{2}{4} \times (a)^{\frac{2}{4}-1}} = \frac{\frac{2}{3} (a)^{-\frac{1}{3}}}{\frac{2}{4} (a)^{-\frac{1}{4}}} = \frac{8}{9} (a)^{-\frac{1}{3} + \frac{1}{4}} = \frac{8}{9} a^{-\frac{1}{12}}$$

$$\text{Hence, } \lim_{x \rightarrow a} \frac{\frac{2}{3} \times \frac{2}{3} - \frac{2}{4} \times \frac{2}{4}}{\frac{2}{3} - \frac{2}{4}} = \frac{8}{9} a^{-\frac{1}{12}}$$

12. Question

If $\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = 108$, find the value of n.

Answer

Given,

$$\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = 108, \text{ we need to find value of } n$$

So we will first find the limit and then equate it with 108 to get the value of n.

$$\text{We need to find the limit for: } \lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3}$$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z matches exactly with the form as described above so we don't need to do any manipulations-

$$Z = \lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

$$\therefore Z = n(3)^{n-1}$$

According to question $Z = 108$

$$\therefore n(3)^{n-1} = 108$$

To solve such equations, factorize the number into prime factors and try to make combinations such that one satisfies with the equation.

$$\Rightarrow n(3)^{n-1} = 4 \times 27 = 4 \times (3)^{4-1}$$

Clearly on comparison we have -

$$n = 4$$

13. Question

if $\lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a} = 9$, find all possible values of a.

Answer

Given,

$$\lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a} = 9, \text{ we need to find value of } n$$

So we will first find the limit and then equate it with 9 to get the value of n.

$$\text{We need to find the limit for: } \lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a}$$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z matches exactly with the form as described above so we don't need to do any manipulations-

$$Z = \lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

$$\therefore Z = 9(a)^{9-1} = 9a^8$$

According to question $Z = 9$

$$\therefore 9(a)^8 = 9$$

$$\Rightarrow a^8 = 1 = 1^8 \text{ or } (-1)^8$$

Clearly on comparison we have -

$$a = 1 \text{ or } -1$$

14. Question

$$\text{If } \lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} = 405, \text{ find all possible values of } a.$$

Answer

Given,

$$\lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} = 405, \text{ we need to find value of } n$$

So we will first find the limit and then equate it with 405 to get the value of n.

$$\text{We need to find the limit for: } \lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a}$$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z matches exactly with the form as described above so we don't need to do any manipulations-

$$Z = \lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a}$$

Use the formula: $\lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$

$$\therefore Z = 5(a)^{5-1} = 5a^4$$

According to question $Z = 405$

$$\therefore 5(a)^4 = 405$$

$$\Rightarrow a^4 = 81 = 3^4 \text{ or } (-3)^4$$

Clearly on comparison we have -

$$a = 3 \text{ or } -3$$

15. Question

If $\lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a} = \lim_{x \rightarrow 5} (4 + x)$, find all possible values of a.

Answer

Given,

$$\lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a} = \lim_{x \rightarrow 5} (4 + x), \text{ we need to find value of } n$$

So we will first find the limit and then equate it with $\lim_{x \rightarrow 5} (4 + x)$ to get the value of n.

We need to find the limit for: $\lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a}$

As limit can't be find out simply by putting $x = a$ because it is taking indeterminate form(0/0) form, so we need to have a different approach.

$$\text{Let, } Z = \lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

As Z matches exactly with the form as described above so we don't need to do any manipulations-

$$Z = \lim_{x \rightarrow a} \frac{x^9 - a^9}{x - a}$$

$$\text{Use the formula: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

$$\therefore Z = 9(a)^{9-1} = 9a^8$$

$$\text{According to question } Z = \lim_{x \rightarrow 5} (4 + x) = 4 + 5 = 9$$

$$\therefore 9(a)^8 = 9$$

$$\Rightarrow a^8 = 1 = 1^8 \text{ or } (-1)^8$$

Clearly on comparison we have -

$$a = 1 \text{ or } -1$$

16. Question

If $\lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} = \lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1}$, find all possible values of a.

Answer

Given,

$$\lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} = \lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} = \lim_{x \rightarrow 1} \frac{x^4 - 1^4}{x - 1}$$

Note: To solve the problems of limit similar to one in our question we use the formula mentioned below which can be derived using binomial theorem.

$$\text{Formula to be used: } \lim_{x \rightarrow a} \frac{(x)^n - (a)^n}{x - a} = na^{n-1}$$

Using the formula we have -

$$3a^{3-1} = 4(1)^{4-1}$$

$$\Rightarrow 3a^2 = 4$$

$$\Rightarrow a^2 = 4/3$$

$$\therefore a = \pm (2/\sqrt{3})$$

Exercise 29.6

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \frac{(3x - 1)(4x - 2)}{(x + 8)(x - 1)}$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \frac{(3x-1)(4x-2)}{(x+8)(x-1)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(3x - 1)(4x - 2)}{(x + 8)(x - 1)} = \lim_{x \rightarrow \infty} \frac{(12x^2 - 10x + 2)}{(x^2 + 9x - 8)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(3x - 1)(4x - 2)}{(x + 8)(x - 1)} = \lim_{x \rightarrow \infty} \left(\frac{12 - \frac{10}{x} + \frac{2}{x^2}}{1 + \frac{9}{x} - \frac{8}{x^2}} \right)$$

$x \rightarrow \infty$ and $\frac{1}{x} \rightarrow 0$ then,

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(3x - 1)(4x - 2)}{(x + 8)(x - 1)} = \frac{12 - 0 + 0}{1}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \frac{(3x-1)(4x-2)}{(x+8)(x-1)} = 12$$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7}$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7} = \lim_{x \rightarrow \infty} \frac{3 - \frac{4}{x} + \frac{6}{x^2} - \frac{1}{x^3}}{2 + \frac{1}{x} - \frac{5}{x^2} + \frac{7}{x^3}}$$

Since, $x \rightarrow \infty$ and $\frac{1}{x} \rightarrow 0$ then

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7} = \frac{3 - 0 + 0 - 0}{2 + 0 - 0 + 0}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \frac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7} = \frac{3}{2}$$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}}$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}} = \lim_{x \rightarrow \infty} \frac{5 - \frac{6}{x^3}}{\sqrt{\frac{9}{x^6} + \frac{4x^6}{x^6}}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}} = \lim_{x \rightarrow \infty} \frac{\left(5 - \frac{6}{x^3}\right)}{\sqrt{\frac{9}{x^6} + 4}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}} = \frac{5}{\sqrt{4}}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \frac{5x^3 - 6}{\sqrt{9 + 4x^6}} = \frac{5}{2}$$

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x$$

Rationalizing the numerator we get,

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x = \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + cx} - x \right) \cdot \frac{\sqrt{x^2 + cx} + x}{\sqrt{x^2 + cx} + x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x = \lim_{x \rightarrow \infty} \frac{(x^2 + cx - x^2)}{\sqrt{x^2 + cx} + x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x = \lim_{x \rightarrow \infty} \frac{cx}{\sqrt{x^2 + cx} + x}$$

Taking x common from both numerator and denominator

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x = \lim_{x \rightarrow \infty} \frac{c}{\sqrt{1 + \frac{c}{x}} + 1}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x = \frac{c}{1 + 1}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \sqrt{x^2 + cx} - x = \frac{c}{2}$$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x}$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x}$$

On rationalizing the numerator we get,

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x} = \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x} \cdot \frac{\sqrt{x+1} + \sqrt{x}}{(\sqrt{x+1} + \sqrt{x})}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x} = \lim_{x \rightarrow \infty} \frac{(x+1-x)}{\sqrt{x+1} + \sqrt{x}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x} = \lim_{x \rightarrow \infty} \left(\frac{1}{\sqrt{x+1} + \sqrt{x}} \right)$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x} = \frac{1}{\infty}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x} = 0$$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x$$

On rationalizing the numerator we get,

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x = \lim_{x \rightarrow \infty} (\sqrt{x^2 + 7x} - x) \cdot \frac{\sqrt{x^2 + 7x} + x}{\sqrt{x^2 + 7x} + x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x = \lim_{x \rightarrow \infty} \frac{(x^2 + 7x - x^2)}{\sqrt{x^2 + 7x} + x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x = \lim_{x \rightarrow \infty} \frac{7x}{\sqrt{x^2 + 7x} + x}$$

Taking x common from both numerator and denominator

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x = \lim_{x \rightarrow \infty} \frac{7x}{\sqrt{\frac{x^2}{x^2} + \frac{7x}{x^2} + 1}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x = \lim_{x \rightarrow \infty} \frac{7x}{\sqrt{1 + \frac{7x}{x} + 1}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x = \frac{7}{\sqrt{1 + \frac{7}{x} + 1}}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \sqrt{x^2 + 7x} - x = \frac{7}{2}$$

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{4x^2 + 1} - 1}$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \frac{x}{\sqrt{4x^2 + 1} - 1}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x}{\sqrt{4x^2 + 1} - 1} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{4 + \frac{1}{x^2} - \frac{1}{x}}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x}{\sqrt{4x^2 + 1} - 1} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{4 + \frac{1}{\infty} - \frac{1}{\infty}}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x}{\sqrt{4x^2 + 1} - 1} = \frac{1}{\sqrt{4}}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \frac{x}{\sqrt{4x^2 + 1} - 1} = \frac{1}{2}$$

8. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \frac{n^2}{1 + 2 + 3 + \dots + n}$$

Answer

$$\text{Given: } \lim_{n \rightarrow \infty} \frac{n^2}{1 + 2 + 3 + \dots + n}$$

We know that,

$$1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$$

By putting this Formula, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2}{1 + 2 + 3 + \dots + n} = \lim_{n \rightarrow \infty} \frac{n^2}{\frac{1}{2}n(n + 1)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{n^2}{1+2+3+\dots+n} = \lim_{x \rightarrow \infty} \frac{2n^2}{n^2+n}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{n^2}{1+2+3+\dots+n} = 2 \cdot \lim_{x \rightarrow \infty} \frac{n^2}{n^2+n}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{n^2}{1+2+3+\dots+n} = 2 \cdot \lim_{x \rightarrow \infty} \frac{n^2}{n^2 \left(1 + \frac{1}{n}\right)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{n^2}{1+2+3+\dots+n} = 2 \cdot \frac{1}{1+0}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \frac{n^2}{1+2+3+\dots+n} = 2$$

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \frac{3x^{-1} + 4x^{-2}}{5x^{-1} + 6x^{-2}}$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \frac{3x^{-1} + 4x^{-2}}{5x^{-1} + 6x^{-2}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{3x^{-1} + 4x^{-2}}{5x^{-1} + 6x^{-2}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} \left(3 + \frac{4}{x}\right)}{\frac{1}{x} \left(5 + \frac{6}{x}\right)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{3x^{-1} + 4x^{-2}}{5x^{-1} + 6x^{-2}} = \lim_{x \rightarrow \infty} \frac{3 + 0}{5 + 0}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \frac{3x^{-1} + 4x^{-2}}{5x^{-1} + 6x^{-2}} = \frac{3}{5}$$

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2}}$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2}} \left[\frac{\infty}{\infty} \text{ form} \right]$$

Rationalizing the numerator and denominator we get,

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2}} \times \frac{\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2}}{\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{((x^2 + a^2) - (x^2 + b^2))}{(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2})\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2}}$$

$$\begin{aligned}
&\Rightarrow \lim_{x \rightarrow \infty} \frac{(a^2 - b^2)}{(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2})(\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2})} \\
&\Rightarrow \lim_{x \rightarrow \infty} \frac{(a^2 - b^2)(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2})}{(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2})(\sqrt{x^2 + c^2} - \sqrt{x^2 + d^2})(\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2})} \\
&\Rightarrow \lim_{x \rightarrow \infty} \frac{(a^2 - b^2)(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2})}{(x^2 + c^2 - x^2 - d^2)(\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2})} \\
&\Rightarrow \lim_{x \rightarrow \infty} \frac{(a^2 - b^2)(\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2})}{(c^2 - d^2)(\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2})} \\
&\Rightarrow \lim_{x \rightarrow \infty} \frac{(a^2 - b^2) \left(x\sqrt{1 + \frac{c^2}{x^2}} + x\sqrt{1 + \frac{d^2}{x^2}} \right)}{(c^2 - d^2) \left(x\sqrt{1 + \frac{a^2}{x^2}} + x\sqrt{1 + \frac{b^2}{x^2}} \right)} \\
&\Rightarrow \lim_{x \rightarrow \infty} \frac{(a^2 - b^2) \left(\sqrt{1 + \frac{c^2}{x^2}} + \sqrt{1 + \frac{d^2}{x^2}} \right)}{(c^2 - d^2) \left(\sqrt{1 + \frac{a^2}{x^2}} + \sqrt{1 + \frac{b^2}{x^2}} \right)} \\
&\Rightarrow \frac{(a^2 - b^2)(1 + 1)}{(c^2 - d^2)(1 + 1)}
\end{aligned}$$

Hence, $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + c^2} + \sqrt{x^2 + d^2}} = \frac{(a^2 - b^2)}{(c^2 - d^2)}$

11. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$$

Answer

Given: $\lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$

We know that,

$$(n+2)! = (n+2) \times (n+1)!$$

By putting the value of $(n+2)!$, we get

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!} = \lim_{x \rightarrow \infty} \frac{(n+2)(n+1)! + (n+1)!}{(n+2)(n+1)! - (n+1)!}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!} = \lim_{x \rightarrow \infty} \frac{(n+1)! [(n+2) + 1]}{(n+1)! [(n+2) - 1]}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!} = \lim_{x \rightarrow \infty} \frac{n+2+1}{n+2-1}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!} = \lim_{x \rightarrow \infty} \frac{n+3}{n+1}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n-1)!} = \lim_{x \rightarrow \infty} \frac{n \left(1 + \frac{3}{n}\right)}{n \left(1 + \frac{1}{n}\right)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n-1)!} = \lim_{x \rightarrow \infty} \frac{1 + \frac{3}{n}}{1 + \frac{1}{n}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n-1)!} = \frac{1+0}{1+0}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n-1)!} = 1$$

12. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\}$$

Answer

$$\text{Given: } \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\}$$

On Rationalizing the Numerator we get,

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} \\ = \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} \times \frac{x\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{x\sqrt{x^2 + 1} + \sqrt{x^2 - 1}} \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} = \lim_{x \rightarrow \infty} x \frac{x(x^2 + 1 - x^2 + 1)}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} = \lim_{x \rightarrow \infty} x \frac{2x}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} = \lim_{x \rightarrow \infty} \frac{2x^2}{x^2 \sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} = \lim_{x \rightarrow \infty} \frac{2x^2}{x^2 \sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} = \lim_{x \rightarrow \infty} \frac{2x^2}{x^2 \sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x^2}}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} = \frac{2}{1+1}$$

$$\text{Hence, } \lim_{x \rightarrow \infty} x \left\{ \sqrt{x^2 + 1} - \sqrt{x^2 - 1} \right\} = 1$$

13. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} x \left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2}$$

Answer

Given: $\lim_{x \rightarrow \infty} x \left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2}$

On Rationalizing the numerator we get,

$$\Rightarrow \lim_{x \rightarrow \infty} \left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2} = \lim_{x \rightarrow \infty} \frac{\left[\left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2} \left\{ \sqrt{x+1} + \sqrt{x} \right\} \right]}{\left\{ \sqrt{x+1} + \sqrt{x} \right\}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x \left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2} = \lim_{x \rightarrow \infty} \frac{(\sqrt{x+2})(x+1-x)}{\left\{ \sqrt{x+1} + \sqrt{x} \right\}}$$

Dividing the numerator and the denominator by $\sqrt{x}$, we get,

$$\Rightarrow \lim_{x \rightarrow \infty} \left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2} = \lim_{x \rightarrow \infty} \frac{\frac{(\sqrt{x+2})}{\sqrt{x}}}{\frac{\left\{ \sqrt{x+1} + \sqrt{x} \right\}}{\sqrt{x}}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2} = \lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{2}{x}}}{\sqrt{1 + \frac{1}{x}} + 1}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2} = \frac{1}{\sqrt{1} + 1}$$

Hence, $\lim_{x \rightarrow \infty} \left\{ \sqrt{x+1} - \sqrt{x} \right\} \sqrt{x+2} = \frac{1}{2}$

14. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3}$$

Answer

Given: $\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3}$

Formula Used:

$$\Rightarrow 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Now, Putting this formula and we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{6} \left[\frac{n(n+1)(2n+1)}{n^3} \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3} = \frac{1}{6} \lim_{n \rightarrow \infty} \left[\frac{(n^2 + n)(2n+1)}{n^3} \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3} = \frac{1}{6} \lim_{n \rightarrow \infty} \left[\frac{(2n^3 + n^2 + 2n^2 + n)}{n^3} \right]$$

Taking x^3 as common and we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3} = \frac{1}{6} \lim_{n \rightarrow \infty} \frac{n^3}{n^3} \left[\frac{\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)}{1} \right] \left(\frac{\infty}{\infty} \text{ form} \right)$$

Since, $n \rightarrow \infty$ and $\frac{1}{n} \rightarrow 0$ then,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3} = \frac{1}{6} \cdot \frac{2 + 0 + 0}{1} = \frac{1}{3}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + \dots + n^2}{n^3} = \frac{1}{3}$$

15. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right)$$

Answer

$$\text{Given: } \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right)$$

Taking LCM then, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right) = \lim_{n \rightarrow \infty} \left(\frac{1 + 2 + 3 + \dots + (n-1)}{n^2} \right)$$

Therefore,

$$\left[\frac{1 + 2 + 3 + \dots + (n-1)}{n^2} = \frac{(n-1)n}{2n^2} \right]$$

By putting this, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right) = \lim_{n \rightarrow \infty} \left(\frac{(n-1)(n)}{2n^2} \right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right) = \lim_{n \rightarrow \infty} \left(\frac{n^2 - n}{2n^2} \right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{n^2}{n^2} \left(\frac{1 - \frac{1}{n}}{2} \right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right) = \frac{1 - 0}{2} = \frac{1}{2}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n-1}{n^2} \right) = \frac{1}{2}$$

16. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4}$$

Answer

$$\text{Given: } \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4}$$

Here we know that,

$$\Rightarrow 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} = \lim_{n \rightarrow \infty} \frac{\left[\frac{1}{2} n(n+1) \right]^2}{n^4}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} = \lim_{n \rightarrow \infty} \frac{\frac{1}{4} n^2 (n+1)^2}{n^4}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} = \lim_{n \rightarrow \infty} \frac{1}{4} \cdot \frac{n^2 (n^2 + 1 + 2n)}{n^4}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} = \lim_{n \rightarrow \infty} \frac{1}{4} \cdot \frac{n^4 + n^2 + 2n}{n^4}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} = \lim_{n \rightarrow \infty} \frac{1}{4} \cdot \frac{n^4}{n^4} \left[1 + \frac{1}{n^2} + \frac{2}{n} \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} = \frac{1}{4} \lim_{n \rightarrow \infty} \left[1 + \frac{1}{n^2} + \frac{2}{n} \right]$$

Since, $n \rightarrow \infty$ and $\frac{1}{n} \rightarrow 0$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} = \frac{1}{4} [1 + 0 + 0]$$

Hence, $\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{n^4} = \frac{1}{4}$

17. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4}$$

Answer

Formula Used:

$$\Rightarrow 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Given: $\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4}$

By putting this, in the given equation, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4} = \lim_{n \rightarrow \infty} \frac{\left[\frac{1}{2} \cdot n \cdot (n+1) \right]^2}{(n-1)^4}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4} = \lim_{n \rightarrow \infty} \left[\frac{\frac{1}{4} n^2 (n^2 + 1 + 2n)}{(n-1)^4} \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4} = \frac{1}{4} \cdot \lim_{n \rightarrow \infty} \left[\frac{n^4 + n^2 + 2n^3}{(n-1)^2 (n-1)^2} \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4} = \frac{1}{4} \cdot \lim_{n \rightarrow \infty} \left[\frac{n^4 + n^2 + 2n^3}{(n^2 + 1 - 2n)(n^2 + 1 - 2n)} \right]$$

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4} \\ = \frac{1}{4} \cdot \lim_{n \rightarrow \infty} \left[\frac{n^4 + n^2 + 2n^3}{n^4 + n^2 - 2n^3 + n^2 + 1 - 2n - 2n^3 - 2n + 4n^2} \right] \end{aligned}$$

Taking x^4 as common,

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4} \\ = \frac{1}{4} \cdot \lim_{n \rightarrow \infty} \frac{n^4}{n^4} \left[\frac{\left(1 + \frac{1}{n^2} + \frac{2}{n}\right)}{1 + \frac{1}{n^2} - \frac{2}{n} + \frac{1}{n^2} + \frac{1}{n^4} - \frac{2}{n^3} - \frac{2}{n} - \frac{2}{n^3} + \frac{4}{n^2}} \right] \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4} = \frac{1}{4} \cdot \left(\frac{1}{1} \right) \end{aligned}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + \dots + n^3}{(n-1)^4} = \frac{1}{4}$$

18. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \}$$

Answer

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \} = \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$$

Now, Rationalizing the Numerator, we get,

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \} = \lim_{x \rightarrow \infty} \left[\sqrt{x^2 + x} - x \times \frac{\sqrt{x^2 + x} + x}{\sqrt{x^2 + x} + x} \right]$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \} = \lim_{x \rightarrow \infty} \left[\frac{(x^2 + x) - x^2}{\sqrt{x^2 + x} + x} \right]$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \} = \lim_{x \rightarrow \infty} \left[\frac{x}{\sqrt{x^2 + x} + x} \right]$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \} = \lim_{x \rightarrow \infty} \left[\frac{1}{\sqrt{\frac{x^2}{x^2} + \frac{x}{x^2}} + 1} \right]$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \} = \lim_{x \rightarrow \infty} \left[\frac{1}{\sqrt{1 + \frac{1}{x}} + 1} \right]$$

$$\Rightarrow \lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \} = \left[\frac{1}{1 + 1} \right]$$

$$\text{Hence, } \lim_{x \rightarrow \infty} \sqrt{x} \{ \sqrt{x+1} - \sqrt{x} \} = \frac{1}{2}$$

19. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} \right)$$

Answer

$$\lim_{n \rightarrow \infty} \left[\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} \right] \dots (1)$$

We can see that this is a geometric progression with the common ratio $1/3$.

And, we know the sum of n terms of GP is $S_n = a \left[\frac{1-r^n}{1-r} \right]$

Let suppose, $a = \frac{1}{3}$ and $r = \frac{1}{3}$, then

$$S_n = \frac{1}{3} \left[\frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}} \right]$$

$$= \frac{1}{3} \left[\frac{\left(1 - \frac{1}{3^n}\right)}{\frac{2}{3}} \right]$$

$$= \frac{1}{3} \times \frac{3}{2} \left[1 - \frac{1}{3^n} \right]$$

$$S_n = \frac{1}{2} \left[1 - \frac{1}{3^n} \right]$$

Now, putting the value of S_n in equation (1), we get

$$\Rightarrow \lim_{n \rightarrow \infty} \left[\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} \right] = \frac{1}{2} \lim_{n \rightarrow \infty} \left[1 - \frac{1}{3^n} \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left[\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} \right] = \frac{1}{2} (1 - 0)$$

$$\text{Hence, } \lim_{n \rightarrow \infty} \left[\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} \right] = \frac{1}{2}$$

20. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \frac{x^4 + 7x^3 + 46x + a}{x^4 + 6}, \text{ where } a \text{ is a non-zero real number.}$$

Answer

$$\text{Give: } \lim_{x \rightarrow \infty} \frac{x^4 + 7x^3 + 46x + a}{x^4 + 6}$$

Now, Taking x^4 as common from both numerator and denominator,

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x^4 + 7x^3 + 46x + a}{x^4 + 6} = \lim_{x \rightarrow \infty} \frac{x^4 \left[1 + \frac{7}{x} + \frac{46}{x^3} + \frac{a}{x^4} \right]}{x^4 \left[1 + \frac{6}{x^4} \right]}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x^4 + 7x^3 + 46x + a}{x^4 + 6} = \frac{1 + \frac{a}{0}}{1}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x^4 + 7x^3 + 46x + a}{x^4 + 6} = \frac{0 + a}{1}$$

Hence, $a = 1$

21. Question

Evaluate the following limits:

$f(x) = \frac{ax^2 + b}{x^2 + 1}$, $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow \infty} f(x) = 1$, then prove that $f(-2) = f(2) = 1$.

Answer

Given: $f(x) = \frac{ax^2 + b}{x^2 + 1}$, $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow \infty} f(x) = 1$

To Prove: $f(-2) = f(2) = 1$.

Proof: we have, $f(x) = \frac{ax^2 + b}{x^2 + 1}$

And, $\lim_{x \rightarrow 0} f(x) = 1$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{ax^2 + b}{x^2 + 1} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{ax^2 + b}{x^2 + 1} = \frac{\lim_{x \rightarrow 0} ax^2 + b}{\lim_{x \rightarrow 0} x^2 + 1}$$

Therefore, $b = 1$

Also, $\lim_{x \rightarrow \infty} f(x) = 1$

$$\Rightarrow \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{ax^2 + b}{x^2 + 1} = 1$$

$$\Rightarrow \lim_{x \rightarrow \infty} f(x) = \frac{\lim_{x \rightarrow \infty} ax^2 + b}{\lim_{x \rightarrow \infty} x^2 + 1}$$

$b = 1$

Thus, $f(x) = \frac{ax^2 + b}{x^2 + 1}$

On substituting the value of a and b we get,

$$f(x) = \frac{ax^2 + b}{x^2 + 1} = \frac{x^2 + 1}{x^2 + 1}$$

$$\Rightarrow \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2 + 1}{x^2 + 1}$$

So, $f(x) = 1$

Then, $f(-2) = 1$

Also, $f(2) = 1$

Hence, $f(2) = f(-2) = 1$

22. Question

Show that $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) \neq \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x)$

To Prove: $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) \neq \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x)$

Answer

$$\text{We have L.H.S} = \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x)$$

Rationalizing the numerator, we get,

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) = \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) \times \frac{\sqrt{x^2 + x + 1} + x}{\sqrt{x^2 + x + 1} + x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) = \lim_{x \rightarrow \infty} \frac{(x^2 + x + 1 - x^2)}{\sqrt{x^2 + x + 1} + x}$$

Taking x as common from both numerator and denominator,

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) = \lim_{x \rightarrow \infty} \frac{x(1 + \frac{1}{x})}{x \left[\sqrt{1 + \frac{1}{x} + \frac{1}{x^2} + 1} \right]}$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) = \lim_{x \rightarrow \infty} \frac{(1 + \frac{1}{x})}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2} + 1}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) = \frac{1}{1 + 1}$$

$$\text{Therefore, } \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) = \frac{1}{2}$$

$$\text{Now, Take R.H.S } \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x)$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow \infty} \sqrt{x^2 + 1} - x \times \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 1} - x)(\sqrt{x^2 + 1} + x)}{\sqrt{x^2 + 1} + x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow \infty} \frac{x^2 + 1 - x^2}{x \sqrt{1 + \frac{1}{x^2} + x}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow \infty} \frac{1}{x \sqrt{1 + \frac{1}{x^2} + 1}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \frac{1}{\sqrt{1 + \frac{1}{x^2} + 1}}$$

Now $x \rightarrow \infty$ and $\frac{1}{x} = 0$ then

Therefore, R.H.S = 0

So, L.H.S $\neq$ R.H.S

$$\text{Hence, } \lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) \neq \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x)$$

23. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -\infty} \left(\sqrt{4x^2 - 7x} + 2x \right)$$

Answer

Rationalizing the numerator, we get

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\sqrt{4x^2 - 7x} + 2x \right) = \lim_{x \rightarrow \infty} \left(\sqrt{4x^2 - 7x} + 2x \right) \times \frac{\sqrt{4x^2 - 7x} - 2x}{\sqrt{4x^2 - 7x} - 2x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\sqrt{4x^2 - 7x} + 2x \right) = \lim_{x \rightarrow \infty} \frac{4x^2 - 7x - 4x^2}{\sqrt{4x^2 - 7x} - 2x}$$

Taking x as common from both numerator and denominator,

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\sqrt{4x^2 - 7x} + 2x \right) = \lim_{x \rightarrow \infty} \frac{-7}{\sqrt{4 - \frac{7}{x} - \frac{1}{x}}}$$

Now $x \rightarrow \infty$ and $\frac{1}{x} = 0$ then

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\sqrt{4x^2 - 7x} + 2x \right) = -\frac{7}{1} = -7$$

Hence, $\lim_{x \rightarrow \infty} \left(\sqrt{4x^2 - 7x} + 2x \right) = -7$.

24. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -\infty} \left(\sqrt{x^2 - 8x} + x \right)$$

Answer

Rationalizing the numerator, we get

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\sqrt{x^2 - 8x} + x \right) = \lim_{x \rightarrow \infty} \left(\sqrt{x^2 - 8x} + x \right) \times \frac{\sqrt{x^2 - 8x} - x}{\sqrt{x^2 - 8x} - x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\sqrt{x^2 - 8x} + x \right) = \lim_{x \rightarrow \infty} \frac{(-8x)}{\sqrt{x^2 - 8x} - x}$$

Taking x as common from both numerator and denominator,

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\sqrt{x^2 - 8x} + x \right) = \lim_{x \rightarrow \infty} \frac{-8}{\sqrt{1 - \frac{8}{x} - \frac{1}{x}}}$$

Now $x \rightarrow \infty$ and $\frac{1}{x} = 0$ then

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\sqrt{x^2 - 8x} + x \right) = -\frac{8}{1} = -8$$

Hence, $\lim_{x \rightarrow \infty} \left(\sqrt{x^2 - 8x} + x \right) = -8$.

25. Question

Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1^4 + 2^4 + 3^4 + \dots + n^4}{n^5} - \lim_{\substack{n \rightarrow \infty \\ x \rightarrow \infty}} \frac{1^3 + 2^3 + \dots + n^3}{n^5}$$

Answer

Formula Used:

$$\Rightarrow 1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(1+n)(1+2n)(-1+3n+3n^2)}{30}$$

$$\Rightarrow 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Now putting these value, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\left(\frac{n(1+n)(1+2n)(-1+3n+3n^2)}{30} \right)}{n^5} - \lim_{n \rightarrow \infty} \frac{\left(\left(\frac{n(n+1)}{2} \right)^2 \right)}{n^5}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} + 1 \right) \left(\frac{1}{n} + 2 \right) \left(-\frac{1}{n^2} + \frac{3}{n} + 3 \right)}{30} - \lim_{n \rightarrow \infty} \frac{1}{n^5} \left(\frac{n^2(n^2 + 2n + 1)}{4} \right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} + 1 \right) \left(\frac{1}{n} + 2 \right) \left(-\frac{1}{n^2} + \frac{3}{n} + 3 \right)}{30} - \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3}}{4} \right)$$

Now $n \rightarrow \infty$ and $\frac{1}{n} = 0$ then,

$$= \frac{1 \times 2 \times 3}{30} - 0$$

$$= \frac{1}{5}$$

26. Question

Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n+1)}{n^3}$$

Answer

Here We know,

$$\Rightarrow 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

By putting these value, we get,

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n-1)}{n^3} \\ = \lim_{n \rightarrow \infty} \frac{\frac{(n(n+1)(2n+1))}{6} + \frac{n(n+1)}{2}}{n^3} \end{aligned}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n-1)}{n^3} \\ = \lim_{n \rightarrow \infty} \frac{(n(n+1)(2n+1) + 3n(n+1))}{6n^3}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n-1)}{n^3} = \lim_{n \rightarrow \infty} \frac{n(n+1) \left[\frac{(2n+1) + 3}{6} \right]}{n^3}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n-1)}{n^3} = \lim_{n \rightarrow \infty} \frac{n(n+1)(2n+4)}{6n^3}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n-1)}{n^3} = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right) \left(2 + \frac{4}{n}\right)}{6}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n-1)}{n^3} = \frac{1 \times 2}{6} = \frac{1}{3}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} \frac{1.2 + 2.3 + 3.4 + \dots + n(n-1)}{n^3} = \frac{1}{3}$$

Exercise 29.7

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$$

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{5x} \\ = \frac{1}{5} \lim_{x \rightarrow 0} \frac{\sin 3x}{x}$$

Multiplying and Dividing by 3:

$$= \frac{1}{5} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 \\ = \frac{3}{5} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}$$

As, $x \rightarrow 0 \Rightarrow 3x \rightarrow 0$

$$= \frac{3}{5} \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x}$$

Now, put $3x = y$

$$= \frac{3}{5} \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 3x}{5x} \\&= \frac{3}{5} \lim_{y \rightarrow 0} \frac{\sin y}{y} \\&= \frac{3}{5} \times 1 \\&= \frac{3}{5}\end{aligned}$$

Hence the value of $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x} = \frac{3}{5}$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x}$

We know, $1^\circ = \frac{\pi}{180}$ radians

$\therefore x^\circ = \frac{\pi x}{180}$ radians

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} \\&= \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{x}\end{aligned}$$

Multiplying and Dividing by $\frac{\pi}{180}$

$$= \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180} \times \frac{\pi}{180}}{x \times \frac{\pi}{180}}$$

$$= \frac{\pi}{180} \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}}$$

As, $x \rightarrow 0 \Rightarrow \frac{\pi x}{180} \rightarrow 0$

$$= \frac{\pi}{180} \lim_{\frac{\pi x}{180} \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}}$$

Now, put $\frac{\pi x}{180} = y$

$$= \frac{\pi}{180} \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} &= \frac{\pi}{180} \lim_{y \rightarrow 0} \frac{\sin y}{y} \\ &= \frac{\pi}{180} \times 1 \\ &= \frac{\pi}{180} \end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x^2}{\sin x^2}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{x^2}{\sin x^2}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^2}{\sin x^2} &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x^2}{x^2}} \end{aligned}$$

$$\text{As, } x \rightarrow 0 \Rightarrow x^2 \rightarrow 0$$

$$= \lim_{x^2 \rightarrow 0} \frac{1}{\frac{\sin x^2}{x^2}}$$

$$= \frac{1}{\lim_{x^2 \rightarrow 0} \frac{\sin x^2}{x^2}}$$

$$\text{Now, put } x^2 = y$$

$$= \frac{1}{\lim_{y \rightarrow 0} \frac{\sin y}{y}}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{x^2}{\sin x^2}$$

$$= \frac{1}{\lim_{y \rightarrow 0} \frac{\sin y}{y}}$$

$$= \frac{1}{1}$$

$$= 1$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{x^3}{\sin x^2} = 1$

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin x \cos x}{3x}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\sin x \cos x}{3x}$$

$$\lim_{x \rightarrow 0} \frac{\sin x \cos x}{3x}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x \cos x}{x}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) \cos x$$

We know,

$$\lim_{x \rightarrow 0} A(x) \cdot B(x) = \lim_{x \rightarrow 0} A(x) \times \lim_{x \rightarrow 0} B(x)$$

Therefore,

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} \times \lim_{x \rightarrow 0} \cos x$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin x \cos x}{3x}$$

$$= \frac{1}{3} \times 1 \times \cos 0$$

$$= \frac{1}{3} \times 1 \times 1$$

$$\{\because \cos 0 = 1\}$$

$$= \frac{1}{3}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin x \cos x}{3x} = \frac{1}{3}$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{3 \sin x - 4 \sin^3 x}{x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{3 \sin x - 4 \sin^3 x}{x}$

We know,

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{3 \sin x - 4 \sin^3 x}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 3x}{x}$$

Multiplying and Dividing by 3:

$$= \lim_{x \rightarrow 0} \frac{\sin 3x \times 3}{3x}$$

$$= 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}$$

As, $x \rightarrow 0 \Rightarrow 3x \rightarrow 0$

$$= 3 \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x}$$

Now, put $3x = y$

$$= 3 \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{3 \sin x - 4 \sin^3 x}{x}$$

$$= 3 \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

$$= 3 \times 1$$

$$= 3$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{3 \sin x - 4 \sin^3 x}{x} = 3$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\tan 8x}{\sin 2x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\tan 8x}{\sin 2x}$

$$\lim_{x \rightarrow 0} \frac{\tan 8x}{\sin 2x}$$

Multiplying and Dividing by $8x$ in numerator & Multiplying and Dividing by $2x$ in the denominator:

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan 8x}{8x} \times 8x}{\frac{\sin 2x}{2x} \times 2x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan 8x}{8x}}{\frac{\sin 2x}{2x}} \times \frac{8x}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan 8x}{8x}}{\frac{\sin 2x}{2x}} \times 4$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x)}{B(x)} = \frac{\lim_{x \rightarrow 0} A(x)}{\lim_{x \rightarrow 0} B(x)}$$

Therefore,

$$= 4 \times \frac{\lim_{x \rightarrow 0} \frac{\tan 8x}{8x}}{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x}}$$

As, $x \rightarrow 0 \Rightarrow 8x \rightarrow 0$ & $2x \rightarrow 0$

$$= 4 \times \frac{\lim_{8x \rightarrow 0} \frac{\tan 8x}{8x}}{\lim_{2x \rightarrow 0} \frac{\sin 2x}{2x}}$$

Now, put $2x = y$ and $8x = t$

$$= 4 \times \frac{\lim_{t \rightarrow 0} \frac{\tan t}{t}}{\lim_{y \rightarrow 0} \frac{\sin y}{y}}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \text{ \& \; } \lim_{t \rightarrow 0} \frac{\tan t}{t} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\tan 8x}{\sin 2x}$$

$$= 4 \times \frac{\lim_{t \rightarrow 0} \frac{\tan t}{t}}{\lim_{y \rightarrow 0} \frac{\sin y}{y}}$$

$$= 4 \times \frac{1}{1}$$

$$= 4$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\tan 8x}{\sin 2x} = 4$

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\tan mx}{\tan nx}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\tan mx}{\tan nx}$

$$\lim_{x \rightarrow 0} \frac{\tan mx}{\tan nx}$$

Multiplying and Dividing by mx in numerator & Multiplying and Dividing by nx in the denominator:

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan mx}{mx} \times mx}{\frac{\tan nx}{nx} \times nx}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan mx}{mx}}{\frac{\tan nx}{nx}} \times \frac{mx}{nx}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan mx}{mx}}{\frac{\tan nx}{nx}} \times \frac{m}{n}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x)}{B(x)} = \frac{\lim_{x \rightarrow 0} A(x)}{\lim_{x \rightarrow 0} B(x)}$$

Therefore,

$$= \frac{m}{n} \times \frac{\lim_{x \rightarrow 0} \frac{\tan mx}{mx}}{\lim_{x \rightarrow 0} \frac{\tan nx}{nx}}$$

As, $x \rightarrow 0 \Rightarrow mx \rightarrow 0$ & $nx \rightarrow 0$

$$= \frac{m}{n} \times \frac{\lim_{mx \rightarrow 0} \frac{\tan mx}{mx}}{\lim_{nx \rightarrow 0} \frac{\tan nx}{nx}}$$

Now, put $mx = y$ and $nx = t$

$$= \frac{m}{n} \times \frac{\lim_{y \rightarrow 0} \frac{\tan y}{y}}{\lim_{t \rightarrow 0} \frac{\tan t}{t}}$$

Formula used:

$$\lim_{t \rightarrow 0} \frac{\tan t}{t} = 1$$

Therefore,

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\tan mx}{\tan nx} \\
&= \frac{m}{n} \times \frac{\lim_{y \rightarrow 0} \frac{\tan y}{y}}{\lim_{t \rightarrow 0} \frac{\tan t}{t}} \\
&= \frac{m}{n} \times \frac{1}{1} \\
&= \frac{m}{n}
\end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\tan mx}{\tan nx} = \frac{m}{n}$

8. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 3x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 3x}$

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 3x}$$

Multiplying and Dividing by 5x in numerator & Multiplying and Dividing by 3x in the denominator:

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 5x}{5x} \times 5x}{\frac{\tan 3x}{3x} \times 3x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 5x}{5x}}{\frac{\tan 3x}{3x}} \times \frac{5x}{3x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 5x}{5x}}{\frac{\tan 3x}{3x}} \times \frac{5}{3}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x)}{B(x)} = \frac{\lim_{x \rightarrow 0} A(x)}{\lim_{x \rightarrow 0} B(x)}$$

Therefore,

$$= \frac{5}{3} \times \frac{\lim_{x \rightarrow 0} \frac{\sin 5x}{5x}}{\lim_{x \rightarrow 0} \frac{\tan 3x}{3x}}$$

As, $x \rightarrow 0 \Rightarrow 5x \rightarrow 0$ & $3x \rightarrow 0$

$$= \frac{5}{3} \times \frac{\lim_{5x \rightarrow 0} \frac{\sin 5x}{5x}}{\lim_{3x \rightarrow 0} \frac{\tan 3x}{3x}}$$

Now, put $5x = y$ and $3x = t$

$$= \frac{5}{3} \times \frac{\lim_{y \rightarrow 0} \frac{\sin y}{y}}{\lim_{t \rightarrow 0} \frac{\tan t}{t}}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \text{ \& } \lim_{t \rightarrow 0} \frac{\tan t}{t} = 1$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 3x} \\ &= \frac{5}{3} \times \frac{\lim_{y \rightarrow 0} \frac{\sin y}{y}}{\lim_{t \rightarrow 0} \frac{\tan t}{t}} \\ &= \frac{5}{3} \times \frac{1}{1} \\ &= \frac{5}{3} \end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 3x} = \frac{5}{3}$

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x^\circ}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x^\circ}$

We know, $1^\circ = \frac{\pi}{180}$ radians

$$\therefore x^\circ = \frac{\pi x}{180} \text{ radians}$$

$$\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x^\circ}$$

$$= \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}}$$

$$\text{As, } x \rightarrow 0 \Rightarrow \frac{\pi x}{180} \rightarrow 0$$

$$= \lim_{\frac{\pi x}{180} \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}}$$

Now, put $\frac{\pi x}{180} = y$

$$= \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x^\circ} \\ = \lim_{y \rightarrow 0} \frac{\sin y}{y} \end{aligned}$$

$$= 1$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x^\circ} = 1$

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$$

$$\lim_{x \rightarrow 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$$

Dividing numerator and denominator by x:

$$= \lim_{x \rightarrow 0} \frac{\frac{7x \cos x - 3 \sin x}{x}}{\frac{4x + \tan x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{7 \cos x - \frac{3 \sin x}{x}}{4 + \frac{\tan x}{x}}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \rightarrow 0} A(x) - \lim_{x \rightarrow 0} B(x)}{\lim_{x \rightarrow 0} C(x) + \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \rightarrow 0} 7 \cos x - \lim_{x \rightarrow 0} \frac{3 \sin x}{x}}{\lim_{x \rightarrow 0} 4 + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ \& } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

Therefore,

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x} \\
&= \frac{\lim_{x \rightarrow 0} 7 \cos x - 3 \lim_{x \rightarrow 0} \frac{\sin x}{x}}{\lim_{x \rightarrow 0} 4 + \lim_{x \rightarrow 0} \frac{\tan x}{x}} \\
&= \frac{7 \cos 0 - 3 \times 1}{4 + 1} \\
&\{\because \cos 0 = 1\} \\
&= \frac{7 - 3}{5} \\
&= \frac{4}{5}
\end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x} = \frac{4}{5}$

11. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx}$$

We know,

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

Therefore,

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} \\
&= \lim_{x \rightarrow 0} \frac{-2 \sin \frac{ax+bx}{2} \sin \frac{ax-bx}{2}}{-2 \sin \frac{cx+dx}{2} \sin \frac{cx-dx}{2}} \\
&= \lim_{x \rightarrow 0} \frac{\sin \frac{(a+b)x}{2} \sin \frac{(a-b)x}{2}}{\sin \frac{(c+d)x}{2} \sin \frac{(c-d)x}{2}}
\end{aligned}$$

Multiplying and Dividing by $\frac{(a+b)x}{2} \times \frac{(a-b)x}{2}$ in numerator &

similarly by $\frac{(c+d)x}{2} \times \frac{(c-d)x}{2}$ in denominator, we get,

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2}} \times \frac{(a+b)x}{2} \right) \left(\frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2}} \times \frac{(a-b)x}{2} \right)}{\left(\frac{\sin \frac{(c+d)x}{2}}{\frac{(c+d)x}{2}} \times \frac{(c+d)x}{2} \right) \left(\frac{\sin \frac{(c-d)x}{2}}{\frac{(c-d)x}{2}} \times \frac{(c-d)x}{2} \right)}
\end{aligned}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x) \times B(x)}{C(x) \times D(x)} = \frac{\lim_{x \rightarrow 0} A(x) \times \lim_{x \rightarrow 0} B(x)}{\lim_{x \rightarrow 0} C(x) \times \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$\begin{aligned} & \lim_{x \rightarrow 0} \left(\frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2}} \times \frac{(a+b)x}{2} \right) \times \lim_{x \rightarrow 0} \left(\frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2}} \times \frac{(a-b)x}{2} \right) \\ &= \frac{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{(c+d)x}{2}}{\frac{(c+d)x}{2}} \times \frac{(c+d)x}{2} \right) \times \lim_{x \rightarrow 0} \left(\frac{\sin \frac{(c-d)x}{2}}{\frac{(c-d)x}{2}} \times \frac{(c-d)x}{2} \right)}{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{(c+d)x}{2}}{\frac{(c+d)x}{2}} \times \frac{(c+d)x}{2} \right) \times \lim_{x \rightarrow 0} \left(\frac{\sin \frac{(c-d)x}{2}}{\frac{(c-d)x}{2}} \times \frac{(c-d)x}{2} \right)} \end{aligned}$$

$$\text{As, } x \rightarrow 0 \Rightarrow \frac{(a+b)x}{2} \rightarrow 0; \frac{(a-b)x}{2} \rightarrow 0; \frac{(c+d)x}{2} \rightarrow 0; \frac{(c-d)x}{2} \rightarrow 0$$

$$\begin{aligned} & \frac{\lim_{\frac{(a+b)x}{2} \rightarrow 0} \left(\frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2}} \times \frac{(a+b)x}{2} \right) \times \lim_{\frac{(a-b)x}{2} \rightarrow 0} \left(\frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2}} \times \frac{(a-b)x}{2} \right)}{\lim_{\frac{(c+d)x}{2} \rightarrow 0} \left(\frac{\sin \frac{(c+d)x}{2}}{\frac{(c+d)x}{2}} \times \frac{(c+d)x}{2} \right) \times \lim_{\frac{(c-d)x}{2} \rightarrow 0} \left(\frac{\sin \frac{(c-d)x}{2}}{\frac{(c-d)x}{2}} \times \frac{(c-d)x}{2} \right)} \\ &= \frac{\lim_{\frac{(a+b)x}{2} \rightarrow 0} \left(\frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2}} \times \frac{(a+b)x}{2} \right) \times \lim_{\frac{(a-b)x}{2} \rightarrow 0} \left(\frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2}} \times \frac{(a-b)x}{2} \right)}{\lim_{\frac{(c+d)x}{2} \rightarrow 0} \left(\frac{\sin \frac{(c+d)x}{2}}{\frac{(c+d)x}{2}} \times \frac{(c+d)x}{2} \right) \times \lim_{\frac{(c-d)x}{2} \rightarrow 0} \left(\frac{\sin \frac{(c-d)x}{2}}{\frac{(c-d)x}{2}} \times \frac{(c-d)x}{2} \right)} \end{aligned}$$

$$\text{Put } \frac{(a+b)x}{2} = m; \frac{(a-b)x}{2} = n; \frac{(c+d)x}{2} = k; \frac{(c-d)x}{2} = l$$

$$\begin{aligned} &= \frac{\lim_{m \rightarrow 0} \left(\frac{\sin m}{m} \times m \right) \times \lim_{n \rightarrow 0} \left(\frac{\sin n}{n} \times n \right)}{\lim_{k \rightarrow 0} \left(\frac{\sin k}{k} \times k \right) \times \lim_{l \rightarrow 0} \left(\frac{\sin l}{l} \times l \right)} \end{aligned}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Therefore,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} \\ &= \frac{\lim_{m \rightarrow 0} (1 \times m) \times \lim_{n \rightarrow 0} (1 \times n)}{\lim_{k \rightarrow 0} (1 \times k) \times \lim_{l \rightarrow 0} (1 \times l)} \end{aligned}$$

Now, put values of m, n, k and l:

$$\begin{aligned} &= \frac{\lim_{m \rightarrow 0} \left(\frac{(a+b)x}{2} \right) \times \lim_{n \rightarrow 0} \left(\frac{(a-b)x}{2} \right)}{\lim_{k \rightarrow 0} \left(\frac{(c+d)x}{2} \right) \times \lim_{l \rightarrow 0} \left(\frac{(c-d)x}{2} \right)} \\ &= \lim_{x \rightarrow 0} \frac{\left(\frac{(a+b)x}{2} \right) \left(\frac{(a-b)x}{2} \right)}{\left(\frac{(c+d)x}{2} \right) \left(\frac{(c-d)x}{2} \right)} \\ &= \lim_{x \rightarrow 0} \frac{(a+b)(a-b)}{(c+d)(c-d)} \end{aligned}$$

$$= \frac{(a+b)(a-b)}{(c+d)(c-d)}$$

$$= \frac{a^2 - b^2}{c^2 - d^2}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} = \frac{a^2 - b^2}{c^2 - d^2}$

12. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\tan^2 3x}{x^2}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\tan^2 3x}{x^2}$

$$\lim_{x \rightarrow 0} \frac{\tan^2 3x}{x^2}$$

$$= \lim_{x \rightarrow 0} \left(\frac{\tan 3x}{x} \right)^2$$

Multiplying and dividing by 3^2 :

$$= \lim_{x \rightarrow 0} \left(\frac{\tan 3x}{x} \right)^2 \times \frac{3^2}{3^2}$$

$$= \lim_{x \rightarrow 0} \left(\frac{\tan 3x}{3x} \right)^2 \times 3^2$$

Now, put $3x = y$

$$= 3^2 \times \lim_{y \rightarrow 0} \left(\frac{\tan y}{y} \right)^2$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\tan y}{y} = 1$$

Therefore,

$$= 3^2 \times \lim_{y \rightarrow 0} \left(\frac{\tan y}{y} \right)^2$$

$$= 9 \times 1$$

$$= 9$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\tan^2 3x}{x^2} = 9$

13. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{1 - \cos mx}{x^2}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{1 - \cos mx}{x^2}$$

We know,

$$\cos 2x = 1 - 2 \sin^2 x$$

$$\Rightarrow \cos mx = 1 - 2 \sin^2 \frac{mx}{2}$$

$$\Rightarrow 1 - \cos mx = 2 \sin^2 \frac{mx}{2}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos mx}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{mx}{2}}{x^2}$$

$$= 2 \times \lim_{x \rightarrow 0} \left(\frac{\sin \frac{mx}{2}}{x} \right)^2$$

Multiplying and dividing by $\left(\frac{m}{2}\right)^2$:

$$= 2 \times \lim_{x \rightarrow 0} \left(\frac{\sin \frac{mx}{2}}{x} \right)^2 \times \frac{\left(\frac{m}{2}\right)^2}{\left(\frac{m}{2}\right)^2}$$

$$= 2 \times \lim_{x \rightarrow 0} \left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right)^2 \times \left(\frac{m}{2}\right)^2$$

$$\text{As, } x \rightarrow 0 \Rightarrow \frac{mx}{2} \rightarrow 0$$

$$= 2 \times \lim_{\frac{mx}{2} \rightarrow 0} \left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right)^2 \times \frac{m^2}{4}$$

$$\text{Put } \frac{mx}{2} = y:$$

$$= \frac{2m^2}{4} \times \lim_{y \rightarrow 0} \left(\frac{\sin y}{y} \right)^2$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$= \frac{m^2}{2} \times \lim_{y \rightarrow 0} \left(\frac{\sin y}{y} \right)^2$$

$$= \frac{m^2}{2} \times 1$$

$$= \frac{m^2}{2}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{1 - \cos mx}{x^2} = \frac{m^2}{2}$

14. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$

$$\lim_{x \rightarrow 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$$

Dividing numerator and denominator by $6x$:

$$= \lim_{x \rightarrow 0} \frac{\frac{3 \sin 2x + 2x}{6x}}{\frac{3x + 2 \tan 3x}{6x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{3 \sin 2x}{6x} + \frac{2x}{6x}}{\frac{3x}{6x} + \frac{2 \tan 3x}{6x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{2x} + \frac{1}{3}}{\frac{1}{2} + \frac{\tan 3x}{3x}}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \rightarrow 0} A(x) - \lim_{x \rightarrow 0} B(x)}{\lim_{x \rightarrow 0} C(x) + \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} + \lim_{x \rightarrow 0} \frac{1}{3}}{\lim_{x \rightarrow 0} \frac{1}{2} + \lim_{x \rightarrow 0} \frac{\tan 3x}{3x}}$$

As, $x \rightarrow 0 \Rightarrow 2x \rightarrow 0$ & $3x \rightarrow 0$

$$= \frac{\lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} + \frac{1}{3}}{\frac{1}{2} + \lim_{3x \rightarrow 0} \frac{\tan 3x}{3x}}$$

Put $2x = y$ and $3x = k$;

$$= \frac{\lim_{y \rightarrow 0} \frac{\sin y}{y} + \frac{1}{3}}{\frac{1}{2} + \lim_{k \rightarrow 0} \frac{\tan k}{k}}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \text{ \& } \lim_{k \rightarrow 0} \frac{\tan k}{k} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$$

$$= \frac{\lim_{y \rightarrow 0} \frac{\sin y}{y} + \frac{1}{3}}{\frac{1}{2} + \lim_{k \rightarrow 0} \frac{\tan k}{k}}$$

$$= \frac{1 + \frac{1}{3}}{\frac{1}{2} + 1}$$

$$= \frac{\frac{3+1}{3}}{\frac{1+2}{2}}$$

$$= \frac{\frac{4}{3}}{\frac{3}{2}}$$

$$= \frac{4}{3} \times \frac{2}{3}$$

$$= \frac{8}{9}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x} = \frac{8}{9}$

15. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2}$$

We know,

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin \frac{3x+7x}{2} \sin \frac{7x-3x}{2}}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin \frac{10x}{2} \sin \frac{4x}{2}}{x^2}$$

$$= 2 \times \lim_{x \rightarrow 0} \frac{\sin 5x}{x} \times \frac{\sin 2x}{x}$$

Multiplying and dividing by 10:

$$= 2 \times 10 \times \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \times \frac{\sin 2x}{2x}$$

As,

$$x \rightarrow 0 \Rightarrow 2x \rightarrow 0 \text{ \& } 5x \rightarrow 0$$

$$\lim_{x \rightarrow 0} A(x) \times B(x) = \lim_{x \rightarrow 0} A(x) \times \lim_{x \rightarrow 0} B(x)$$

$$= 20 \times \lim_{5x \rightarrow 0} \frac{\sin 5x}{5x} \times \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x}$$

Put $2x = y$ and $5x = k$;

$$= 20 \times \lim_{k \rightarrow 0} \frac{\sin k}{k} \times \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2}$$

$$= 20 \times \lim_{k \rightarrow 0} \frac{\sin k}{k} \times \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

$$= 20 \times 1$$

$$= 20$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2} = 20$

16. Question

Evaluate the following limits:

$$\lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\tan 2\theta}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\sin 3\theta}{\tan 2\theta}$$

$$\lim_{x \rightarrow 0} \frac{\sin 3\theta}{\tan 2\theta}$$

Multiplying and Dividing by 3θ in numerator & Multiplying and Dividing by 2θ in the denominator:

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 3\theta}{3\theta} \times 3\theta}{\frac{\tan 2\theta}{2\theta} \times 2\theta}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 3\theta}{3\theta}}{\frac{\tan 2\theta}{2\theta}} \times \frac{3\theta}{2\theta}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 3\theta}{3\theta}}{\frac{\tan 2\theta}{2\theta}} \times \frac{3}{2}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x)}{B(x)} = \frac{\lim_{x \rightarrow 0} A(x)}{\lim_{x \rightarrow 0} B(x)}$$

Therefore,

$$= \frac{3}{2} \times \frac{\lim_{x \rightarrow 0} \frac{\sin 3\theta}{3\theta}}{\lim_{x \rightarrow 0} \frac{\tan 2\theta}{2\theta}}$$

As, $x \rightarrow 0 \Rightarrow 3\theta \rightarrow 0$ & $2\theta \rightarrow 0$

$$= \frac{3}{2} \times \frac{\lim_{3\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta}}{\lim_{2\theta \rightarrow 0} \frac{\tan 2\theta}{2\theta}}$$

Now, put $3\theta = y$ and $2\theta = t$

$$= \frac{3}{2} \times \frac{\lim_{y \rightarrow 0} \frac{\sin y}{y}}{\lim_{t \rightarrow 0} \frac{\tan t}{t}}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \text{ \& \; } \lim_{t \rightarrow 0} \frac{\tan t}{t} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin 3\theta}{\tan 2\theta}$$

$$= \frac{3}{2} \times \frac{\lim_{y \rightarrow 0} \frac{\sin y}{y}}{\lim_{t \rightarrow 0} \frac{\tan t}{t}}$$

$$= \frac{3}{2} \times \frac{1}{1}$$

$$= \frac{3}{2}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin 3\theta}{\tan 2\theta} = \frac{3}{2}$

17. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin x^2 (1 - \cos x^2)}{x^6}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\sin x^2 (1 - \cos x^2)}{x^6}$$

We know,

$$\cos 2x = 1 - 2 \sin^2 x$$

$$\Rightarrow \cos x^2 = 1 - 2 \sin^2 \frac{x^2}{2}$$

$$\Rightarrow 1 - \cos x^2 = 2 \sin^2 \frac{x^2}{2}$$

$$\lim_{x \rightarrow 0} \frac{\sin x^2 (1 - \cos x^2)}{x^6}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \times \frac{1 - \cos x^2}{x^4}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \times \frac{2 \sin^2 \frac{x^2}{2}}{x^4}$$

$$= 2 \times \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \times \left(\frac{\sin \frac{x^2}{2}}{\frac{x^2}{2}} \right)^2 \times \frac{1}{4}$$

$$= 2 \times \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \times \left(\frac{\sin \frac{x^2}{2}}{\frac{x^2}{2}} \right)^2 \times \frac{1}{4}$$

$$= \frac{2}{4} \times \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \times \left(\frac{\sin \frac{x^2}{2}}{\frac{x^2}{2}} \right)^2$$

$$\lim_{x \rightarrow 0} A(x) \times B(x) = \lim_{x \rightarrow 0} A(x) \times \lim_{x \rightarrow 0} B(x)$$

$$= \frac{1}{2} \times \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x^2}{2}}{\frac{x^2}{2}} \right)^2$$

$$\text{As, } x \rightarrow 0 \Rightarrow x^2 \rightarrow 0 \text{ \& } \frac{x^2}{2} \rightarrow 0$$

$$= \frac{1}{2} \times \lim_{x^2 \rightarrow 0} \frac{\sin x^2}{x^2} \times \left(\lim_{\frac{x^2}{2} \rightarrow 0} \frac{\sin \frac{x^2}{2}}{\frac{x^2}{2}} \right)^2$$

$$\text{Put } x^2 = y; \frac{x^2}{2} = t$$

$$= \frac{1}{2} \times \lim_{y \rightarrow 0} \frac{\sin y}{y} \times \left(\lim_{t \rightarrow 0} \frac{\sin t}{t} \right)^2$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\begin{aligned} &= \frac{1}{2} \times \lim_{y \rightarrow 0} \frac{\sin y}{y} \times \left(\lim_{t \rightarrow 0} \frac{\sin t}{t} \right)^2 \\ &= \frac{1}{2} \times 1 \\ &= \frac{1}{2} \end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin x^2 (1 - \cos x^2)}{x^6} = \frac{1}{2}$

18. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin^2 4x^2}{x^4}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\sin^2 4x^2}{x^4}$$

$$\lim_{x \rightarrow 0} \frac{\sin^2 4x^2}{x^4}$$

$$= \lim_{x \rightarrow 0} \left(\frac{\sin 4x^2}{x^2} \right)^2 \times \frac{16}{16}$$

$$= \lim_{x \rightarrow 0} \left(\frac{\sin 4x^2}{4x^2} \right)^2 \times 16$$

$$= 16 \times \lim_{x \rightarrow 0} \left(\frac{\sin 4x^2}{4x^2} \right)^2$$

$$\text{As, } x \rightarrow 0 \Rightarrow x^2 \rightarrow 0 \Rightarrow 4x^2 \rightarrow 0$$

$$= 16 \times \lim_{4x^2 \rightarrow 0} \left(\frac{\sin 4x^2}{4x^2} \right)^2$$

$$\text{Put } x^2 = y$$

$$= 16 \times \lim_{y \rightarrow 0} \left(\frac{\sin y}{y} \right)^2$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$= 16 \times (1)^2$$

$$= 16$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin^2 4x^2}{x^4} = 16$

19. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$

$$\lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$$

Dividing numerator and denominator by x:

$$= \lim_{x \rightarrow 0} \frac{\frac{x \cos x + 2 \sin x}{x}}{\frac{x^2 + \tan x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x + \frac{2 \sin x}{x}}{x + \frac{\tan x}{x}}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \rightarrow 0} A(x) - \lim_{x \rightarrow 0} B(x)}{\lim_{x \rightarrow 0} C(x) + \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \rightarrow 0} \cos x + \lim_{x \rightarrow 0} \frac{2 \sin x}{x}}{\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ \& } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$$

$$= \frac{\lim_{x \rightarrow 0} \cos x + 2 \lim_{x \rightarrow 0} \frac{\sin x}{x}}{\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

$$= \frac{\cos 0 + 2 \times 1}{0 + 1}$$

$$\{\because \cos 0 = 1\}$$

$$= \frac{1 + 2}{1}$$

$$= \frac{3}{1}$$

$$= 3$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x} = 3$

20. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{2x - \sin x}{\tan x + x}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{2x - \sin x}{\tan x + x}$$

$$\lim_{x \rightarrow 0} \frac{2x - \sin x}{\tan x + x}$$

Dividing numerator and denominator by x:

$$= \lim_{x \rightarrow 0} \frac{\frac{2x - \sin x}{x}}{\frac{\tan x + x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{2x}{x} - \frac{\sin x}{x}}{\frac{\tan x}{x} + \frac{x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{2 - \frac{\sin x}{x}}{\frac{\tan x}{x} + 1}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \rightarrow 0} A(x) - \lim_{x \rightarrow 0} B(x)}{\lim_{x \rightarrow 0} C(x) + \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \rightarrow 0} 2 - \lim_{x \rightarrow 0} \frac{\sin x}{x}}{\lim_{x \rightarrow 0} \frac{\tan x}{x} + \lim_{x \rightarrow 0} 1}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ \& } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{2x - \sin x}{\tan x + x}$$

$$= \frac{2 - \lim_{x \rightarrow 0} \frac{\sin x}{x}}{\lim_{x \rightarrow 0} \frac{\tan x}{x} + 1}$$

$$= \frac{2 - 1}{1 + 1}$$

$$= \frac{1}{2}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{2x - \sin x}{\tan x + x} = \frac{1}{2}$

21. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{5x \cos x + 3 \sin x}{3x^2 + \tan x}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{5x \cos x + 3 \sin x}{3x^2 + \tan x}$$

$$\lim_{x \rightarrow 0} \frac{5x \cos x + 3 \sin x}{3x^2 + \tan x}$$

Dividing numerator and denominator by x:

$$= \lim_{x \rightarrow 0} \frac{\frac{5x \cos x + 3 \sin x}{x}}{\frac{3x^2 + \tan x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{5 \cos x + \frac{3 \sin x}{x}}{3x + \frac{\tan x}{x}}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \rightarrow 0} A(x) - \lim_{x \rightarrow 0} B(x)}{\lim_{x \rightarrow 0} C(x) + \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$= \frac{\lim_{x \rightarrow 0} 5 \cos x + \lim_{x \rightarrow 0} \frac{3 \sin x}{x}}{\lim_{x \rightarrow 0} 3x + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ \& } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{5x \cos x + 3 \sin x}{3x^2 + \tan x}$$

$$= \frac{\lim_{x \rightarrow 0} 5 \cos x + 3 \lim_{x \rightarrow 0} \frac{\sin x}{x}}{\lim_{x \rightarrow 0} 3x + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

$$= \frac{5 \cos 0 + 3 \times 1}{3 \times 0 + 1}$$

$$\{\because \cos 0 = 1\}$$

$$= \frac{5 + 3}{0 + 1}$$

$$= \frac{8}{1}$$

$$= 8$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{5x \cos x + 3 \sin x}{3x^2 + \tan x} = 8$

22. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin 3x - \sin x}{\sin x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\sin 3x - \sin x}{\sin x}$

We know,

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin 3x - \sin x}{\sin x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \cos \frac{3x+x}{2} \sin \frac{3x-x}{2}}{\sin x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \cos \frac{4x}{2} \sin \frac{2x}{2}}{\sin x}$$

$$= 2 \times \lim_{x \rightarrow 0} \frac{\cos 2x \sin x}{\sin x}$$

$$= 2 \times \lim_{x \rightarrow 0} \cos 2x$$

$$= 2 \times \cos(2 \times 0)$$

$$= 2 \times \cos 0$$

$$\{\because \cos 0 = 1\}$$

$$= 2 \times 1$$

$$= 2$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin 3x - \sin x}{\sin x} = 2$

23. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin 5x - \sin 3x}{\sin x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\sin 5x - \sin 3x}{\sin x}$

We know,

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

Therefore,

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\sin 5x - \sin 3x}{\sin x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \cos \frac{5x+3x}{2} \sin \frac{5x-3x}{2}}{\sin x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \cos \frac{8x}{2} \sin \frac{2x}{2}}{\sin x} \\
 &= 2 \times \lim_{x \rightarrow 0} \frac{\cos 4x \sin x}{\sin x} \\
 &= 2 \times \lim_{x \rightarrow 0} \cos 4x \\
 &= 2 \times \cos(4 \times 0) \\
 &= 2 \times \cos 0 \\
 &\{\because \cos 0 = 1\} \\
 &= 2 \times 1 \\
 &= 2
 \end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin 5x - \sin 3x}{\sin x} = 2$

24. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 5x}{x^2}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 5x}{x^2}$

We know,

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}$$

Therefore,

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{\cos 3x - \cos 5x}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \frac{3x+5x}{2} \sin \frac{5x-3x}{2}}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sin \frac{8x}{2} \sin \frac{2x}{2}}{x^2} \\
 &= 2 \times \lim_{x \rightarrow 0} \frac{\sin 4x}{x} \times \frac{\sin x}{x}
 \end{aligned}$$

Multiplying and dividing by 10:

$$= 2 \times 4 \times \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \times \frac{\sin x}{x}$$

As,

$$X \rightarrow 0 \Rightarrow 4x \rightarrow 0$$

$$\lim_{x \rightarrow 0} A(x) \times B(x) = \lim_{x \rightarrow 0} A(x) \times \lim_{x \rightarrow 0} B(x)$$

$$= 8 \times \lim_{4x \rightarrow 0} \frac{\sin 4x}{4x} \times \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

Put $4x = k$;

$$= 8 \times \lim_{k \rightarrow 0} \frac{\sin k}{k} \times \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

Formula used:

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2}$$

$$= 8 \times \lim_{k \rightarrow 0} \frac{\sin k}{k} \times \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$= 8 \times 1$$

$$= 8$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 5x}{x^2} = 8$

25. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$$

$$\lim_{x \rightarrow 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$$

Dividing numerator and denominator by x :

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan 3x - 2x}{x}}{\frac{3x - \sin^2 x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan 3x}{x} - \frac{2x}{x}}{\frac{3x}{x} - \frac{\sin^2 x}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan 3x}{x} - 2}{3 - \frac{\sin^2 x}{x}}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x) - B(x)}{C(x) + D(x)} = \frac{\lim_{x \rightarrow 0} A(x) - \lim_{x \rightarrow 0} B(x)}{\lim_{x \rightarrow 0} C(x) + \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$\begin{aligned} &= \frac{\lim_{x \rightarrow 0} \frac{\tan 3x}{x} - \lim_{x \rightarrow 0} 2}{\lim_{x \rightarrow 0} 3 + \lim_{x \rightarrow 0} \frac{\sin^2 x}{x}} \\ &= \frac{\lim_{x \rightarrow 0} \left(\frac{\tan 3x}{3x} \right) \times 3 - \lim_{x \rightarrow 0} 2}{\lim_{x \rightarrow 0} 3 + \lim_{x \rightarrow 0} \left(\frac{\sin^2 x}{x^2} \right) \times x} \\ &= \frac{3 \lim_{x \rightarrow 0} \frac{\tan 3x}{3x} - 2}{3 + \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \times x} \end{aligned}$$

$$\text{As, } x \rightarrow 0 \Rightarrow 3x \rightarrow 0$$

$$= \frac{3 \lim_{3x \rightarrow 0} \frac{\tan 3x}{3x} - 2}{3 + \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \times x}$$

Put $3x = y$:

$$= \frac{3 \lim_{y \rightarrow 0} \frac{\tan y}{y} - 2}{3 + \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \times x}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ \& } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

Therefore,

$$\begin{aligned} &\lim_{x \rightarrow 0} \frac{\tan 3x - 2x}{3x - \sin^2 x} \\ &= \frac{3 \lim_{y \rightarrow 0} \frac{\tan y}{y} - 2}{3 + \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \times x} \\ &= \frac{3 - 2}{3 + \lim_{x \rightarrow 0} x} \\ &= \frac{3 - 2}{3 + 0} \\ &= \frac{1}{3} \end{aligned}$$

Hence , the value of $\lim_{x \rightarrow 0} \frac{\tan 3x - 2x}{3x - \sin^2 x} = \frac{1}{3}$

26. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x}$

We know,

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

Therefore,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{2 \cos \frac{2+x+2-x}{2} \sin \frac{2+x-(2-x)}{2}}{x} \\ &= \lim_{x \rightarrow 0} \frac{2 \cos \frac{4}{2} \sin \frac{2+x-2+x}{2}}{x} \\ &= \lim_{x \rightarrow 0} \frac{2 \cos \frac{4}{2} \sin \frac{2x}{2}}{x} \\ &= \lim_{x \rightarrow 0} \frac{2 \cos 2 \sin x}{x} \\ &= 2 \cos 2 \times \lim_{x \rightarrow 0} \frac{\sin x}{x} \end{aligned}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Therefore,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x} \\ &= 2 \cos 2 \times \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= 2 \cos 2 \times 1 \\ &= 2 \cos 2 \end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x} = 2 \cos 2$

27. Question

Evaluate the following limits:

$$\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$$

Answer

To find: $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

We know,

$$(a + b)^2 = a^2 + b^2 + 2ab$$

Therefore,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{(a + h)^2 \sin(a + h) - a^2 \sin a}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a^2 + h^2 + 2ah) \sin(a + h) - a^2 \sin a}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 \sin(a + h) + h^2 \sin(a + h) + 2ah \sin(a + h) - a^2 \sin a}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 \{\sin(a + h) - \sin a\}}{h} + \frac{h^2 \sin(a + h)}{h} + \frac{2ah \sin(a + h)}{h} \end{aligned}$$

Now,

$$\lim_{x \rightarrow 0} A(x) + B(x) + C(x) = \lim_{x \rightarrow 0} A(x) + \lim_{x \rightarrow 0} B(x) + \lim_{x \rightarrow 0} C(x) \text{ \&}$$

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}$$

We get,

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{a^2 \left\{ 2 \cos \frac{a + h + a}{2} \sin \frac{a + h - a}{2} \right\}}{h} + \lim_{h \rightarrow 0} \frac{h^2 \sin(a + h)}{h} + \lim_{h \rightarrow 0} \frac{2ah \sin(a + h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 \left\{ 2 \cos \frac{2a + h}{2} \sin \frac{h}{2} \right\}}{h} + \lim_{h \rightarrow 0} h \sin(a + h) + \lim_{h \rightarrow 0} 2a \sin(a + h) \\ &= \lim_{h \rightarrow 0} 2a^2 \cos \left(\frac{2a + h}{2} \right) \times \frac{\sin \frac{h}{2}}{2 \times \frac{h}{2}} + 0 \times \sin(a + 0) + 2a \sin(a + 0) \\ &= \lim_{h \rightarrow 0} a^2 \cos \left(\frac{2a + h}{2} \right) \times \frac{\sin \frac{h}{2}}{\frac{h}{2}} + 0 + 2a \sin a \end{aligned}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Therefore,

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{(a + h)^2 \sin(a + h) - a^2 \sin a}{h} \\ &= a^2 \cos \left(\frac{2a + 0}{2} \right) \times 1 + 2a \sin a \\ &= a^2 \cos \left(\frac{2a}{2} \right) + 2a \sin a \\ &= a^2 \cos a + 2a \sin a \end{aligned}$$

$$\text{Hence, the value of } \lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h} = a^2 \cos a + 2a \sin a$$

28. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin 3x - 3 \sin x}$$

Answer

To find: $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin 3x - 3 \sin x}$

We know,

$$\tan x = \frac{\sin x}{\cos x} \text{ \& sin 3x = 3 sin x - 4 sin}^3 \text{ x}$$

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin 3x - 3 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{3 \sin x - 4 \sin^3 x - 3 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x \left(\frac{1}{\cos x} - 1 \right)}{-4 \sin^3 x}$$

$$= -\frac{1}{4} \times \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} - 1}{\sin^2 x}$$

$$= -\frac{1}{4} \times \lim_{x \rightarrow 0} \frac{1 - \cos x}{\cos x (1 - \cos^2 x)}$$

$$\{\because \sin^2 x = 1 - \cos^2 x\}$$

$$= -\frac{1}{4} \times \lim_{x \rightarrow 0} \frac{(1 - \cos x)}{\cos x (1 - \cos x) (1 + \cos x)}$$

$$\{\because a^2 - b^2 = (a - b) (a+b)\}$$

$$= -\frac{1}{4} \times \lim_{x \rightarrow 0} \frac{1}{\cos x (1 + \cos x)}$$

$$= -\frac{1}{4} \times \frac{1}{\cos 0 (1 + \cos 0)}$$

$$\{\because \cos 0 = 1\}$$

$$= -\frac{1}{4} \times \frac{1}{(1 + 1)}$$

$$= -\frac{1}{4} \times \frac{1}{2}$$

$$= -\frac{1}{8}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin 3x - 3 \sin x} = -\frac{1}{8}$

29. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x}$$

We know,

$$\sec x = \frac{1}{\cos x}$$

$$\lim_{x \rightarrow 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{\cos 5x} - \frac{1}{\cos 3x}}{\frac{1}{\cos 3x} - \frac{1}{\cos x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\cos 3x - \cos 5x}{\cos 5x \cos 3x}}{\frac{\cos x - \cos 3x}{\cos 3x \cos x}}$$

$$= \lim_{x \rightarrow 0} \frac{\cos 3x - \cos 5x}{\cos x - \cos 3x} \times \frac{\cos 5x \cos 3x}{\cos 3x \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\cos 3x - \cos 5x}{\cos x - \cos 3x} \times \frac{\cos 5x}{\cos x}$$

We know,

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin \frac{3x+5x}{2} \sin \frac{5x-3x}{2}}{2 \sin \frac{x+3x}{2} \sin \frac{3x-x}{2}} \times \frac{\cos 5x}{\cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin \frac{8x}{2} \sin \frac{2x}{2}}{\sin \frac{4x}{2} \sin \frac{2x}{2}} \times \frac{\cos 5x}{\cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 4x \sin x}{\sin 2x \sin x} \times \frac{\cos 5x}{\cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 4x \cos 5x}{\sin 2x \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin 4x}{4x}\right) \times 4x \times \cos 5x}{\left(\frac{\sin 2x}{2x}\right) \times 2x \times \cos x}$$

$$= 2 \times \lim_{x \rightarrow 0} \frac{\left(\frac{\sin 4x}{4x}\right) \times \cos 5x}{\left(\frac{\sin 2x}{2x}\right) \times \cos x}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x) \times B(x)}{C(x) \times D(x)} = \frac{\lim_{x \rightarrow 0} A(x) \times \lim_{x \rightarrow 0} B(x)}{\lim_{x \rightarrow 0} C(x) \times \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$= 2 \times \frac{\lim_{x \rightarrow 0} \left(\frac{\sin 4x}{4x} \right) \times \lim_{x \rightarrow 0} \cos 5x}{\lim_{x \rightarrow 0} \left(\frac{\sin 2x}{2x} \right) \times \lim_{x \rightarrow 0} \cos x}$$

As, $x \rightarrow 0 \Rightarrow 3x \rightarrow 0$ & $4x \rightarrow 0$

$$= 2 \times \frac{\lim_{4x \rightarrow 0} \left(\frac{\sin 4x}{4x} \right) \times \lim_{x \rightarrow 0} \cos 5x}{\lim_{2x \rightarrow 0} \left(\frac{\sin 2x}{2x} \right) \times \lim_{x \rightarrow 0} \cos x}$$

Put $2x = y$ & $4x = t$:

$$= 2 \times \frac{\lim_{t \rightarrow 0} \left(\frac{\sin t}{t} \right) \times \lim_{x \rightarrow 0} \cos 5x}{\lim_{y \rightarrow 0} \left(\frac{\sin y}{y} \right) \times \lim_{x \rightarrow 0} \cos x}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Therefore,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x} \\ &= 2 \times \frac{\lim_{t \rightarrow 0} \left(\frac{\sin t}{t} \right) \times \lim_{x \rightarrow 0} \cos 5x}{\lim_{y \rightarrow 0} \left(\frac{\sin y}{y} \right) \times \lim_{x \rightarrow 0} \cos x} \\ &= 2 \times \frac{1 \times \cos(5 \times 0)}{1 \times \cos 0} \\ &= 2 \times \frac{1 \times \cos 0}{1 \times \cos 0} \\ &= 2 \end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x} = 2$

30. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x}$$

Answer

$$\text{To find: } \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x}$$

We know,

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\Rightarrow 2\sin^2 x = 1 - \cos 2x$$

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x} \\
&= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{2 \sin \frac{2x + 8x}{2} \sin \frac{8x - 2x}{2}} \\
&= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin \frac{10x}{2} \sin \frac{6x}{2}} \\
&= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin 5x \sin 3x}
\end{aligned}$$

Dividing numerator and denominator by x^2 :

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{\frac{\sin^2 x}{x^2}}{\frac{\sin 5x \sin 3x}{x^2}} \\
&= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin x}{x}\right)^2}{\frac{\sin 5x}{x} \times \frac{\sin 3x}{x}}
\end{aligned}$$

We know,

$$\lim_{x \rightarrow 0} \frac{A(x)}{C(x) \times D(x)} = \frac{\lim_{x \rightarrow 0} A(x)}{\lim_{x \rightarrow 0} C(x) \times \lim_{x \rightarrow 0} D(x)}$$

Therefore,

$$\begin{aligned}
&= \frac{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^2}{\lim_{x \rightarrow 0} \frac{\sin 5x}{x} \times \lim_{x \rightarrow 0} \frac{\sin 3x}{x}} \\
&= \frac{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^2}{\lim_{x \rightarrow 0} \left(\frac{\sin 5x}{5x}\right) \times 5 \times \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x}\right) \times 3} \\
&= \frac{1}{15} \times \frac{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^2}{\lim_{x \rightarrow 0} \left(\frac{\sin 5x}{5x}\right) \times \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x}\right)}
\end{aligned}$$

As, $x \rightarrow 0 \Rightarrow 3x \rightarrow 0$ & $5x \rightarrow 0$

$$= \frac{1}{15} \times \frac{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^2}{\lim_{5x \rightarrow 0} \left(\frac{\sin 5x}{5x}\right) \times \lim_{3x \rightarrow 0} \left(\frac{\sin 3x}{3x}\right)}$$

Put $3x = y$ & $5x = t$:

$$= \frac{1}{15} \times \frac{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^2}{\lim_{t \rightarrow 0} \left(\frac{\sin t}{t}\right) \times \lim_{y \rightarrow 0} \left(\frac{\sin y}{y}\right)}$$

Formula used:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Therefore,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x} \\ &= \frac{1}{15} \times \frac{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2}{\lim_{t \rightarrow 0} \left(\frac{\sin t}{t} \right) \times \lim_{y \rightarrow 0} \left(\frac{\sin y}{y} \right)} \\ &= \frac{1}{15} \times \frac{(1)^2}{1} \\ &= \frac{1}{15} \end{aligned}$$

Hence, the value of $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x} = \frac{1}{15}$

31. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$$

Now, $1 - \cos 2x = 2 \sin^2 x$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x + \tan^2 x}{x \sin x} \\ &= \frac{2 \lim_{x \rightarrow 0} \sin^2 x + \lim_{x \rightarrow 0} \tan^2 x}{\lim_{x \rightarrow 0} x \sin x} \\ &= \frac{2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 + \lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^2 \times x^2}{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) \times x^2} \\ &= \frac{(2 \times 1 \times x^2) + (1 \times x^2)}{(1 \times x^2)} \end{aligned}$$

Since, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x} = \frac{3x^2}{x^2}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x} = 3$$

Hence, $\lim_{x \rightarrow 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x} = 3$

32. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin(a+x) + \sin(a-x) - 2\sin a}{x \sin x}$$

Answer

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(a+x) + \sin(a-x) - 2\sin a}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin\left(\frac{a+x+a-x}{2}\right) \cos\left(\frac{a+x-a-x}{2}\right) - 2\sin a}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin a (\cos x - 1)}{x \sin x} \\ &= -2 \sin a \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x \left(2 \sin \frac{x}{2} \cos \frac{x}{2}\right)} \\ &= -2 \sin a \lim_{x \rightarrow 0} \frac{2 \sin \frac{x}{2}}{x \left(\cos \frac{x}{2}\right)} \\ &= -2 \sin a \lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{2} \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{\sin(a+x) + \sin(a-x) - 2\sin a}{x \sin x} = -2 \sin a \times 1 \times \frac{1}{2} \end{aligned}$$

$$\text{Since, } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin(a+x) + \sin(a-x) - 2\sin a}{x \sin x} = -\sin a$$

33. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x^2 - \tan 2x}{\tan x}$$

Answer

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^2 - \tan 2x}{\tan x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{x^2}{2x} - \frac{\tan 2x}{2x}}{\frac{\tan x}{x}} \\ &= \lim_{x \rightarrow 0} \frac{\left[\frac{x^2}{2x} - \frac{\tan 2x}{2x}\right] 2x}{\frac{\tan x}{x} x} \\ &= \lim_{x \rightarrow 0} \frac{\left[\frac{x^2}{2x} - \frac{\tan 2x}{2x}\right] 2}{\frac{\tan x}{x}} \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{x^2 - \tan 2x}{\tan x} = 2 \left[\frac{0 - 1}{1} \right] \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{x^2 - \tan 2x}{\tan x} = -2 \end{aligned}$$

34. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$$

Rationalize the numerator, we get $\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{\sqrt{2} + \sqrt{1 + \cos x}}{\sin^2 x} \times \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sqrt{2} - \sqrt{1 + \cos x}}$

$$= \lim_{x \rightarrow 0} \frac{2 - 1 - \cos x}{\sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{1 - \cos^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{(1 + \cos x)(1 - \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{1}{(1 + \cos x)}$$

$$= \frac{1}{1 + \cos 0}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \frac{1}{1 + 1}$$

Hence, $\lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \frac{1}{2}$

35. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{x \frac{\sin x}{\cos x}}{1 - \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{x \sin x}{\cos x (1 - \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{x \left(2 \sin \frac{x}{2} \cos \frac{x}{2} \right)}{\cos x \left(2 \sin^2 \frac{x}{2} \right)}$$

$$= \lim_{x \rightarrow 0} \frac{x \cos \frac{x}{2}}{\cos x \left(\sin \frac{x}{2} \right)}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\cos x \left(\frac{\tan x}{2} \right) \frac{1}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\cos x} \times \frac{1}{\lim_{x \rightarrow 0} \frac{\tan x}{2} \times \frac{1}{\frac{x}{2}}}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x} = 1 \times 2 \times 1$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x} = 2$$

36. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x^2 + 1 - \cos x}{x \sin x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{x^2 + 1 - \cos x}{x \sin x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^2 + 1 - \cos x}{x \sin x} = \lim_{x \rightarrow 0} \frac{x^2 + 2 \sin^2 \frac{x}{2}}{x \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \left[1 + 2 \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \right]}{x \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\left[1 + 2 \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4} \right]}{\frac{\sin x}{x}}$$

$$= \frac{\left[1 + 2 \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{1}{4} \right]}{\lim_{x \rightarrow 0} \frac{\sin x}{x}}$$

$$= \frac{1 + 2 \times 1 \times \frac{1}{4}}{1} = 1 + \frac{1}{2}$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{x^2 + 1 - \cos x}{x \sin x} = \frac{1 + 2 \times 1 \times \frac{1}{4}}{1} = \frac{3}{2}$$

37. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin 2x(\cos 3x - \cos x)}{x^3}$$

Answer

$$\lim_{x \rightarrow 0} \frac{\sin 2x(\cos 3x - \cos x)}{x^3}$$

$$\text{Since, } \cos a - \cos b = 2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

$$= \lim_{x \rightarrow 0} \frac{\sin 2x \left(-2 \sin\left(\frac{3x+x}{2}\right) \sin\left(\frac{3x-x}{2}\right) \right)}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 2x (-2 \sin 2x \sin x)}{x^3}$$

$$= \frac{-2 \lim_{x \rightarrow 0} \sin 2x \times \lim_{x \rightarrow 0} \sin 2x \times \lim_{x \rightarrow 0} \sin x}{x^3}$$

$$= -2 \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 \right) \times \left(2 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)$$

$$= -2(1 \times 2) \times 2 \times 1$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{\sin 2x(\cos 3x - \cos x)}{x^3} = -8$$

38. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{2 \sin x^0 - \sin 2x^0}{x^3}$$

Answer

$$\lim_{x \rightarrow 0} \frac{2 \sin x^0 - \sin 2x^0}{x^3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin x^0 - \sin 2x^0}{x^3} = \lim_{x \rightarrow 0} \frac{2 \sin \frac{\pi x}{180} - \sin \frac{2x\pi}{180}}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin \frac{\pi x}{180} - 2 \sin \frac{x\pi}{180} \cos \frac{\pi x}{180}}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin \frac{\pi x}{180} \left(1 - \cos \frac{\pi x}{180} \right)}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin \frac{\pi x}{180} \left(2 \sin^2 \frac{x\pi}{360} \right)}{x^3}$$

$$= 4 \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{180}}{x} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{360}}{x} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{360}}{x} \right)$$

$$= 4 \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{180}}{x \frac{\pi}{180}} \times \frac{\pi}{180} \right) \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{360}}{x \frac{\pi}{360}} \times \frac{\pi}{360} \right) \\ \times \left(\lim_{x \rightarrow 0} \frac{\sin \frac{x\pi}{360}}{x \frac{\pi}{360}} \times \frac{\pi}{360} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin x^0 - \sin 2x^0}{x^3} = 4 \times \frac{\pi}{180} \times \frac{\pi}{360} \times \frac{\pi}{360}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin x^0 - \sin 2x^0}{x^3} = \left(\frac{\pi}{180} \right)^3$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{2 \sin x^0 - \sin 2x^0}{x^3} = \left(\frac{\pi}{180} \right)^3$$

39. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x^3 \cot x}{1 - \cos x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{x^3 \cot x}{1 - \cos x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3 \cot x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{x^3 \frac{1}{\tan x}}{1 - \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{x^3}{\tan x (1 - \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{x^3}{\tan x (2 \sin^2 \frac{x}{2})}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\frac{\tan x}{x} \times \frac{2 \sin^2 \frac{x}{2}}{x^2}}$$

$$= \frac{1}{\lim_{x \rightarrow 0} \frac{\tan x}{x} \left[\lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right]^2} \times \frac{1}{4}$$

$$\text{Since, } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3 \cot x}{1 - \cos x} = \frac{1}{1 \times 2 \times \frac{1}{4}}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3 \cot x}{1 - \cos x} = 2$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{x^3 \cot x}{1 - \cos x} = 2$$

40. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos 2x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos 2x}$$

Since, $1 - \cos 2x = 2\sin^2 x$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos 2x} = \lim_{x \rightarrow 0} \frac{x \tan x}{2 \sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\tan x}{x}}{\frac{2 \sin^2 x}{x^2}}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{\tan x}{x}}{2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2}$$

Since, $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos 2x} = \frac{1}{2 \times 1}$$

Hence, $\lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos 2x} = \frac{1}{2}$

41. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin(3+x) - \sin(3-x)}{x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{\sin(3+x) - \sin(3-x)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \cos \left(\frac{3+x+3-x}{2} \right) \sin \left(\frac{3+x-3-x}{2} \right)}{x}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\cos \left(\frac{3+x+3-x}{2} \right) \sin \left(\frac{3+x-3-x}{2} \right)}{x}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\cos 3 \cdot \sin x}{x}$$

$$= 2 \cos 3 \cdot \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$= 2 \cos 3$$

Hence, $\lim_{x \rightarrow 0} \frac{\sin(3+x) - \sin(3-x)}{x} = 2 \cos 3$

42. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

Answer

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

We know that, $\cos 2x = 1 - 2\sin^2 x$

Therefore,

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 - 2\sin^2 x - 1}{\cos x - 1}$$

$$= \lim_{x \rightarrow 0} \frac{(-2\sin^2 x)}{\cos x - 1}$$

$$= \lim_{x \rightarrow 0} \left(-\frac{2(1 - \cos^2 x)}{\cos x - 1} \right)$$

$$[\cos^2 x - 1 = (\cos x + 1)(\cos x - 1)]$$

$$= \lim_{x \rightarrow 0} 2(1 + \cos x)$$

$$= 2(1 + 0)$$

$$= 2$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} = 2$$

43. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2}$$

Answer

$$\lim_{x \rightarrow 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2} = \lim_{x \rightarrow 0} \frac{3\sin^2 x}{3x^2} - \lim_{x \rightarrow 0} \frac{2\sin x^2}{3x^2}$$

$$= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 - \frac{2}{3} \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2$$

$$\text{Since, } \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1$$

$$= 1 - \frac{2}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2} = \frac{1}{3}$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{3\sin^2 x - 2\sin x^2}{3x^2} = \frac{1}{3}$$

44. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x}$$

Answer

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x} \times \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}} \\ &= \lim_{x \rightarrow 0} \frac{(1 + \sin x) - (1 - \sin x)}{x(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin x}{x(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})} \\ &= 2 \cdot \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\lim_{x \rightarrow 0} (\sqrt{1 + \sin x} + \sqrt{1 - \sin x})} \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x} = 2 \times 1 \times \frac{1}{2} \end{aligned}$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x} = 1$$

45. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2}$$

Answer

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 2x}{x^2} \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} = 2 \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{x} \right)^2 \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} = 2 \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{2x} \right)^2 \times 2^2 \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} = 2 \times 1 \times 4 \end{aligned}$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} = 8$$

46. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x} = \lim_{x \rightarrow 0} \frac{\cos x + \frac{\sin x}{x}}{x + \frac{\tan x}{x}}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x} = \frac{\lim_{x \rightarrow 0} \cos x + \lim_{x \rightarrow 0} \frac{\sin x}{x}}{\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x} = \frac{1 + 1}{0 + 1}$$

Hence, $\lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x} = 2$

47. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{3 \tan^2 x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{3 \tan^2 x}$$

Since, $1 - \cos 2x = 2 \sin^2 x$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{3 \tan^2 x} = \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{3 \tan^2 x}$$

$$= \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{2}{3} \lim_{x \rightarrow 0} \cos^2 x$$

$$= \frac{2}{3} \lim_{x \rightarrow 0} \cos^2 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{3 \tan^2 x} = \frac{2}{3}$$

Hence, $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{3 \tan^2 x} = \frac{2}{3}$

48. Question

Evaluate the following limits:

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$$

Answer

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$$

$$\Rightarrow \lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta \rightarrow 0} \frac{2 \sin^2 2\theta}{2 \sin^2 3\theta}$$

$$\begin{aligned}
&= \lim_{\theta \rightarrow 0} \frac{(\sin 2\theta)^2}{(\sin 3\theta)^2} \\
&= \frac{\lim_{\theta \rightarrow 0} \left(\frac{\sin 2\theta}{2\theta}\right)^2 \times 4\theta^2}{\lim_{\theta \rightarrow 0} \left(\frac{\sin 3\theta}{3\theta}\right)^2 \times 9\theta^2} \\
&= \frac{1^2 \times 4\theta^2}{1 \times 9\theta^2} \\
&= \frac{4}{9}
\end{aligned}$$

Hence, $\lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta} = \frac{4}{9}$

49. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$$

Answer

$$\begin{aligned}
&\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} \\
&\Rightarrow \lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} = \lim_{x \rightarrow 0} \frac{a + \cos x}{\frac{b \sin x}{x}} \\
&\Rightarrow \lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} = \frac{\lim_{x \rightarrow 0} (a + \cos x)}{\lim_{x \rightarrow 0} \frac{b \sin x}{x}} \\
&\Rightarrow \lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} = \frac{a + 1}{b}
\end{aligned}$$

Hence, $\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} = \frac{a + 1}{b}$

50. Question

Evaluate the following limits:

$$\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\tan 3\theta}$$

Answer

$$\begin{aligned}
&\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\tan 3\theta} \\
&\Rightarrow \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\tan 3\theta} = \frac{\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{4\theta} \times 4\theta}{\lim_{\theta \rightarrow 0} \frac{\tan 3\theta}{3\theta} \times 3\theta} \\
&\Rightarrow \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\tan 3\theta} = \frac{1 \times 4\theta}{1 \times 3\theta} \\
&\Rightarrow \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\tan 3\theta} = \frac{4}{3}
\end{aligned}$$

Hence, $\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\tan 3\theta} = \frac{4}{3}$

51. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3}$$

Answer

$$\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3}$$

Since, $\sin 2x = 2 \sin x \cdot \cos x$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} = \lim_{x \rightarrow 0} \frac{2 \sin x - (2 \sin x \cdot \cos x)}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin x (1 - \cos x)}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin x (1 - \cos x)}{x^3} \times \frac{(1 + \cos x)}{(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin x (1 - \cos x)^2}{x^3 ((1 + \cos x))}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin x (\sin^2 x)}{x^3 ((1 + \cos x))}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^3 x}{x^3 ((1 + \cos x))}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin^3 x}{x^3 ((1 + \cos x))}$$

$$= 2 \lim_{x \rightarrow 0} \left(\frac{\sin}{x} \right)^3 \times \lim_{x \rightarrow 0} \frac{1}{(1 + \cos x)}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} = 2 \times 1 \times \frac{1}{2}$$

Hence, $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} = 1$

52. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{1 - \cos 5x}{1 - \cos 6x}$$

Answer

$$\lim_{x \rightarrow 0} \frac{1 - \cos 5x}{1 - \cos 6x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{5x}{2}}{2 \sin^2 \frac{3x}{2}}$$

$$\begin{aligned}
& \lim_{x \rightarrow 0} \left(\frac{\sin \frac{5x}{2}}{\frac{5x}{2}} \right)^2 \times \frac{25}{4} x^2 \\
&= \frac{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{3x}{2}}{\frac{3x}{2}} \right)^2 \times \frac{9}{4} x^2}{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{3x}{2}}{\frac{3x}{2}} \right)^2 \times \frac{9}{4} x^2} \\
&= \frac{2 \times \frac{25}{4} x^2}{2 \times 1 \times 9x^2} \\
&\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 5x}{1 - \cos 6x} = \frac{25}{4 \times 9}
\end{aligned}$$

Hence, $\lim_{x \rightarrow 0} \frac{1 - \cos 5x}{1 - \cos 6x} = \frac{25}{36}$

53. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\operatorname{cosec} x - \cot x}{x}$$

Answer

Given, $\lim_{x \rightarrow 0} \frac{\operatorname{cose} x - \cot x}{x}$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\operatorname{cose} x - \cot x}{x} = \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) \times \frac{1}{x}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} \left(\frac{1 - \cos x}{x} \right) \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} \left(\frac{2 \sin^2 \frac{x}{2}}{x} \right) \right)$$

$$= 2 \lim_{x \rightarrow 0} \left(\frac{1}{\frac{\sin x}{x}} \times x \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \frac{x}{4} \right)$$

$$= 2 \times \frac{1}{x} \times \frac{x}{4}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\operatorname{cose} x - \cot x}{x} = \frac{1}{2}$$

Hence, $\lim_{x \rightarrow 0} \frac{\operatorname{cose} x - \cot x}{x} = \frac{1}{2}$

54. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin 3x + 7x}{4x + \sin 2x}$$

Answer

Given, $\lim_{x \rightarrow 0} \frac{\sin 3x + 7x}{4x + \sin 2x}$

Now, divide by x

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{x} + \frac{7x}{x}}{\frac{4x}{x} + \frac{\sin 2x}{x}} \\
 &= \frac{\lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 + 7}{4 + \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2} \\
 &= \frac{3 + 7}{4 + 2} \\
 &= \frac{10}{6}
 \end{aligned}$$

Hence, $\lim_{x \rightarrow 0} \frac{\sin 3x + 7x}{4x + \sin 2x} = \frac{10}{6}$

55. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{5x + 4 \sin 3x}{4 \sin 2x + 7x}$$

Answer

Given, $\lim_{x \rightarrow 0} \frac{5x + 4 \sin 3x}{4 \sin 2x + 7x}$

$$\begin{aligned}
 \Rightarrow \lim_{x \rightarrow 0} \frac{5x + 4 \sin 3x}{4 \sin 2x + 7x} &= \lim_{x \rightarrow 0} \frac{5 + \frac{4 \sin 3x}{x}}{\frac{4 \sin 2x}{x} + 7} \\
 &= \frac{5 + \left[\lim_{x \rightarrow 0} \frac{4 \sin 3x}{3x} \times 3 \right]}{\left[\lim_{x \rightarrow 0} \frac{4 \sin 2x}{2x} \times 2 \right] + 7} \\
 &= \frac{5 + 4 \times 1 \times 3}{4 \times 2 + 7} \\
 &= \frac{5 + 12}{8 + 7} \\
 &= \frac{17}{15}
 \end{aligned}$$

Hence, $\lim_{x \rightarrow 0} \frac{5x + 4 \sin 3x}{4 \sin 2x + 7x} = \frac{17}{15}$

56. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{3 \sin x - \sin 3x}{x^3}$$

Answer

Given, $\lim_{x \rightarrow 0} \frac{3 \sin x - \sin 3x}{x^3}$

Since, $\sin 3x = 3 \sin x - 4 \sin^3 x$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{3 \sin x - (3 \sin x - 4 \sin^3 x)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{4 \sin^3 x}{x^3} \\
&= 4 \times \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \\
&= 4 \times 1
\end{aligned}$$

Hence, $\lim_{x \rightarrow 0} \frac{3 \sin x - \sin 3x}{x^3} = 4$

57. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\tan 2x - \sin 2x}{x^3}$$

Answer

$$\lim_{x \rightarrow 0} \frac{\tan 2x - \sin 2x}{x^3}$$

Put $\tan x = \frac{\sin x}{\cos x}$

$$= \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{\cos 2x} - \sin 2x}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 2x \left(\frac{1}{\cos 2x} - 1 \right)}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 2x (1 - \cos 2x)}{x^3 (\cos 2x)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 2x (2 \sin^2 x)}{x^3 (\cos 2x)}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{\sin 2x}{x} \left(\lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \right)}{\lim_{x \rightarrow 0} \cos 2x}$$

$$= \frac{\left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2 \right) 2 \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2}{\lim_{x \rightarrow 0} \cos 2x}$$

$$= \frac{(2 \times 1)(2 \times 1)}{1}$$

$$= 4$$

Hence, $\lim_{x \rightarrow 0} \frac{\tan 2x - \sin 2x}{x^3} = 4$

58. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$$

Answer

Given, $\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$

Taking x as common, we get

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} = \lim_{x \rightarrow 0} \frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{\sin ax}{x} \times a + b}{a + \lim_{x \rightarrow 0} \frac{\sin bx}{x} \times b}$$

$$= \frac{a + b}{a + b}$$

$$= 1$$

Hence, $\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} = 1$

59. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$$

Answer

Given, $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{\sin x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{\tan \frac{x}{2}}{\frac{x}{2}} \right) \times \frac{x}{2}$$

$$= \lim_{x \rightarrow 0} (1) \times \frac{x}{2}$$

$$= 0$$

Hence, $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x) = 0$

60. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{(\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x)}{\cos^2 \beta x - \cos^2 \alpha x}$$

Answer

$$\text{Here, } \lim_{x \rightarrow 0} \frac{(\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x)}{\cos^2 \beta x - \cos^2 \alpha x}$$

$$= \lim_{x \rightarrow 0} \frac{(2\sin \frac{(\alpha + \beta + \alpha - \beta)}{2} x + \cos \frac{(\alpha + \beta - \alpha + \beta)}{2} x + 2\sin \alpha x \cos \alpha x)}{(\cos \beta x - \cos \alpha x)(\cos \beta x + \cos \alpha x)}$$

$$= \lim_{x \rightarrow 0} \frac{\{2\sin \alpha x \cos \beta x + 2\sin \alpha x \cos \alpha x\}}{(\cos \beta x - \cos \alpha x)(\cos \beta x + \cos \alpha x)}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin \alpha x (\cos \beta x + \cos \alpha x)}{(\cos \beta x - \cos \alpha x)(\cos \beta x + \cos \alpha x)}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin \alpha x}{(\cos \beta x - \cos \alpha x)}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin \alpha x}{(1 - 2\sin^2 \left(\frac{\beta x}{2}\right) - 2\sin^2 \left(\frac{\alpha x}{2}\right))}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin \alpha x}{(2\sin^2 \left(\frac{\alpha x}{2}\right) - 2\sin^2 \left(\frac{\beta x}{2}\right))}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x)}{\cos^2 \beta x - \cos^2 \alpha x} = \frac{2\alpha}{\alpha^2 - \beta^2}$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{(\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x)}{\cos^2 \beta x - \cos^2 \alpha x} = \frac{2\alpha}{\alpha^2 - \beta^2}$$

61. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$$

Answer

$$\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$$

$$\text{Explanation: Here, } \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$$

$$= \lim_{x \rightarrow 0} \frac{1 - 2\sin^2 \left(\frac{ax}{2}\right) - 1 + 2\sin^2 \left(\frac{bx}{2}\right)}{1 - 2\sin^2 \left(\frac{cx}{2}\right) - 1}$$

$$= \lim_{x \rightarrow 0} \frac{-2\sin^2 \left(\frac{ax}{2}\right) + 2\sin^2 \left(\frac{bx}{2}\right)}{-2\sin^2 \left(\frac{cx}{2}\right)}$$

$$= \lim_{x \rightarrow 0} \frac{-\sin^2 \left(\frac{ax}{2}\right) 4a^2 x^2 + \sin^2 \left(\frac{bx}{2}\right) 4b^2 x^2}{-\sin^2 \left(\frac{cx}{2}\right) 4c^2 x^2}$$

$$= \frac{-a^2 + b^2}{-c^2}$$

$$= \frac{b^2 - a^2}{c^2}$$

Hence, $\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - 1} = \frac{b^2 - a^2}{c^2}$

62. Question

Evaluate the following limits:

$$\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$$

Answer

Given, $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

Explanation: $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

$$= \lim_{h \rightarrow 0} \frac{(a+h)^2 (\sin a \cos h + \cos a \sin h) - a^2 \sin a}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(a+h)^2 (\sin a \cos h) - a^2 \sin a + (a+h)^2 \cos a \sin h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a^2 \sin a (\cos h - 1) + 2ah \sin a \cos h + h^2 \sin a \cos h + (a+h)^2 \cos a \sin h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a^2 \sin a (\cos h - 1)}{h} + \lim_{h \rightarrow 0} \frac{2ah \sin a \cos h}{h} + \lim_{h \rightarrow 0} \frac{h^2 \sin a \cos h}{h} + \lim_{h \rightarrow 0} \frac{(a+h)^2 \cos a \sin h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-a^2 \sin a \sin^2\left(\frac{h}{2}\right)}{\frac{h}{2}} + 2a \sin a + 0 + a^2 \cos a$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h} = 0 + 2a \sin a + a^2 \cos a$$

Hence, $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h} = 2a \sin a + a^2 \cos a$

63. Question

Evaluate the following limits:

If $\lim_{x \rightarrow 0} kx \operatorname{cosec} x = \lim_{x \rightarrow 0} x \operatorname{cosec} kx$, find k.

Answer

Given, $\lim_{x \rightarrow 0} kx \operatorname{cosec} x = \lim_{x \rightarrow 0} x \operatorname{cosec} kx$

To Find: Value of k?

Explanation: Here, $\lim_{x \rightarrow 0} kx \operatorname{cosec} x = \lim_{x \rightarrow 0} x \operatorname{cosec} kx$

$$\lim_{x \rightarrow 0} kx \frac{1}{\sin x} = \lim_{x \rightarrow 0} x \frac{1}{\sin kx}$$

Taking k common from L.H.S and multiply and divide by k in R.H.S, we get

$$k \lim_{x \rightarrow 0} x \frac{1}{\sin x} = \frac{1}{k} \lim_{x \rightarrow 0} \frac{kx}{\sin kx}$$

$$k = \frac{1}{k}$$

$$k^2 = 1$$

$$k = \pm 1$$

Hence, The value of k is 1, - 1.

Exercise 29.8

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x$$

Answer

$$\text{Given: } \lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x$$

$$\text{Assumption: Let } y = \frac{\pi}{2} - x$$

$$\text{So, } x \rightarrow \frac{\pi}{2}, y \rightarrow 0$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x = \lim_{y \rightarrow 0} y \tan \left(\frac{\pi}{2} - y \right)$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x = \lim_{y \rightarrow 0} y \frac{\sin \left(\frac{\pi}{2} - y \right)}{\cos \left(\frac{\pi}{2} - y \right)}$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x = \lim_{y \rightarrow 0} y \frac{\cos y}{\sin y}$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x = \lim_{y \rightarrow 0} \cos y - \lim_{y \rightarrow 0} \frac{y}{\sin y}$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \left(\frac{\pi}{2} - x \right) \tan x = \cos 0 - \frac{0}{\sin 0}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{2} - x \right) \tan x = 1 - 0$$

$$\text{Hence, } \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{2} - x \right) \tan x = 1$$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi/2} \frac{\sin 2x}{\cos x}$$

Answer

$$\text{Given, } \lim_{x \rightarrow \pi/2} \frac{\sin 2x}{\cos x}$$

We know, $\sin 2x = 2 \sin x \cdot \cos x$

By putting this value, we get

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\sin 2x}{\cos x} = \lim_{x \rightarrow \pi/2} \frac{2 \sin x \cos x}{\cos x}$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\sin 2x}{\cos x} = \lim_{x \rightarrow \pi/2} 2 \sin x$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\sin 2x}{\cos x} = 2 \sin \frac{\pi}{2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin 2x}{\cos x} = 2 \times 1$$

$$\text{Hence } \lim_{x \rightarrow \pi/2} \frac{\sin 2x}{\cos x} = 2$$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x}$$

Answer

$$\text{Given, } \lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x}$$

$$\text{Here, } \cos^2 x = 1 - \sin^2 x$$

By putting this we get,

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x} = \lim_{x \rightarrow \pi/2} \frac{1 - \sin^2 x}{1 - \sin x}$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x} = \lim_{x \rightarrow \pi/2} \frac{(1 - \sin x)(1 + \sin x)}{1 - \sin x}$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x} = \lim_{x \rightarrow \pi/2} 1 + \sin x$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x} = 1 + \sin \frac{\pi}{2}$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x} = 1 + 1$$

$$\text{Hence, } \lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{1 - \sin x} = 2$$

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi/3} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}(\pi/3 - x)}$$

Answer

$$\text{Given, } \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}(\frac{\pi}{3} - x)}$$

[Applying the formula $1 - \cos 2x = 2\sin^2 x$]

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)} = \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{2 \sin^2 3x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)} = \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{2} \sin 3x}{\sqrt{2} \left(\frac{\pi}{3} - x \right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)} = \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sin 3x}{\left(\frac{\pi}{3} - x \right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)} = \lim_{x \rightarrow \frac{\pi}{3}} \frac{3 \sin 3x}{\pi - 3x}$$

We know that, $\sin x = \sin(\pi - x)$

Therefore,

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)} = \lim_{x \rightarrow \frac{\pi}{3}} \frac{3 \sin(\pi - 3x)}{\pi - 3x}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)} = 3$$

$$\text{Hence, } \Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)} = 3$$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a}$$

Answer

$$\text{Given, } \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a} = \lim_{x \rightarrow a} \frac{\left(-2 \sin \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right) \right)}{x - a}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a} = -2 \lim_{x \rightarrow a} \sin \left(\frac{x+a}{2} \right) \lim_{x \rightarrow a} \sin \left(\frac{x-a}{2} \right)$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a} = -2 \sin \left(\frac{a+a}{2} \right) \left(\lim_{x \rightarrow a} \sin \frac{\left(\frac{x-a}{2} \right)}{x-a} \right) \times \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a} = -2 \sin a \times 1 \times \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a} = -\sin a$$

$$\text{Hence, } \lim_{x \rightarrow a} \frac{\cos x - \cos a}{x - a} = -\sin a$$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}}$$

Answer

Given, $\lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}}$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}} = \lim_{y \rightarrow 0} \frac{1 - \tan\left(y + \frac{\pi}{4}\right)}{y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}} = \lim_{y \rightarrow 0} \frac{\left(\frac{\tan y + \tan \frac{\pi}{4}}{1 - \tan y \tan \frac{\pi}{4}} \right)}{y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}} = \lim_{y \rightarrow 0} \frac{(1 - \tan y - \tan y - 1)}{y(1 - \tan y)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}} = \lim_{y \rightarrow 0} \frac{-2 \tan y}{y(1 - \tan y)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}} = -2 \lim_{y \rightarrow 0} \frac{\tan y}{y} \times \lim_{y \rightarrow 0} \frac{1}{(1 - \tan y)}$$

We know, $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}} = -2 \times \frac{1}{(1 - 0)}$$

Hence, $\lim_{x \rightarrow \frac{\pi}{3}} \frac{1 - \tan x}{x - \frac{\pi}{4}} = -2$

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$$

Answer

We have Given, If $\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$

If $x \rightarrow \frac{\pi}{2}$, $\frac{\pi}{2} - x \rightarrow 0$, $\pi - 3x \rightarrow 0$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{1 - \sin\left(\frac{\pi}{2} - y\right)}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = 2 \lim_{y \rightarrow 0} \left(\frac{\sin^2 \frac{y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4}$$

Since, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = 2 \times 1 \times \frac{1}{4}$$

Hence, $\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = \frac{1}{2}$

8. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x}$$

Answer

We have $\lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x}$

If $x \rightarrow \frac{\pi}{3}$, $\frac{\pi}{3} - x \rightarrow 0$, $\pi - 3x \rightarrow 0$

Let $\frac{\pi}{3} - x = y$ then $y \rightarrow 0$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x} = \lim_{y \rightarrow 0} \frac{\sqrt{3} - \frac{\tan \frac{\pi}{3} - \tan y}{1 + \tan \frac{\pi}{3} \cdot \tan y}}{3 \left(\frac{\pi}{3} - x \right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x} = \lim_{y \rightarrow 0} \frac{\sqrt{3} - \frac{\sqrt{3} - \tan y}{1 + \sqrt{3} \tan y}}{3y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x} = \lim_{y \rightarrow 0} \frac{(\sqrt{3} + 3 \tan y - \sqrt{3} + \tan y)}{3(1 + \sqrt{3} \tan y)y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x} = \lim_{y \rightarrow 0} \frac{4 \tan y}{3(1 + \sqrt{3} \tan y)y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x} = \frac{4}{3} \lim_{y \rightarrow 0} \frac{\tan y}{y} \times \frac{1}{\left(1 + \sqrt{3} \frac{\tan y}{y} y\right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x} = \frac{4 \times 1}{3} \times \frac{1}{1 + 0}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x} = \frac{4}{3}$$

Hence, $\lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x} = \frac{4}{3}$

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2}$$

Answer

Given, $\lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2}$

$$\Rightarrow \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} = \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax(x-a)}$$

Let $t = x - a$

Then, as $x \rightarrow a$, $t \rightarrow 0$

$$\Rightarrow \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} = \lim_{t \rightarrow 0} \frac{(a \sin(t+a) - (t+a) \sin a)}{a(t+a)t}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} = \lim_{t \rightarrow 0} \frac{a \sin t \cos a + a \sin a (\cos t - 1) - t \sin a}{a(t+a)t}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} = \lim_{t \rightarrow 0} \frac{a \sin t \cos a + a \sin a \left(2 \sin^2 \left(\frac{t}{2}\right)\right) - t \sin a}{a(t+a)t}$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} &= \lim_{t \rightarrow 0} \frac{a \sin t \cos a}{a(t+a)t} + \lim_{t \rightarrow 0} \frac{a \sin a \left(2 \sin^2 \left(\frac{t}{2}\right)\right)}{a(t+a)t} - \lim_{t \rightarrow 0} \frac{t \sin a}{a(t+a)t} \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} = \frac{a \cos a}{a^2} + -0 - \frac{\sin a}{a^2}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} = \frac{a \cos a - \sin a}{a^2}$$

Hence, $\lim_{x \rightarrow a} \frac{a \sin x - x \sin a}{ax^2 - xa^2} = \frac{a \cos a - \sin a}{a^2}$

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$$

Answer

We have $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$

Rationalise the numerator, we get

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1 + \sin x}}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 - 1 - \sin x}{\cos^2 x (\sqrt{2} + \sqrt{1 + \sin x})}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{(1 - \sin^2 x) (\sqrt{2} + \sqrt{1 + \sin x})}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{(1 - \sin x)(1 + \sin x) (\sqrt{2} + \sqrt{1 + \sin x})}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{(1 + \sin x) (\sqrt{2} + \sqrt{1 + \sin x})}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = \frac{1}{(1 + 1)(\sqrt{2} + \sqrt{2})}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = \frac{1}{4\sqrt{2}}$$

$$\text{Hence, } \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = \frac{1}{4\sqrt{2}}$$

11. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2}$$

Answer

$$\text{Given, } \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2 - \sin\left(\frac{\pi}{2} - y\right)} - 1}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2 - \cos y} - 1}{y^2}$$

Now, rationalize the Numerator, we get,

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2 - \cos y} - 1}{y^2} \times \frac{\sqrt{2 - \cos y} + 1}{\sqrt{2 - \cos y} + 1}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2 - \cos y} - 1}{y^2 (\sqrt{2 - \cos y} + 1)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{2 - \cos y - 1}{y^2 (\sqrt{2 - \cos y} + 1)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2 (\sqrt{2 - \cos y} + 1)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2 (\sqrt{2 - \cos y} + 1)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = 2 \times \lim_{y \rightarrow 0} \left(\frac{\sin \frac{y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4} \times \frac{1}{\lim_{y \rightarrow 0} \sqrt{2 - \cos y} + 1}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = 2 \times 1 \times \frac{1}{4} \times \frac{1}{2}$$

$$\text{Hence, } \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2} = \frac{1}{4}$$

12. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2 - \cos x} - \sin x}{\left(\frac{\pi}{4} - x\right)^2}$$

Answer

$$\text{Given, } \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2 - \cos x} - \sin x}{\left(\frac{\pi}{4} - x\right)^2}$$

$$\text{Now, } x \rightarrow \frac{\pi}{4}, \frac{\pi}{4} - x \rightarrow 0, \text{ let } \frac{\pi}{4} - x = y$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2 - \cos x} - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2 - \cos\left(\frac{\pi}{4} - y\right)} - \sin\left(\frac{\pi}{4} - y\right)}{y^2}$$

$$= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \left[\cos \frac{\pi}{4} \cos y + \sin \frac{\pi}{4} \sin y + \sin \frac{\pi}{4} \cos y - \cos \frac{\pi}{4} \sin y \right]}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2 - \cos x} - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2} - \left[\frac{\cos y}{\sqrt{2}} + \frac{\sin y}{\sqrt{2}} + \frac{\cos y}{\sqrt{2}} - \frac{\sin y}{\sqrt{2}} \right]}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2 - \cos x} - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2} - \left[\frac{2 \cos y}{\sqrt{2}} \right]}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2 - \cos x} - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2} - \sqrt{2} \cos y}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2 - \cos x} - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \sqrt{2} \lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \sqrt{2} \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{\frac{y^2}{4}} \times \frac{1}{4}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \sqrt{2} \times \frac{1}{4} \times \left(\lim_{y \rightarrow 0} \frac{\sin \frac{y}{2}}{\frac{y}{2}} \right)^2$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \sqrt{2} \times 2 \times \frac{1}{4} \times 1$$

Hence, $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{\left(\frac{\pi}{4} - x\right)^2} = \frac{1}{\sqrt{2}}$

13. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$$

Answer

Given, $\lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$

Where $x \rightarrow \frac{\pi}{8}$, $\frac{\pi}{8} - x \rightarrow 0$, let $\frac{\pi}{8} - x = y$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{8^3 \left(\frac{\pi}{8} - x\right)^3}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \lim_{y \rightarrow 0} \frac{\cot\left(\frac{\pi}{8} - x\right)4 - \cos\left(\frac{\pi}{8} - x\right)4}{8^3 y^3}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \lim_{y \rightarrow 0} \frac{\tan 4y - \sin 4y}{8^3 y^3}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \lim_{y \rightarrow 0} \frac{\frac{\sin 4y}{\cos 4y} - \sin 4y}{8^3 y^3}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \lim_{y \rightarrow 0} \frac{\sin 4y - \sin 4y \cdot \cos 4y}{8^3 y^3 \cos 4y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \lim_{y \rightarrow 0} \frac{\sin 4y(1 - \cos 4y)}{8^3 y^3 \cos 4y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \lim_{y \rightarrow 0} \frac{\sin 4y \cdot (2 \sin^2 2y)}{8^3 y^3 \cos 4y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \frac{2}{8^3} \lim_{y \rightarrow 0} \frac{\sin 4y}{y} \times \frac{\sin^2 2y}{y^2} \times \frac{1}{\cos 4y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$$

$$= \frac{2}{8^3} \left(\lim_{y \rightarrow 0} \frac{\sin 4y}{4y} \times 4 \right) \times \left(\lim_{y \rightarrow 0} \frac{\sin 2y}{2y} \times 2 \right)^2 \times 4 \times \frac{1}{\lim_{y \rightarrow 0} \cos 4y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \frac{2}{8^3} (1 \times 4) \times (1) \times 4$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \frac{2 \times 4 \times 4}{8 \times 8 \times 8}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \frac{1}{16}$$

Hence, $\lim_{x \rightarrow \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3} = \frac{1}{16}$

14. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}}$$

Answer

We have Given, $\lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}}$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = \lim_{x \rightarrow a} \frac{\left(-2 \sin \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right) \right)}{\sqrt{x} - \sqrt{a}}$$

Now, Rationalize the Denominator

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2 \lim_{x \rightarrow a} \frac{\left(\sin \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right) \right)}{\sqrt{x} - \sqrt{a}(\sqrt{x} + \sqrt{a})} \cdot \sqrt{x} + \sqrt{a}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2 \lim_{x \rightarrow a} \sin \left(\frac{x+a}{2} \right) \cdot \lim_{x \rightarrow a} \frac{\sin \frac{x-a}{2} \times \frac{1}{2}}{\frac{x-a}{2}} \lim_{x \rightarrow a} \sqrt{x} + \sqrt{a}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2 \sin(a) \times \frac{1}{2} \times 2\sqrt{a}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2\sqrt{a} \sin a$$

Hence, $\lim_{x \rightarrow a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}} = -2\sqrt{a} \sin a$

15. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2}$$

Answer

Given, $\lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2}$

If $x \rightarrow \pi$, then $\pi - x \rightarrow 0$, let $\pi - x = y$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{5 + \cos(\pi - y)} - 2}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{5 + \cos y} - 2}{y^2}$$

Rationalize the Numerator

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{5 + \cos y} - 2 \times (\sqrt{5 + \cos y} + 2)}{y^2(\sqrt{5 + \cos y} + 2)}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{5 - \cos y - 4}{y^2(\sqrt{5 + \cos y} + 2)}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2(\sqrt{5 + \cos y} + 2)}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = 2 \times \lim_{y \rightarrow 0} \left(\frac{\frac{\sin y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4} \times \frac{1}{\lim_{y \rightarrow 0} (\sqrt{5 + \cos y} + 2)}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = 2 \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{4} + 2}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = 2 \times \frac{1}{4} \times \frac{1}{4}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = \frac{1}{8}$$

Hence, $\lim_{x \rightarrow \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2} = \frac{1}{8}$

16. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a}$$

Answer

We have Given, $\lim_{x \rightarrow a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a}$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a} = \lim_{x \rightarrow a} \frac{\left(-2 \sin \left(\frac{\sqrt{x} + \sqrt{a}}{2} \right) \sin \left(\frac{\sqrt{x} - \sqrt{a}}{2} \right) \right)}{\sqrt{x} - \sqrt{a}(\sqrt{x} + \sqrt{a})}$$

Now, Rationalize the Denominator

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a} = -2 \lim_{x \rightarrow a} \frac{\left(\sin \left(\frac{\sqrt{x} + \sqrt{a}}{2} \right) \sin \left(\frac{\sqrt{x} - \sqrt{a}}{2} \right) \right)}{\frac{(\sqrt{x} - \sqrt{a})}{2}(\sqrt{x} + \sqrt{a})} \cdot \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a} = -2 \sin(\sqrt{a}) \times \frac{1}{2\sqrt{a}} \times \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a} = -\sin(\sqrt{a}) \times \frac{1}{2\sqrt{a}}$$

$$\text{Hence, } \lim_{x \rightarrow a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a} = -\sin(\sqrt{a}) \times \frac{1}{2\sqrt{a}}$$

17. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{x - a}$$

Answer

$$\text{we have } \lim_{x \rightarrow a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}$$

$$= \lim_{x \rightarrow a} \frac{2 \sin \left(\frac{\sqrt{x} - \sqrt{a}}{2} \right) \cos \left(\frac{\sqrt{x} + \sqrt{a}}{2} \right)}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}$$

$$= 2 \lim_{x \rightarrow a} \left[\frac{\sin \left(\frac{\sqrt{x} - \sqrt{a}}{2} \right)}{\frac{\sqrt{x} - \sqrt{a}}{2}} \right] \times \frac{1}{2} \times \lim_{x \rightarrow a} \left[\cos \frac{\left(\frac{\sqrt{x} + \sqrt{a}}{2} \right)}{\frac{\sqrt{x} + \sqrt{a}}{2}} \right]$$

$$= 2 \times 1 \times \frac{1}{2} \times \cos \sqrt{a} \times \frac{1}{2\sqrt{a}}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{x - a} = \frac{\cos \sqrt{a}}{2\sqrt{a}}$$

18. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x}$$

Answer

$$\text{We have Given, } \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x}$$

Here, $x \rightarrow 1$, then $x - 1 \rightarrow 0$, let $x - 1 = y$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = \lim_{x-1 \rightarrow 0} \frac{(1-x)(1+x)}{\sin 2\pi x}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = \lim_{x-1 \rightarrow 0} \frac{(1-x)(1+x)}{\sin 2\pi x}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = \lim_{y \rightarrow 0} \frac{-y(1+y+1)}{\sin 2\pi(y+1)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = \lim_{y \rightarrow 0} \frac{-y(1 + y + 1)}{\sin 2\pi(y + 1)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = \lim_{y \rightarrow 0} \frac{y(y + 2)}{\sin 2\pi y + 2\pi}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = \lim_{y \rightarrow 0} \frac{y(y + 2)}{\sin 2\pi y}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = \lim_{y \rightarrow 0} (y + 2) \times \frac{y}{\sin \frac{2\pi y}{2\pi y} \times 2\pi y}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = -2 \times \frac{1}{2\pi}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = -\frac{1}{\pi}$$

$$\text{Hence, } \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = -\frac{1}{\pi}$$

19. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}}, \text{ where } f(x) = \sin 2x$$

Answer

Given, $f(x) = \sin 2x$

$$\text{Since, } \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{4}}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{4}}$$

Now, $x \rightarrow \frac{\pi}{4}$ and $x - \frac{\pi}{4} \rightarrow 0$, let $x - \frac{\pi}{4} = y$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{y \rightarrow 0} \frac{\sin 2\left(y + \frac{\pi}{4}\right) - 1}{y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{y \rightarrow 0} \frac{\sin\left(\frac{\pi}{2} + 2y\right) - 1}{y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{y \rightarrow 0} \frac{\cos 2y - 1}{y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{y \rightarrow 0} \frac{1 - \cos 2y}{y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = -\lim_{y \rightarrow 0} \frac{2 \sin^2 y}{y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = -2 \lim_{y \rightarrow 0} \left(\frac{\sin y}{y}\right)^2 \times y$$

Since, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = -2 \times 0$$

$$\text{Hence, } \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \sin 2\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = 0$$

20. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2}$$

Answer

$$\text{Given, } \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2}$$

Now, $x \rightarrow 1$, then $x - 1 \rightarrow 0$, let $x - 1 = y$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \lim_{y \rightarrow 0} \frac{1 + \cos \pi(y + 1)}{-y^2}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \lim_{y \rightarrow 0} \frac{1 + \cos \pi(y + 1)}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \lim_{y \rightarrow 0} \frac{1 - \cos(\pi y)}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{\pi y}{2}}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2} = 2 \lim_{y \rightarrow 0} \left(\frac{\sin \frac{\pi y}{2}}{\frac{\pi y}{2}} \right)^2 \times \frac{\pi^2}{4}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2} = 2 \times 1 \times \frac{\pi^2}{4}$$

$$\text{Hence, } \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{(1 - x)^2} = \frac{\pi^2}{2}$$

21. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x}$$

Answer

We have Given, $\lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x}$

Here, $x \rightarrow 1$, then $x - 1 \rightarrow 0$, let $x - 1 = y$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x} = \lim_{x-1 \rightarrow 0} \frac{(1 - x)(1 + x)}{\sin \pi x}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x} = \lim_{x-1 \rightarrow 0} \frac{(1 - x)(1 + x)}{\sin \pi x}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x} = \lim_{y \rightarrow 0} \frac{-y(1 + y + 1)}{\sin \pi(y + 1)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x} = \lim_{y \rightarrow 0} \frac{y(y + 2)}{\sin \pi y + \pi}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x} = \lim_{y \rightarrow 0} \frac{y(y + 2)}{\frac{\sin \pi y}{2}}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x} = \lim_{y \rightarrow 0} \frac{y + 2}{\frac{\sin \pi y}{\pi y} \cdot \pi y}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x} = \frac{2}{\pi}$$

Hence, $\lim_{x \rightarrow 1} \frac{1 - x^2}{\sin 2\pi x} = \frac{2}{\pi}$

22. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x}$$

Answer

We have $\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x}$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} = \lim_{y \rightarrow 0} \frac{(1 - \sin 2(y + \frac{\pi}{4}))}{1 + \cos 4(y + \frac{\pi}{4})}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} = \lim_{y \rightarrow 0} \frac{(1 - \sin(\frac{\pi}{2} + 2y))}{1 + \cos(\pi + 4y)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} = \lim_{y \rightarrow 0} \frac{1 - \cos 2y}{1 - \cos 4y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} = \lim_{y \rightarrow 0} \frac{2 \sin^2 y}{2 \sin^2 2y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} = \lim_{y \rightarrow 0} \frac{\left(\frac{2 \sin^2 y}{y}\right)^2 y^2}{\left(\frac{2 \sin^2 2y}{2y}\right)^2 4y^2}$$

Since, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, then

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} = \frac{1 \times y^2}{1 \times 4y^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} = \frac{1}{4}$$

Hence, $\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x} = \frac{1}{4}$

23. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x}$$

Answer

$$\text{Given, } \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x}$$

As we know, $\tan^2 x = \sec^2 x - 1$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\sec^2 x - 1}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\frac{1}{\cos^2 x} - 1}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \lim_{x \rightarrow \pi} \frac{\cos^2 x \cdot (1 + \cos x)}{1 - \cos^2 x}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \lim_{x \rightarrow \pi} \frac{\cos^2 x \cdot (1 + \cos x)}{(1 + \cos x)(1 - \cos x)}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \lim_{x \rightarrow \pi} \frac{\cos^2 x}{1 - \cos x}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \frac{1}{1 - (-1)}$$

Hence, $\Rightarrow \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \frac{1}{2}$

24. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right)$$

Answer

$$\lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right)$$

Divide and multiply by 2, we get

$$\Rightarrow \lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) = \lim_{n \rightarrow \infty} 2 \left[n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) \right] \times \frac{1}{2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) = \lim_{n \rightarrow \infty} n \sin \frac{\pi}{2n} \times \frac{1}{2}$$

Now, $n \rightarrow \infty$, then $\frac{1}{n} \rightarrow 0$, let $\frac{1}{n} = y$

$$\Rightarrow \lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) = \lim_{y \rightarrow 0} \frac{1}{y} \sin \frac{\pi}{2} \times \frac{1}{n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) = \frac{1}{2} \lim_{y \rightarrow 0} \frac{\left(\sin\left(\frac{\pi}{2}\right)y\right)}{y}$$

$$\Rightarrow \lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) = \frac{1}{2} \lim_{y \rightarrow 0} \frac{\left(\sin\left(\frac{\pi}{2}\right)y\right)}{\frac{\pi y}{2}} \times \frac{\pi}{2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) = \frac{1}{2} \times \frac{1\pi}{2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) = \frac{\pi}{4}$$

Hence, $\lim_{n \rightarrow \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right) = \frac{\pi}{4}$

25. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right)$$

Answer

We have $\lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right)$

$$\Rightarrow \lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right) = \lim_{n \rightarrow \infty} \frac{2^n}{2^1} \sin\left(\frac{a}{2^n}\right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right) = \lim_{n \rightarrow \infty} \frac{2^n}{2^1} \sin \frac{a}{2^n}$$

Now, $n \rightarrow \infty$, $\frac{1}{n} \rightarrow 0$ and let $h = 1/n$

$$\Rightarrow \lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right) = \lim_{h \rightarrow 0} \frac{2^{\frac{1}{h}}}{2^1} \cdot \sin \frac{a}{2^{\frac{1}{h}}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right) = \lim_{h \rightarrow 0} \frac{2^{\frac{1}{h}}}{2^1} \cdot \frac{\left(\sin \frac{a}{2^{\frac{1}{h}}}\right)}{\frac{a}{2^{\frac{1}{h}}}} \cdot \frac{a}{2^{\frac{1}{h}}}$$

We know, $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ then, we get

$$\Rightarrow \lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right) = \frac{a}{2}$$

Hence, $\lim_{n \rightarrow \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right) = \frac{a}{2}$

26. Question

Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)}$$

Answer

We have $\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)}$

Now, $n \rightarrow \infty, \frac{1}{n} = h \rightarrow 0$

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} &= \frac{\lim_{h \rightarrow 0} \sin\left(\frac{a}{2^{\frac{1}{h}}}\right)}{\lim_{h \rightarrow 0} \sin\left(\frac{b}{2^{\frac{1}{h}}}\right)} \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} &= \frac{\lim_{h \rightarrow 0} \sin\left(\frac{a}{2^{\frac{1}{h}}}\right) \cdot \frac{a}{2^{\frac{1}{h}}}}{\lim_{h \rightarrow 0} \sin\left(\frac{b}{2^{\frac{1}{h}}}\right) \cdot \frac{b}{2^{\frac{1}{h}}}} \end{aligned}$$

We know, $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ then , we get

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} = \frac{1 \times \frac{a}{2^{\frac{1}{h}}}}{1 \times \frac{b}{2^{\frac{1}{h}}}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} = \frac{a}{b}$$

Hence, $\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)} = \frac{a}{b}$

27. Question

Evaluate the following limits:

$$\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)}$$

Answer

We have $\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)}$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{x(x + 1) + \sin(x + 1)}$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \lim_{x \rightarrow -1} \frac{(x - 2)(x + 1)}{x(x + 1) + \sin(x + 1)}$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \lim_{x \rightarrow -1} \frac{1}{\frac{x(x + 1)}{(x - 2)(x + 1)} + \frac{\sin(x + 1)}{(x - 2)(x + 1)}}$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \lim_{x \rightarrow -1} \frac{1}{\frac{x}{(x - 2)} + \frac{\sin(x + 1)}{(x - 2)(x + 1)}}$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \lim_{x \rightarrow -1} \frac{1}{(x - 2)} \left[\frac{1}{x + \frac{\sin(x + 1)}{(x + 1)}} \right]$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \lim_{x \rightarrow -1} \frac{1}{(x - 2)} \left[\frac{1}{\lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} \frac{\sin(x + 1)}{(x + 1)}} \right]$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \left(\frac{1}{-1 - 2} \right) \times \left(\frac{1}{(-1) + 1} \right)$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \frac{1}{0} = \infty$$

$$\text{Hence, } \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)} = \infty$$

28. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)}$$

Answer

$$\text{We have } \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)}$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 1)}{x^2 - 2x + \sin(x - 2)}$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)} = \lim_{x \rightarrow 2} \frac{1}{\frac{x}{x + 1} + \frac{\sin(x - 2)}{(x - 2)(x + 1)}}$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)} = \lim_{x \rightarrow 2} (x + 1) \left[\frac{1}{x + \frac{\sin(x - 2)}{(x - 2)}} \right]$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)} = \lim_{x \rightarrow 2} (x + 1) \left[\frac{1}{\lim_{x \rightarrow 2} (x) + \lim_{x \rightarrow 2} \frac{\sin(x - 2)}{(x - 2)}} \right]$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x-2)} = (2+1) \left[\frac{1}{2 + \lim_{x \rightarrow 2} \frac{\sin(x-2)}{(x-2)}} \right]$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x-2)} = 3 \left[\frac{1}{2+1} \right]$$

$$\text{Hence, } \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x-2)} = 1$$

29. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$$

Answer

$$\text{We have } \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$$

When, $x \rightarrow 1, x-1 \rightarrow 0$, let $x-1 = y$, then $y \rightarrow 0$

$$\Rightarrow \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \lim_{(x-1) \rightarrow 0} -(x-1) \tan\left(\frac{\pi x}{2}\right)$$

$$\Rightarrow \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = -\lim_{y \rightarrow 0} y \tan\left(\frac{\pi}{2}\right) (y+1)$$

$$\Rightarrow \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = -\lim_{y \rightarrow 0} y \tan\left(\frac{\pi}{2} + \frac{\pi}{2} y\right)$$

$$\Rightarrow \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \lim_{y \rightarrow 0} y \left(\cot \frac{\pi}{2} y\right)$$

$$\Rightarrow \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \lim_{y \rightarrow 0} \frac{y}{\tan \frac{\pi y}{2}}$$

$$\Rightarrow \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \lim_{y \rightarrow 0} \frac{\frac{\pi y}{2} \times \frac{2}{\pi}}{\tan \frac{\pi y}{2}}$$

$$\Rightarrow \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \frac{2}{\pi}$$

$$\text{Hence, } \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \frac{2}{\pi}$$

30. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x}$$

Answer

$$\text{We have } \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x}$$

$$\text{If } x \rightarrow \frac{\pi}{4}, \text{ then } x - \frac{\pi}{4} = 0, \text{ let } x - \frac{\pi}{4} \rightarrow y$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{1 - \tan y}{1 - \sqrt{2} \sin y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{1 - \tan\left(y + \frac{\pi}{4}\right)}{1 - \sqrt{2} \sin\left(y + \frac{\pi}{4}\right)}$$

$$\text{Since, } \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$$

$$\sin(a + b) = \sin a \cdot \cos b + \cos a \cdot \sin b$$

By putting these , we get

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{1 - \left(\frac{\tan \frac{\pi}{4} + \tan y}{1 + \tan \frac{\pi}{4} \cdot \tan y} \right)}{1 - \sqrt{2} \left(\cos \frac{\pi}{4} + \cos y \cdot \sin \frac{\pi}{4} \right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{\left(1 - \left(\frac{1 + \tan y}{1 - \tan y} \right) \right)}{1 - \sqrt{2} \left(\frac{\sin y}{\sqrt{2}} + \frac{\cos y}{\sqrt{2}} \right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{1 - \tan y - 1 - \tan y}{(1 - \tan y)(1 - \sin y + \cos y)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{-2 \tan y}{(1 - \tan y)(1 - \sin y + \cos y)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = -2 \lim_{y \rightarrow 0} \frac{\tan y \times 1}{(1 - \tan y)(1 - \sin y + \cos y)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{\tan y \times 1}{\lim_{y \rightarrow 0} (1 - \tan y) \lim_{y \rightarrow 0} (1 - \sin y - \cos y)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{\lim_{y \rightarrow 0} \left(\frac{\tan y}{y} \right) \times y}{\lim_{y \rightarrow 0} (1 - \tan y) \lim_{y \rightarrow 0} (1 - \sin y - \cos y)}$$

$$\text{Since, } \frac{\tan y}{y} = 1$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = -\lim_{y \rightarrow 0} \frac{2y}{(1 - y)(1 - y - 1)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = \lim_{y \rightarrow 0} \frac{2}{1 - y}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = 2$$

$$\text{Hence, } \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = 2$$

31. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$$

Answer

We have Given, $\lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$

If $x \rightarrow \pi$, then $x - \pi = 0$, let $x - \pi \rightarrow y$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \lim_{x - \pi \rightarrow 0} \frac{\sqrt{2 + \cos x} - 1}{(-1)^2(x - \pi)^2}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2 + \cos(\pi + y)} - 1}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2 - \cos y} - 1}{y^2}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{(\sqrt{2 - \cos y} - 1)(\sqrt{2 - \cos y} + 1)}{y^2(\sqrt{2 - \cos y} + 1)}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2(\sqrt{2 - \cos y} + 1)}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{y^2(\sqrt{2 - \cos y} + 1)}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = 2 \cdot \lim_{y \rightarrow 0} \left(\frac{\sin}{\frac{y}{2}} \right)^2 \times \frac{1}{4} \times \frac{1}{\lim_{y \rightarrow 0} \sqrt{2 - \cos 0} + 1}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = 2 \times \frac{1}{4} \times \frac{1}{\sqrt{2 - 1} + 1}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \frac{1}{4}$$

Hence, $\lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \frac{1}{4}$

32. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi/4} \frac{\sqrt{\cos x} - \sqrt{\sin x}}{x - \pi/4}$$

Answer

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{\cos x} - \sqrt{\sin x}}{x - \frac{\pi}{4}}$$

Rationalizing we get,

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{\cos x} - \sqrt{\sin x}}{x - \frac{\pi}{4}} \times \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\left(x - \frac{\pi}{4}\right) (\sqrt{\cos x} + \sqrt{\sin x})}$$

As, $x \rightarrow \frac{\pi}{4}$, $x - \frac{\pi}{4} \rightarrow 0$, let $x - \frac{\pi}{4} = y$

Therefore, $y \rightarrow 0$,

Now,

$$\begin{aligned} &= \lim_{y \rightarrow 0} \frac{\left(\cos\left(\frac{\pi}{4} + y\right) - \sin\left(\frac{\pi}{4} + y\right)\right)}{y \left(\sqrt{\cos\left(\frac{\pi}{4} + y\right)} + \sqrt{\sin\left(\frac{\pi}{4} + y\right)}\right)} \\ &= \lim_{y \rightarrow 0} \frac{\frac{1}{\sqrt{2}} \cos y - \frac{1}{\sqrt{2}} \sin y - \left[\frac{1}{\sqrt{2}} \cos y + \frac{1}{\sqrt{2}} \sin y\right]}{y \left(\sqrt{\frac{1}{\sqrt{2}} \cos y - \frac{1}{\sqrt{2}} \sin y} + \sqrt{\frac{1}{\sqrt{2}} \cos y + \frac{1}{\sqrt{2}} \sin y}\right)} \\ &= \lim_{y \rightarrow 0} \frac{-\sqrt{2} \sin y}{y \left[\sqrt{\frac{1}{\sqrt{2}} \cos y - \frac{1}{\sqrt{2}} \sin y} + \sqrt{\frac{1}{\sqrt{2}} \cos y + \frac{1}{\sqrt{2}} \sin y}\right]} \\ &= \frac{-1}{\frac{1}{2^{\frac{1}{4}}}} \end{aligned}$$

Hence, $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{\cos x} - \sqrt{\sin x}}{x - \frac{\pi}{4}} = -2^{\frac{1}{4}}$

33. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi(x-1)}$$

Answer

We have Given, $\lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi(x-1)}$

if $x \rightarrow 1$ then, $x - 1 \rightarrow 0$ let $x - 1 = y$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi(x-1)} = \lim_{x-1 \rightarrow 0} \frac{x-1}{x \sin \pi(x-1)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi(x-1)} = \lim_{y \rightarrow 0} \frac{y}{\sin \pi y(y+1)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi(x-1)} = \lim_{y \rightarrow 0} \frac{1}{\frac{\sin \pi y(y+1)}{y}}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi (x-1)} = \frac{1}{\lim_{y \rightarrow 0} (y+1) \times \lim_{y \rightarrow 0} \left(\frac{\sin \pi y}{y \times \pi} \cdot \pi \right)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi (x-1)} = \frac{1}{(1)(1 \times \pi)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi (x-1)} = \frac{1}{\pi}$$

$$\text{Hence, } \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\sin \pi (x-1)} = \frac{1}{\pi}$$

34. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$$

Answer

$$[\operatorname{cosec}^2 x - \cot^2 x = 1]$$

$$\lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x \rightarrow \frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 1}$$

$$[\text{Applying, } a^2 - b^2 = (a + b)(a - b)]$$

$$\lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x \rightarrow \frac{\pi}{6}} \frac{(\operatorname{cosec} x + 2)(\operatorname{cosec} x - 2)}{\operatorname{cosec} x - 2}$$

$$\text{Hence, } \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = 2 + 2 = 4$$

35. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2}$$

Answer

$$\text{We have Given, } \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2}$$

$$\text{Now, if } x \rightarrow \frac{\pi}{4} \text{ then } x - \frac{\pi}{4} \rightarrow 0 \text{ let } x - \frac{\pi}{4} \rightarrow y$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} = \lim_{x - \frac{\pi}{4} \rightarrow 0} \frac{\sqrt{2} - \cos x - \sin x}{(4)^2 \left(x - \frac{\pi}{4} \right)^2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2} - \cos \left(y + \frac{\pi}{4} \right) - \sin \left(y + \frac{\pi}{4} \right)}{16y^2}$$

$$\text{Here, } \cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$$

And, $\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$

$$\begin{aligned}
 &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} \\
 &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - (\cos y \cos \frac{\pi}{4} - \sin y \sin \frac{\pi}{4}) - (\sin y \cos \frac{\pi}{4} + \cos y \sin \frac{\pi}{4})}{16y^2} \\
 &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} \\
 &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \left(\cos y \cdot \frac{1}{\sqrt{2}} - \sin y \cdot \frac{1}{\sqrt{2}} \right) - \left(\sin y \cdot \frac{1}{\sqrt{2}} + \cos y \cdot \frac{1}{\sqrt{2}} \right)}{16y^2} \\
 &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} \\
 &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \frac{1}{\sqrt{2}}(\cos y - \sin y) - \frac{1}{\sqrt{2}}(\sin y + \cos y)}{16y^2} \\
 &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} \\
 &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \frac{1}{\sqrt{2}}[(\cos y - \sin y) - (\sin y + \cos y)]}{16y^2} \\
 &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} \\
 &= \lim_{y \rightarrow 0} \frac{\sqrt{2} - \frac{1}{\sqrt{2}}[(\cos y - \sin y) - (\sin y + \cos y)]}{16y^2} \\
 &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} = \lim_{y \rightarrow 0} \frac{\sqrt{2}(1 - \cos y)}{16y^2} \\
 &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} = \frac{\sqrt{2}}{8} \lim_{y \rightarrow 0} \left(\frac{\sin \frac{y}{2}}{\frac{y}{2}} \right)^2 \times \frac{1}{4} \\
 &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} = \frac{1}{16\sqrt{2}}
 \end{aligned}$$

Hence, $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2} = \frac{1}{16\sqrt{2}}$

36. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x \right) \sin x - 2 \cos x}{\left(\frac{\pi}{2} - x \right) + \cot x}$$

Answer

We have Given, $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \sin x-2 \cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \sin x-2 \cos x}{\left(\frac{\pi}{2}-x\right)+\cot x} = \lim_{y \rightarrow 0} \frac{\left(y \sin \left(\frac{\pi}{2}-y\right)-2 \cos \left(\frac{\pi}{2}-y\right)\right)}{y+\cot \left(\frac{\pi}{2}-y\right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \sin x-2 \cos x}{\left(\frac{\pi}{2}-x\right)+\cot x} = \lim_{y \rightarrow 0} \left(\frac{y \cos y-2 \sin y}{1+\tan y} \right)$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \sin x-2 \cos x}{\left(\frac{\pi}{2}-x\right)+\cot x} = \lim_{y \rightarrow 0} \left(\frac{\cos y-2 \cdot \frac{\sin y}{y}}{1+\frac{\tan y}{y}} \right)$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \sin x-2 \cos x}{\left(\frac{\pi}{2}-x\right)+\cot x} = \frac{1-2}{1+1} = -\frac{1}{2}$$

Hence, $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \sin x-2 \cos x}{\left(\frac{\pi}{2}-x\right)+\cot x} = -\frac{1}{2}$

37. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}$$

Answer

We have Given, $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}$

if $x \rightarrow \frac{\pi}{4}$ then $x - \frac{\pi}{4} \rightarrow 0$ let $x - \frac{\pi}{4} = y$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)} = \lim_{y \rightarrow 0} \frac{\cos \left(\frac{\pi}{4}+y\right)-\sin \left(\frac{\pi}{4}+y\right)}{-y\left(\cos \left(\frac{\pi}{4}+y\right)+\sin \left(\frac{\pi}{4}+y\right)\right)}$$

$$\begin{aligned} &\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)} \\ &= \lim_{y \rightarrow 0} \frac{\left(\cos \frac{\pi}{4} \cos y-\sin \frac{\pi}{4} \sin y\right)-\left(\sin \frac{\pi}{4} \cos y+\cos \frac{\pi}{4} \sin y\right)}{-y\left(\cos \left(\frac{\pi}{4}+y\right)+\sin \left(\frac{\pi}{4}+y\right)\right)} \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)} = \lim_{y \rightarrow 0} \frac{\frac{\cos}{\sqrt{2}}-\frac{\sin}{\sqrt{2}}-\frac{\cos}{\sqrt{2}}-\frac{\sin}{\sqrt{2}}}{-y\left(\cos \left(\frac{\pi}{4}+y\right)+\sin \left(\frac{\pi}{4}+y\right)\right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)} = \lim_{y \rightarrow 0} \frac{-\frac{2 \sin y}{\sqrt{2}}}{-y\left(\cos \left(\frac{\pi}{4}+y\right)+\sin \left(\frac{\pi}{4}+y\right)\right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)}$$

$$= \sqrt{2} \lim_{y \rightarrow 0} \left(\frac{\sin y}{y}\right) \frac{1}{\lim_{y \rightarrow 0} \left(\cos\left(\frac{\pi}{4} + y\right) + \sin\left(\frac{\pi}{4} + y\right)\right)}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)} = \sqrt{2} \times 1 \times \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)} = \sqrt{2} \times \frac{1}{\frac{2}{\sqrt{2}}}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)} = \frac{\sqrt{2} \times \sqrt{2}}{2}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos x - \sin x}{\left(\frac{\pi}{4} - x\right)(\cos x + \sin x)} = 1$$

Hence, the answer is 1.

38. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi} \frac{1 - \sin \frac{x}{2}}{\cos \frac{x}{2} \left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)}$$

Answer

$$= \lim_{x \rightarrow \pi} \frac{1 - \sin\left(\frac{x}{2}\right)}{\left(\cos^2\left(\frac{x}{4}\right) - \sin^2\frac{x}{4}\right)\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)}$$

$$= \lim_{x \rightarrow \pi} \frac{1 - \sin\left(\frac{x}{2}\right)}{\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)^2 \left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)}$$

$$= \lim_{x \rightarrow \pi} \frac{1 - \sin\left(\frac{x}{2}\right)}{\left(1 - \sin\frac{x}{2}\right)\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)}$$

$$= \lim_{x \rightarrow \pi} \frac{1}{\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)}$$

$$= \frac{\sqrt{2}}{2}$$

$$= \frac{1}{\sqrt{2}}$$

Hence,

$$\lim_{x \rightarrow \pi} \frac{1 - \sin\left(\frac{x}{2}\right)}{\left(\cos\frac{x}{2}\right)\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)} = \frac{1}{\sqrt{2}}$$

Exercise 29.9

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x}$$

Answer

As we need to find $\lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \lim_{x \rightarrow \pi} \frac{1 + \cos \pi}{\tan^2 \pi} = \frac{1 - 1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

Tip: Similar limit problems involving trigonometric ratios are mostly solved using sandwich theorem.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

So to solve this problem we need to have a sin term so that we can make use of sandwich theorem.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

$$\text{As, } Z = \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x}$$

Multiplying numerator and denominator by $1 - \cos x$, We have-

$$Z = \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} \times \frac{1 - \cos x}{1 - \cos x}$$

$$\Rightarrow Z = \lim_{x \rightarrow \pi} \frac{1 - \cos^2 x}{\tan^2 x (1 - \cos x)}$$

$$\{\text{As } 1 - \cos^2 x = \sin^2 x\}$$

$$\Rightarrow Z = \lim_{x \rightarrow \pi} \frac{\sin^2 x}{\tan^2 x (1 - \cos x)}$$

$$\Rightarrow Z = \lim_{x \rightarrow \pi} \frac{1}{1 - \cos x} \times \lim_{x \rightarrow \pi} \frac{\sin^2 x}{\tan^2 x}$$

$$\Rightarrow Z = \lim_{x \rightarrow \pi} \frac{1}{1 - \cos \pi} \times \lim_{x \rightarrow \pi} \frac{\sin^2 x}{\tan^2 x}$$

$$\Rightarrow Z = \frac{1}{2} \lim_{x \rightarrow \pi} \frac{\sin^2 x}{\tan^2 x}$$

To apply sandwich theorem, we need to have limit such that variable tends to 0 and following forms should be there $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

Here $x \rightarrow \pi$ so we need to do modifications before applying the theorem.

$$\text{As, } \sin(\pi - x) = \sin x \text{ or } \sin(x - \pi) = -\sin x \text{ and } \tan(\pi - x) = -\tan x$$

$\therefore$ we can say that-

$$\sin^2 x = \sin^2(x - \pi) \text{ and } \tan^2 x = \tan^2(x - \pi)$$

$$\text{As } x \rightarrow \pi$$

$$\therefore (x - \pi) \rightarrow 0$$

Let us represent $x - \pi$ with y

$$\therefore Z = \frac{1}{2} \lim_{(x-\pi) \rightarrow 0} \frac{\sin^2(x-\pi)}{\tan^2(x-\pi)} = \frac{1}{2} \lim_{y \rightarrow 0} \frac{\sin^2 y}{\tan^2 y}$$

Dividing both numerator and denominator by y^2

$$Z = \frac{1}{2} \lim_{y \rightarrow 0} \frac{\frac{(\sin^2 y)}{y^2}}{\frac{\tan^2 y}{y^2}}$$

$$\Rightarrow Z = \frac{1}{2} \frac{\lim_{y \rightarrow 0} \left(\frac{\sin y}{y}\right)^2}{\lim_{y \rightarrow 0} \left(\frac{\tan y}{y}\right)^2} \text{ {Using basic limits algebra}}$$

$$\text{As, } \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\therefore Z = \frac{1}{2} \times \frac{1}{1} = \frac{1}{2}$$

$$\therefore \lim_{x \rightarrow \pi} \frac{1 + \cos x}{\tan^2 x} = \frac{1}{2}$$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2\left(\frac{\pi}{4}\right) - 2}{\cot \frac{\pi}{4} - 1} = \frac{(\sqrt{2})^2 - 2}{1 - 1} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

$$\text{As } Z = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$$

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x \left(1 - \frac{2}{\operatorname{cosec}^2 x}\right)}{\cot x \left(1 - \frac{1}{\cot x}\right)} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x (1 - 2 \sin^2 x)}{\cot x (1 - \tan x)}$$

$$\therefore \cot x = \frac{\operatorname{cosec} x}{\sec x}$$

$$\therefore Z = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sec x \operatorname{cosec} x (1 - 2 \sin^2 x)}{1 - \tan x}$$

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{4}} (\sec x \operatorname{cosec} x) \times \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{1 - 2 \sin^2 x}{1 - \tan x}\right)$$

{Using basic limits algebra}

$$\Rightarrow Z = \sec \frac{\pi}{4} \operatorname{cosec} \frac{\pi}{4} \times \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{1 - 2 \sin^2 x}{1 - \tan x}\right) = 2 \times \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{1 - 2 \sin^2 x}{1 - \tan x}\right)$$

$$\therefore (1 - 2\sin^2 x) = \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$\therefore Z = 2 \times \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{\frac{1 - \tan^2 x}{1 + \tan^2 x}}{1 - \tan x} \right)$$

$$\Rightarrow Z = 2 \times \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{1 - \tan^2 x}{(1 - \tan x)(1 + \tan^2 x)} \right)$$

$$\text{As, } a^2 - b^2 = (a+b)(a-b)$$

$$\Rightarrow Z = 2 \times \lim_{x \rightarrow \frac{\pi}{4}} \frac{(1 - \tan x)(1 + \tan x)}{(1 - \tan x)(1 + \tan^2 x)} = 2 \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 + \tan x}{1 + \tan^2 x}$$

Now put the value of x, we have-

$$\therefore Z = 2 \left(\frac{1 + \tan \frac{\pi}{4}}{1 + \tan^2 \frac{\pi}{4}} \right) = 2 \times \left(\frac{2}{2} \right) = 2$$

Hence,

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1} = 2 \quad \dots \text{ans}$$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc)

$$\text{Let } Z = \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 \frac{\pi}{6} - 3}{\operatorname{cosec} \frac{\pi}{6} - 2} = \frac{(\sqrt{3})^2 - 3}{2 - 2} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

$$\text{As } Z = \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$$

$$\text{As, } a^2 - b^2 = (a+b)(a-b)$$

$$\therefore Z = \lim_{x \rightarrow \frac{\pi}{6}} \frac{(\cot x - \sqrt{3})(\cot x + \sqrt{3})}{\operatorname{cosec} x - 2}$$

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{6}} (\cot x + \sqrt{3}) \lim_{x \rightarrow \frac{\pi}{6}} \left(\frac{\cot x - \sqrt{3}}{\operatorname{cosec} x - 2} \right)$$

$$\Rightarrow Z = \left(\cot \frac{\pi}{6} + \sqrt{3} \right) \lim_{x \rightarrow \frac{\pi}{6}} \left(\frac{\cot x - \sqrt{3}}{\operatorname{cosec} x - 2} \right)$$

$$\Rightarrow Z = 2\sqrt{3} \lim_{x \rightarrow \frac{\pi}{6}} \left(\frac{\cot x - \sqrt{3}}{\operatorname{cosec} x - 2} \right)$$

Multiplying cosec x + 2 to both numerator and denominator-

$$Z = 2\sqrt{3} \lim_{x \rightarrow \frac{\pi}{6}} \left(\frac{\cot x - \sqrt{3}}{\csc x - 2} \right) \left(\frac{\csc x + 2}{\csc x + 2} \right) = 2\sqrt{3} \lim_{x \rightarrow \frac{\pi}{6}} \frac{(\cot x - \sqrt{3})(\csc x + 2)}{\csc^2 x - 4}$$

$$Z = 2\sqrt{3} \lim_{x \rightarrow \frac{\pi}{6}} (\csc x + 2) \times \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot x - \sqrt{3}}{\csc^2 x - 1 - 3}$$

As, $\csc^2 x - 1 = \cot^2 x$

$$\therefore Z = 2\sqrt{3} \left(\csc \frac{\pi}{6} + 2 \right) \times \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot x - \sqrt{3}}{\cot^2 x - 3} = 8\sqrt{3} \times \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot x - \sqrt{3}}{(\cot x - \sqrt{3})(\cot x + \sqrt{3})}$$

$$\Rightarrow Z = 8\sqrt{3} \lim_{x \rightarrow \frac{\pi}{6}} \frac{1}{\cot x + \sqrt{3}} = 8\sqrt{3} \times \frac{1}{\cot \frac{\pi}{6} + \sqrt{3}} = \frac{8\sqrt{3}}{2\sqrt{3}} = 4$$

$$\therefore \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\csc x - 2} = 4$$

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - \csc^2 x}{1 - \cot x}$$

Answer

As we need to find $\lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - \csc^2 x}{1 - \cot x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - \csc^2 x}{1 - \cot x} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - \csc^2 \frac{\pi}{4}}{1 - \cot \frac{\pi}{4}} = \frac{2 - (\sqrt{2})^2}{1 - 1} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

$$\text{As } Z = \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - \csc^2 x}{1 - \cot x}$$

$$\because \csc^2 x - 1 = \cot^2 x$$

$$\therefore Z = \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - (\csc^2 x - 1)}{1 - \cot x} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \cot^2 x}{1 - \cot x}$$

$$\text{As, } a^2 - b^2 = (a+b)(a-b)$$

Thus,

$$Z = \lim_{x \rightarrow \frac{\pi}{4}} \frac{(1 - \cot x)(1 + \cot x)}{1 - \cot x} = \lim_{x \rightarrow \frac{\pi}{4}} (1 + \cot x)$$

$$\therefore Z = 1 + \cot \frac{\pi}{4} = 1 + 1 = 2$$

Hence,

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{2 - \csc^2 x}{1 - \cot x} = 2 \quad \dots \text{ans}$$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$$

Answer

As we need to find $\lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos \pi} - 1}{(\pi - \pi)^2} = \frac{\sqrt{2 - 1} - 1}{(\pi - \pi)^2} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

Note: While modifying be careful that you don't introduce any zero terms in the denominator

$$\text{As } Z = \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$$

Multiplying numerator and denominator by $\sqrt{2 + \cos x} + 1$, we have-

$$Z = \lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} \times \frac{\sqrt{2 + \cos x} + 1}{\sqrt{2 + \cos x} + 1}$$

$$\Rightarrow Z = \lim_{x \rightarrow \pi} \frac{(\sqrt{2 + \cos x})^2 - 1^2}{(\pi - x)^2 \sqrt{2 + \cos x} + 1}$$

{using $a^2 - b^2 = (a+b)(a-b)$ }

$$\Rightarrow Z = \lim_{x \rightarrow \pi} \frac{2 + \cos x - 1}{(\pi - x)^2} \lim_{x \rightarrow \pi} \frac{1}{\sqrt{2 + \cos x} + 1}$$

{using basic algebra of limits}

$$\Rightarrow Z = \frac{1}{\sqrt{2 + \cos \pi} + 1} \lim_{x \rightarrow \pi} \frac{1 + \cos x}{(\pi - x)^2} = \frac{1}{2} \lim_{x \rightarrow \pi} \frac{1 + \cos x}{(\pi - x)^2}$$

$$\text{As, } 1 + \cos x = 2 \cos^2(x/2)$$

$$\therefore Z = \frac{1}{2} \lim_{x \rightarrow \pi} \frac{2 \cos^2\left(\frac{x}{2}\right)}{(\pi - x)^2}$$

Tip: Similar limit problems involving trigonometric ratios along with algebraic equations are mostly solved using sandwich theorem. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

So to solve this problem we need to have a sin term so that we can make use of sandwich theorem.

$$\therefore \sin(\pi/2 - x) = \cos x$$

$$\therefore Z = \frac{1}{2} \lim_{x \rightarrow \pi} \frac{2 \sin^2\left(\frac{\pi - x}{2}\right)}{(\pi - x)^2}$$

$$\text{As } x \rightarrow \pi \Rightarrow \pi - x \rightarrow 0$$

$$\text{Let } y = \pi - x$$

$$Z = \frac{1}{2} \lim_{y \rightarrow 0} \frac{2 \sin^2\left(\frac{y}{2}\right)}{y^2}$$

To apply sandwich theorem we have to get the similar form as described below-

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\therefore Z = \frac{1}{2} \lim_{y \rightarrow 0} \frac{2 \sin^2\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)^2 \times 4} = \frac{1}{4} \lim_{y \rightarrow 0} \left(\frac{\sin\left(\frac{y}{2}\right)}{\frac{y}{2}}\right)^2$$

$$\Rightarrow Z = \frac{1}{4} \times 1 = \frac{1}{4}$$

Hence,

$$\lim_{x \rightarrow \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \frac{1}{4} \quad \dots \text{ans}$$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{3\pi}{2}} \frac{1 + \operatorname{cosec}^2 x}{\cot^2 x}$$

Answer

As we need to find $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{1 + \operatorname{cosec}^2 x}{\cot^2 x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \frac{3\pi}{2}} \frac{1 + \operatorname{cosec}^2 x}{\cot^2 x} = \lim_{x \rightarrow \frac{3\pi}{2}} \frac{1 + \operatorname{cosec}^2\left(\frac{3\pi}{2}\right)}{\cot^2\left(\frac{3\pi}{2}\right)} = \frac{1+1}{0} = \frac{2}{0} = \infty$$

$\therefore Z$ is not taking an indeterminate form.

$\therefore$ Limiting the value of Z is not defined.

Hence,

$$\lim_{x \rightarrow \frac{3\pi}{2}} \frac{1 + \operatorname{cosec}^2 x}{\cot^2 x} = \infty$$

Exercise 29.10

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{5^x - 1}{\sqrt{4 + x} - 2}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{5^x - 1}{\sqrt{4 + x} - 2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{5^x - 1}{\sqrt{4 + x} - 2} = \lim_{x \rightarrow 0} \frac{5^0 - 1}{\sqrt{4 + 0} - 2} = \frac{1 - 1}{2 - 2} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

Note: While modifying be careful that you don't introduce any zero terms in the denominator

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$$

Multiplying both numerator and denominator by $\sqrt{4+x} + 2$ so that we can remove the indeterminate form.

$$\therefore Z = \lim_{x \rightarrow 0} \frac{5^x - 1}{\sqrt{4+x} - 2} \times \frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{(5^x - 1)\sqrt{4+x} + 2}{(\sqrt{4+x})^2 - 2^2}$$

{using $a^2 - b^2 = (a + b)(a - b)$ }

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{(5^x - 1)\sqrt{4+x} + 2}{4+x-4} = \lim_{x \rightarrow 0} \frac{(5^x - 1)\sqrt{4+x} + 2}{x}$$

Using basic algebra of limits-

$$Z = \lim_{x \rightarrow 0} \frac{(5^x - 1)}{x} \times \lim_{x \rightarrow 0} \sqrt{4+x} + 2 = \{\sqrt{4+0} + 2\} \lim_{x \rightarrow 0} \frac{(5^x - 1)}{x}$$

$$\Rightarrow Z = 4 \lim_{x \rightarrow 0} \frac{(5^x - 1)}{x}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = 4 \log 5$$

$$\text{Or, } \lim_{x \rightarrow 0} \frac{5^x - 1}{\sqrt{4+x} - 2} = 4 \log 5$$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{3^x - 1}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{\log(1+x)}{3^x - 1}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{\log(1+x)}{3^x - 1} = \lim_{x \rightarrow 0} \frac{\log(1+0)}{3^0 - 1} = \frac{\log 1}{1-1} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

This question is a direct application of limits formula of exponential and logarithmic limits.

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a \text{ and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{\log(1+x)}{3^x - 1}$$

To get the above forms, we need to divide numerator and denominator by x .

$$\therefore Z = \lim_{x \rightarrow 0} \frac{\frac{\log(1+x)}{x}}{\frac{3^x - 1}{x}} = \frac{\lim_{x \rightarrow 0} \frac{\log(1+x)}{x}}{\lim_{x \rightarrow 0} \frac{3^x - 1}{x}} \text{ {using basic limit algebra}}$$

$$\Rightarrow Z = \frac{1}{\log 3} \text{ {using the formulae described above}}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{3^x - 1} = \frac{1}{\log 3}$$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{a^x + a^{-x} - 2}{x^2}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{a^x + a^{-x} - 2}{x^2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{a^x + a^{-x} - 2}{x^2} = \lim_{x \rightarrow 0} \frac{a^0 + a^{-0} - 2}{0^2} = \frac{1+1-2}{0^2} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{a^x + a^{-x} - 2}{x^2} = \lim_{x \rightarrow 0} \frac{a^{-x}(a^{2x} - 2a^x + 1)}{x^2}$$

$$\therefore Z = \lim_{x \rightarrow 0} \frac{(a^{2x} - 2a^x + 1)}{a^x x^2} = \lim_{x \rightarrow 0} \frac{(a^x - 1)^2}{a^x x^2} \text{ \{using } (a+b)^2 = a^2 + b^2 + 2ab\}}$$

Using algebra of limit, we can write that

$$Z = \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right)^2 \times \lim_{x \rightarrow 0} \frac{1}{a^x}$$

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$$

$$\therefore Z = (\log a)^2 \frac{1}{a^0} = (\log a)^2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{a^x + a^{-x} - 2}{x^2} = (\log a)^2$$

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{a^{nx} - 1}{b^{nx} - 1}, n \neq 0$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{a^{nx} - 1}{b^{nx} - 1}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{a^{nx} - 1}{b^{nx} - 1} = \lim_{x \rightarrow 0} \frac{a^{n \cdot 0} - 1}{b^{n \cdot 0} - 1} = \frac{1-1}{1-1} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get the desired forms, we need to include mx and nx as follows:

$$\therefore Z = \lim_{x \rightarrow 0} \frac{a^{mx} - 1}{b^{nx} - 1} = \lim_{x \rightarrow 0} \frac{\frac{a^{mx} - 1}{mx} \times mx}{\frac{b^{nx} - 1}{nx} \times nx}$$

$$\Rightarrow Z = \frac{m}{n} \lim_{x \rightarrow 0} \frac{\frac{a^{mx} - 1}{mx}}{\frac{b^{nx} - 1}{nx}}$$

Using algebra of limits-

$$Z = \frac{m \lim_{x \rightarrow 0} \frac{a^{mx} - 1}{mx}}{n \lim_{x \rightarrow 0} \frac{b^{nx} - 1}{nx}}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \frac{m}{n} \frac{\log a}{\log b}, n \neq 0$$

Hence,

$$\lim_{x \rightarrow 0} \frac{a^{mx} - 1}{b^{nx} - 1} = \frac{m}{n} \frac{\log a}{\log b}, n \neq 0$$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{a^x + b^x - 2}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{a^x + b^x - 2}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{a^x + b^x - 2}{x} = \lim_{x \rightarrow 0} \frac{a^0 + b^0 - 2}{x} = \frac{1+1-2}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in a formula, we move as follows-

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{a^x + b^x - 2}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{a^x - 1 + b^x - 1}{x}$$

Using algebra of limits we have-

$$Z = \lim_{x \rightarrow 0} \frac{a^x - 1}{x} + \lim_{x \rightarrow 0} \frac{b^x - 1}{x}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \log a + \log b = \log ab$$

Hence,

$$\lim_{x \rightarrow 0} \frac{a^x + b^x - 2}{x} = \log ab$$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{9^x - 2 \cdot 6^x + 4^x}{x^2}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{9^x - 2 \cdot 6^x + 4^x}{x^2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{9^x - 2 \cdot 6^x + 4^x}{x^2} = \lim_{x \rightarrow 0} \frac{9^0 - 2 \cdot 6^0 + 4^0}{x^2} = \frac{1 + 1 - 2}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{9^x - 2 \cdot 6^x + 4^x}{x^2} = \lim_{x \rightarrow 0} \frac{(3^x)^2 - 2 \cdot 3^x \cdot 2^x + (2^x)^2}{x^2}$$

$$\therefore Z = \lim_{x \rightarrow 0} \frac{(3^x - 2^x)^2}{x^2}$$

{using $(a-b)^2 = a^2 + b^2 - 2ab$ }

$$Z = \lim_{x \rightarrow 0} \left(\frac{3^x - 2^x}{x} \right)^2$$

To apply the formula we need to bring the exact form present in the formula, so-

$$Z = \lim_{x \rightarrow 0} \left(\frac{3^x - 1 - 2^x + 1}{x} \right)^2$$

{Adding and subtracting 1 in numerator}

$$\Rightarrow Z = \lim_{x \rightarrow 0} \left(\frac{3^x - 1}{x} - \frac{2^x - 1}{x} \right)^2$$

Using algebra of limits-

$$Z = \left(\lim_{x \rightarrow 0} \frac{3^x - 1}{x} - \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \right)^2$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = (\log 3 - \log 2)^2 = \left(\log \frac{3}{2} \right)^2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{9^x - 2 \cdot 6^x + 4^x}{x^2} = \left(\log \frac{3}{2} \right)^2$$

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{8^x - 4^x - 2^x + 1}{x^2}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{8^x - 4^x - 2^x + 1}{x^2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{8^x - 4^x - 2^x + 1}{x^2} = \lim_{x \rightarrow 0} \frac{8^0 - 4^0 - 2^0 + 1}{x^2} = \frac{2 - 2}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{8^x - 4^x - 2^x + 1}{x^2} = \lim_{x \rightarrow 0} \frac{4^x(2^x - 1) - 1(2^x - 1)}{x^2} = \lim_{x \rightarrow 0} \frac{(4^x - 1)(2^x - 1)}{x^2}$$

Using Algebra of limits-

We have-

$$Z = \lim_{x \rightarrow 0} \frac{(4^x - 1)}{x} \times \lim_{x \rightarrow 0} \frac{(2^x - 1)}{x}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \log 4 \times \log 2$$

$$\because \log 4 = \log 2^2 = 2 \log 2$$

{using properties of log}

$$\therefore Z = 2(\log 2)^2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{8^x - 4^x - 2^x + 1}{x^2} = 2(\log 2)^2$$

8. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{x} = \lim_{x \rightarrow 0} \frac{a^{m \cdot 0} - b^{n \cdot 0}}{x} = \frac{1-1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get the desired forms, we need to include mx and nx as follows:

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{a^{mx} - 1 - b^{nx} + 1}{x}$$

{Adding and subtracting 1 in numerator}

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{a^{mx} - 1}{x} - \lim_{x \rightarrow 0} \frac{b^{nx} - 1}{x}$$

{using algebra of limits}

To get the form as present in the formula we multiply and divide m and n into both terms respectively:

$$\therefore Z = \lim_{x \rightarrow 0} \frac{a^{mx} - 1}{mx} \times m - \lim_{x \rightarrow 0} \frac{b^{nx} - 1}{nx} \times n$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = m \log a - n \log b = \log \left(\frac{a^m}{b^n} \right)$$

{using properties of log}

Hence,

$$\lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{x} = \log \left(\frac{a^m}{b^n} \right)$$

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{a^x + b^x + c^x - 3}{x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{a^x + b^x + c^x - 3}{x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or ∞-∞, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{a^x + b^x + c^x - 3}{x} = \lim_{x \rightarrow 0} \frac{a^0 + b^0 + c^0 - 3}{x} = \frac{1+1+1-3}{0} = \frac{0}{0}$$

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in a formula, we move as follows-

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{a^x + b^x + c^x - 3}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{a^x - 1 + b^x - 1 + c^x - 1}{x}$$

Using algebra of limits we have-

$$Z = \lim_{x \rightarrow 0} \frac{a^x - 1}{x} + \lim_{x \rightarrow 0} \frac{b^x - 1}{x} + \lim_{x \rightarrow 0} \frac{c^x - 1}{x}$$

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$$

$$\therefore Z = \log a + \log b + \log c = \log abc$$

Hence,

$$\lim_{x \rightarrow 0} \frac{a^x + b^x + c^x - 3}{x} = \log abc$$

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{x - 2}{\log_a(x - 1)}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 2} \frac{x - 2}{\log_a(x - 1)}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 2} \frac{x - 2}{\log_a(x - 1)} = \lim_{x \rightarrow 2} \frac{2 - 2}{\log_a(2 - 1)} = \frac{2 - 2}{\log 1} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in a formula, we move as follows-

$$Z = \lim_{x \rightarrow 2} \frac{x - 2}{\log_a(1 + x - 2)}$$

$$\text{As } x \rightarrow 2 \therefore x - 2 \rightarrow 0$$

$$\text{Let } x - 2 = y$$

$$\therefore Z = \lim_{y \rightarrow 0} \frac{y}{\log_a(1+y)} = \lim_{y \rightarrow 0} \frac{1}{\frac{\log_a(1+y)}{y}}$$

We can't use the formula directly as the base of log is we need to change this to e.

Applying the formula for change of base-

$$\text{We have- } \log_a(1+y) = \frac{\log_e(1+y)}{\log_e a}$$

$$\therefore Z = \lim_{y \rightarrow 0} \frac{1}{\frac{\log_e(1+y)}{\log_e a}} = \frac{\log_e a}{\lim_{y \rightarrow 0} \frac{\log_e(1+y)}{y}}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\therefore Z = \log_e a = \log a$$

Hence,

$$\lim_{x \rightarrow 2} \frac{x-2}{\log_a(x-1)} = \log a$$

11. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{5^x + 3^x + 2^x - 3}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{5^x + 3^x + 2^x - 3}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{5^x + 3^x + 2^x - 3}{x} = \lim_{x \rightarrow 0} \frac{5^0 + 3^0 + 2^0 - 3}{x} = \frac{1+1+1-3}{0} = \frac{0}{0}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in formula, we move as follows-

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{5^x + 3^x + 2^x - 3}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{5^x - 1 + 3^x - 1 + 2^x - 1}{x}$$

Using algebra of limits we have-

$$Z = \lim_{x \rightarrow 0} \frac{5^x - 1}{x} + \lim_{x \rightarrow 0} \frac{3^x - 1}{x} + \lim_{x \rightarrow 0} \frac{2^x - 1}{x}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \log 5 + \log 3 + \log 2 = \log (5 \times 3 \times 2)$$

Hence,

$$\lim_{x \rightarrow 0} \frac{5^x + 3^x + 2^x - 3}{x} = \log 30$$

12. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} (a^{1/x} - 1)x$$

Answer

As we need to find $\lim_{x \rightarrow \infty} (a^{1/x} - 1)x$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \infty} \left(a^{\frac{1}{x}} - 1 \right) x = \lim_{x \rightarrow \infty} \left(a^{\frac{1}{\infty}} - 1 \right) x \infty = 0 \times \infty = (\text{indeterminate})$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in formula, we move as follows-

$$\text{Let } 1/x = y$$

$$\text{As } x \rightarrow \infty \Rightarrow y \rightarrow 0$$

$\therefore$ Z can be rewritten as-

$$Z = \lim_{y \rightarrow 0} \frac{(a^y - 1)}{y}$$

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$$

$$\therefore Z = \log a$$

Hence,

$$\lim_{x \rightarrow \infty} \left(a^{\frac{1}{x}} - 1 \right) x = \log a$$

13. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{\sin kx}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{\sin kx}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{\sin kx} = \lim_{x \rightarrow 0} \frac{a^{m \cdot 0} - b^{n \cdot 0}}{\sin 0} = \frac{1-1}{0} = \frac{0}{0} (\text{indeterminate form})$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits and also use of sandwich theorem - $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

To get the desired forms, we need to include mx and nx as follows:

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{\sin kx}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{a^{mx} - 1 - b^{nx} + 1}{\sin kx} \text{ \{Adding and subtracting 1 in numerator\}}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{a^{mx}-1}{\sin kx} - \lim_{x \rightarrow 0} \frac{b^{nx}-1}{\sin kx}$$

{using algebra of limits}

To get the form as present in the formula we multiply and divide x into both terms respectively:

$$\therefore Z = \lim_{x \rightarrow 0} \frac{a^{mx}-1}{\frac{\sin kx}{x}} - \lim_{x \rightarrow 0} \frac{b^{nx}-1}{\frac{\sin kx}{x}}$$

{manipulating to get the forms present in formulae}

$$Z = \lim_{x \rightarrow 0} \frac{a^{mx}-1}{\frac{\sin kx}{x}} \times \frac{x}{kx} - \lim_{x \rightarrow 0} \frac{b^{nx}-1}{\frac{\sin kx}{x}} \times \frac{x}{kx}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x-1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = \frac{m \log a}{k} - \frac{n \log b}{k} = \frac{1}{k} (m \log a - n \log b)$$

Hence,

$$\lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{\sin kx} = \frac{1}{k} \log \left(\frac{a^m}{b^n} \right)$$

14. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{a^x + b^x - c^x - d^x}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{a^x + b^x - c^x - d^x}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{a^x + b^x - c^x - d^x}{x} = \lim_{x \rightarrow 0} \frac{a^0 + b^0 - c^0 - d^0}{x} = \frac{1+1-1-1}{0} = \frac{0}{0}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x-1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get similar forms as in a formula, we move as follows-

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{a^x + b^x - c^x - d^x}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{a^x-1 + b^x-1 - c^x+1 - d^x+1}{x}$$

Using algebra of limits we have-

$$Z = \lim_{x \rightarrow 0} \frac{a^x-1}{x} + \lim_{x \rightarrow 0} \frac{b^x-1}{x} - \lim_{x \rightarrow 0} \frac{c^x-1}{x} - \lim_{x \rightarrow 0} \frac{d^x-1}{x}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x-1)}{x} = \log a$

$$\therefore Z = \log a + \log b - \log c - \log d = \log \frac{ab}{cd}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{a^x + b^x - c^x - d^x}{x} = \log\left(\frac{ab}{cd}\right)$$

15. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^x - 1 + \sin x}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^x - 1 + \sin x}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^x - 1 + \sin x}{x} = \lim_{x \rightarrow 0} \frac{e^0 - 1 + \sin 0}{0} = \frac{1 - 1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^x - 1 + \sin x}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^x - 1}{x} + \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = \log e + 1$$

$$\{\because \log e = 1\}$$

$$\Rightarrow Z = 1 + 1 = 2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^x - 1 + \sin x}{x} = 2$$

16. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{e^x - 1}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{\sin 2x}{e^x - 1}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{\sin 2x}{e^x - 1} = \lim_{x \rightarrow 0} \frac{\sin 0}{e^0 - 1} = \frac{0}{1 - 1} = \frac{0}{0} \text{ (indeterminate form)}$$

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{\sin 2x}{e^x - 1}$$

To get the desired form to apply the formula we need to divide numerator and denominator by x.

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{x}}{\frac{e^x - 1}{x}}$$

Using algebra of limits, we have-

$$Z = \frac{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2}{\lim_{x \rightarrow 0} \frac{e^x - 1}{x}}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = \frac{2}{\log e}$$

$$\{\because \log e = 1\}$$

$$\Rightarrow Z = 2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{e^x - 1} = 2$$

17. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty \cdot \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x} = \lim_{x \rightarrow 0} \frac{e^{\sin 0} - 1}{0} = \frac{1-1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x}$$

To get rid of indeterminate form we will divide numerator and denominator by sin x

$$\therefore Z = \lim_{x \rightarrow 0} \frac{\frac{e^{\sin x} - 1}{\sin x}}{\frac{x}{\sin x}}$$

Using Algebra of limits we have-

$$Z = \frac{\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{\sin x}}{\lim_{x \rightarrow 0} \frac{x}{\sin x}} = \frac{A}{B}$$

$$\text{Where, } A = \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{\sin x}$$

$$\text{and } B = \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

{from sandwich theorem}

$$\text{As } A = \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{\sin x}$$

Let, $\sin x = y$

As $x \rightarrow 0 \Rightarrow y \rightarrow 0$

$$\therefore A = \lim_{y \rightarrow 0} \frac{e^y - 1}{y}$$

$$\text{Using } \lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$$

$$A = \log e = 1$$

$$\therefore Z = \frac{A}{B} = \frac{1}{1} = 1$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x} = 1$$

18. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^{2x} - e^x}{\sin 2x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{e^{2x} - e^x}{\sin 2x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^{2x} - e^x}{\sin 2x} = \lim_{x \rightarrow 0} \frac{e^0 - e^0}{\sin 0} = \frac{1-1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^{2x} - e^x}{\sin 2x}$$

Adding and subtracting 1 in the numerator to get the desired form

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^{2x} - 1 - e^x + 1}{\sin 2x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin 2x} - \lim_{x \rightarrow 0} \frac{e^x - 1}{\sin 2x}$$

{using algebra of limits}

To get the desired form to apply the formula we need to divide numerator and denominator by x.

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\frac{e^{2x} - 1}{2x}}{\frac{\sin 2x}{2x}} - \lim_{x \rightarrow 0} \frac{\frac{e^x - 1}{x}}{\frac{\sin 2x}{2x} \times 2}$$

Using algebra of limits, we have-

$$Z = \frac{\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{2x}}{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x}} - \frac{\lim_{x \rightarrow 0} \frac{e^x - 1}{x}}{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times 2}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = \frac{\log e}{1} - \frac{\log e}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

{ $\because \log e = 1$ }

$$\Rightarrow Z = 1/2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^{2x} - e^x}{\sin 2x} = \frac{1}{2}$$

19. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \frac{\log x - \log a}{x - a}$$

Answer

As we need to find $\lim_{x \rightarrow a} \frac{\log x - \log a}{x - a}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow a} \frac{\log x - \log a}{x - a} = \lim_{x \rightarrow a} \frac{\log a - \log a}{a - a} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\text{As } Z = \lim_{x \rightarrow a} \frac{\log x - \log a}{x - a}$$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

$\therefore$ We proceed as follows-

$$Z = \lim_{x \rightarrow a} \frac{\log x - \log a}{x - a} = \lim_{x \rightarrow a} \frac{\log\left(\frac{x}{a}\right)}{\frac{x}{a} - 1}$$

$$\Rightarrow Z = \lim_{x \rightarrow a} \frac{\log\left(\frac{x}{a}\right)}{\left(\frac{x}{a} - 1\right)}$$

$$\Rightarrow Z = \lim_{x \rightarrow a} \frac{\log\left(1 + \frac{x}{a} - 1\right)}{a\left(\frac{x}{a} - 1\right)}$$

$$\because x \rightarrow a \Rightarrow x/a \rightarrow 1$$

$$\Rightarrow x/a - 1 \rightarrow 0$$

$$\text{Let, } (x/a) - 1 = y$$

$$\therefore y \rightarrow 0$$

Hence, Z can be rewritten as-

$$Z = \lim_{y \rightarrow 0} \frac{\log(1+y)}{a(y)}$$

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\therefore Z = \frac{1}{a} \lim_{y \rightarrow 0} \frac{\log(1+y)}{(y)} = \frac{1}{a}$$

Hence,

$$\lim_{x \rightarrow a} \frac{\log x - \log a}{x - a} = \frac{1}{a}$$

20. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a-x)}{x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a-x)}{x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a-x)}{x} = \lim_{x \rightarrow 0} \frac{\log a - \log a}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a-x)}{x}$$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

$\therefore$ We proceed as follows-

$$Z = \lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a-x)}{x} = \lim_{x \rightarrow 0} \frac{\log\left(\frac{a+x}{a-x}\right)}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\log\left(\frac{a+x}{a-x}\right)}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{a-x}\right)}{x}$$

To apply the formula of logarithmic limit we need $\frac{2x}{a-x}$ denominator

∴ multiplying $\frac{2}{a-x}$ in numerator and denominator

Hence, Z can be rewritten as-

$$Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{a-x}\right)}{\frac{2x}{a-x}} \times \frac{2}{a-x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{a-x}\right)}{\frac{2x}{a-x}} \times \lim_{x \rightarrow 0} \frac{2}{a-x}$$

{Using algebra of limits}

$$\Rightarrow Z = \frac{2}{a} \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{a-x}\right)}{\frac{2x}{a-x}}$$

$$\text{As, } x \rightarrow 0 \Rightarrow \frac{2x}{a-x} \rightarrow 0$$

$$\text{Let, } \frac{2x}{a-x} = y$$

$$\therefore Z = \frac{2}{a} \lim_{y \rightarrow 0} \frac{\log(1+y)}{y}$$

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\therefore Z = \frac{2}{a} \lim_{y \rightarrow 0} \frac{\log(1+y)}{(y)} = \frac{2}{a}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a-x)}{x} = \frac{2}{a}$$

21. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\log(2+x) + \log 0.5}{x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{\log(2+x) + \log 0.5}{x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{\log(2+x) + \log 0.5}{x} = \frac{\log(2+0) + \log 0.5}{0} = \frac{0}{0} \text{ (indeterminate)}$$

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{\log(2+x) + \log 0.5}{x}$$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

∴ We proceed as follows-

$$Z = \lim_{x \rightarrow 0} \frac{\log(2+x) + \log 0.5}{x} = \lim_{x \rightarrow 0} \frac{\log\{(2+x) \times 0.5\}}{x}$$

{using properties of log}

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{2}\right)}{x}$$

To apply the formula of logarithmic limit, we need the $x/2$ denominator

$\therefore$ multiplying $1/2$ in numerator and denominator

Hence, Z can be rewritten as-

$$Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{2}\right)}{\frac{x}{2}} \times \frac{1}{2}$$

$$\Rightarrow Z = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{2}\right)}{\frac{x}{2}}$$

{Using algebra of limits}

$$\text{As } x \rightarrow 0 \Rightarrow \frac{x}{2} \rightarrow 0$$

$$\text{Let, } \frac{x}{2} = y$$

$$\therefore Z = \frac{1}{2} \lim_{y \rightarrow 0} \frac{\log(1+y)}{y}$$

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\therefore Z = \frac{1}{2} \lim_{y \rightarrow 0} \frac{\log(1+y)}{(y)} = \frac{1}{2}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\log(2+x) + \log 0.5}{x} = \frac{1}{2}$$

22. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\log(a+x) - \log a}{x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{\log(a+x) - \log a}{x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{\log(a+x) - \log a}{x} = \lim_{x \rightarrow 0} \frac{\log a - \log a}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a)}{x}$$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

$\therefore$ We proceed as follows-

$$Z = \lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a)}{x} = \lim_{x \rightarrow 0} \frac{\log\left(\frac{a+x}{a}\right)}{x} \text{ {using properties of log}}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{a}\right)}{x}$$

To apply the formula of logarithmic limit, we need x/a in the denominator

$\therefore$ multiplying $1/a$ in numerator and denominator

Hence, Z can be rewritten as-

$$Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{a}\right)}{\frac{x}{a}} \times \frac{1}{a}$$

$$\Rightarrow Z = \frac{1}{a} \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{a}\right)}{\frac{x}{a}}$$

{Using algebra of limits}

$$\text{As } x \rightarrow 0 \Rightarrow \frac{x}{a} \rightarrow 0$$

$$\text{Let, } \frac{x}{a} = y$$

$$\therefore Z = \frac{1}{a} \lim_{y \rightarrow 0} \frac{\log(1+y)}{y}$$

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\therefore Z = \frac{1}{a} \lim_{y \rightarrow 0} \frac{\log(1+y)}{(y)} = \frac{1}{a}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\log(a+x) - \log(a)}{x} = \frac{1}{a}$$

23. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = \lim_{x \rightarrow 0} \frac{\log 3 - \log 3}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x}$$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

$\therefore$ We proceed as follows-

$$Z = \lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = \lim_{x \rightarrow 0} \frac{\log\left(\frac{3+x}{3-x}\right)}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\log\left(\frac{3+x}{3-x}\right)}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{3-x}\right)}{x}$$

To apply the formula of logarithmic limit we need $\frac{2x}{3-x}$ denominator

$\therefore$ multiplying $\frac{2}{3-x}$ in numerator and denominator

Hence, Z can be rewritten as-

$$Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{3-x}\right)}{\frac{2x}{3-x}} \times \frac{2}{3-x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{3-x}\right)}{\frac{2x}{3-x}} \times \lim_{x \rightarrow 0} \frac{2}{3-x}$$

{Using algebra of limits}

$$\Rightarrow Z = \frac{2}{3} \lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{2x}{3-x}\right)}{\frac{2x}{3-x}}$$

$$\text{As, } x \rightarrow 0 \Rightarrow \frac{2x}{3-x} \rightarrow 0$$

$$\text{Let, } \frac{2x}{3-x} = y$$

$$\therefore Z = \frac{2}{3} \lim_{y \rightarrow 0} \frac{\log(1+y)}{y}$$

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\therefore Z = \frac{2}{3} \lim_{y \rightarrow 0} \frac{\log(1+y)}{(y)} = \frac{2}{3}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = \frac{2}{3}$$

24. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{8^x - 2^x}{x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{8^x - 2^x}{x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{8^x - 2^x}{x} = \lim_{x \rightarrow 0} \frac{8^0 - 2^0}{0} = \frac{1-1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

This question is a direct application of limits formula of exponential and logarithmic limits.

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{8^x - 2^x}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{8^x - 1 - 2^x + 1}{x}$$

{Adding and subtracting 1 in numerator}

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{8^x - 1}{x} - \lim_{x \rightarrow 0} \frac{2^x - 1}{x}$$

{using algebra of limits}

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$$

$$\therefore Z = \log 8 - \log 2 = \log\left(\frac{8}{2}\right) = \log 4$$

{using properties of log}

Hence,

$$\lim_{x \rightarrow 0} \frac{8^x - 2^x}{x} = \log 4$$

25. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x(2^x - 1)}{1 - \cos x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{1 - \cos x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{0(2^0 - 1)}{1 - \cos 0} = \frac{0}{1 - 1} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{1 - \cos x}$$

$$\text{As, } 1 - \cos x = 2\sin^2(x/2)$$

$$\therefore Z = \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{2\sin^2\left(\frac{x}{2}\right)}$$

$$\Rightarrow Z = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{\sin^2\left(\frac{x}{2}\right)}$$

To get the desired form to apply the formula we need to divide numerator and denominator by x^2 .

$$\Rightarrow Z = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\frac{x(2^x - 1)}{x^2}}{\frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2 \times 4}} = \frac{4}{2} \lim_{x \rightarrow 0} \frac{\frac{(2^x - 1)}{x}}{\left(\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\right)^2}$$

Using algebra of limits, we have-

$$Z = 2 \frac{\lim_{x \rightarrow 0} \frac{(2^x - 1)}{x}}{\lim_{x \rightarrow 0} \left(\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\right)^2}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = 2 \frac{\log 2}{1^2}$$

$$\Rightarrow Z = 2 \log 2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{x(2^x - 1)}{1 - \cos x} = 2 \log 2$$

26. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\log(1+x)}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\log(1+x)}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\log(1+x)} = \lim_{x \rightarrow 0} \frac{\sqrt{1+0} - 1}{\log(1+0)} = \frac{1-1}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\log(1+x)}$$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

$\therefore$ multiplying numerator and denominator by $\sqrt{1+x} + 1$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\log(1+x)} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x})^2 - 1^2}{\log(1+x) \times (\sqrt{1+x} + 1)}$$

{using $(a+b)(a-b) = a^2 - b^2$ }

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{1+x-1}{\log(1+x)} \times \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x} + 1}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{x}{\log(1+x)} \times \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x}{\log(1+x)}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\therefore Z = 1/2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{\log(1+x)} = \frac{1}{2}$$

27. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{\log|1+x^3|}{\sin^3 x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{\log|1+x^3|}{\sin^3 x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{\log|1+x^3|}{\sin^3 x} = \lim_{x \rightarrow 0} \frac{\log|1+0^3|}{\sin^3 0} = \frac{\log 1}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{\log|1+x^3|}{\sin^3 x}$$

To apply the formula of logarithmic limits we need to get the form that matches with one in formula

$\therefore$ dividing numerator and denominator by x^3

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\frac{\log|1+x^3|}{x^3}}{\frac{\sin^3 x}{x^3}}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\frac{\log|1+x^3|}{x^3}}{\left(\frac{\sin x}{x}\right)^3}$$

$$\Rightarrow Z = \frac{\lim_{x \rightarrow 0} \frac{\log|1+x^3|}{x^3}}{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^3}$$

{using algebra of limits}

Use the formula: $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = 1/1$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\log|1+x^3|}{\sin^3 x} = 1$$

28. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$$

Answer

As we need to find $\lim_{x \rightarrow \frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} = \frac{a^{\cot \frac{\pi}{2}} - a^{\cos \frac{\pi}{2}}}{\cot \frac{\pi}{2} - \cos \frac{\pi}{2}} = \frac{1-1}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

$$\text{As } Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$$

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{a^{\cos x} \left(\frac{a^{\cot x}}{a^{\cos x}} - 1 \right)}{\cot x - \cos x}$$

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{a^{\cos x} (a^{(\cot x - \cos x)} - 1)}{\cot x - \cos x}$$

{using properties of exponents}

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{(a^{(\cot x - \cos x)} - 1)}{\cot x - \cos x} \times \lim_{x \rightarrow \frac{\pi}{2}} a^{\cos x}$$

{using algebra of limits}

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{(a^{(\cot x - \cos x)} - 1)}{\cot x - \cos x} \times a^{\cos \frac{\pi}{2}} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{(a^{(\cot x - \cos x)} - 1)}{\cot x - \cos x} \times a^0$$

$$\therefore Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{(a^{(\cot x - \cos x)} - 1)}{\cot x - \cos x}$$

As, $x \rightarrow (\pi/2)$

$$\therefore \cot(\pi/2) - \cos(\pi/2) \rightarrow 0$$

Let, $y = \cot x - \cos x$

$$\therefore \text{if } x \rightarrow \pi/2 \Rightarrow y \rightarrow 0$$

Hence, Z can be rewritten as-

$$Z = \lim_{y \rightarrow 0} \frac{(a^y - 1)}{y}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \log a$$

Hence,

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} = \log a$$

29. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}} = \frac{e^0 - 1}{\sqrt{1 - \cos 0}} = \frac{1 - 1}{\sqrt{1 - 1}} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}}$$

To apply the formula we need to get the form as present in the formula. So we proceed as follows-

$$\therefore Z = \lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}}$$

Multiplying numerator and denominator by $\sqrt{1 + \cos x}$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}} \times \frac{\sqrt{1 + \cos x}}{\sqrt{1 + \cos x}}$$

Using $(a+b)(a-b) = a^2 - b^2$

$$Z = \lim_{x \rightarrow 0} \frac{(e^x - 1)\sqrt{1 + \cos x}}{\sqrt{1 - \cos^2 x}}$$

$$\because \sqrt{1 - \cos^2 x} = \sin x$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{(e^x - 1)}{\sin x} \times \lim_{x \rightarrow 0} \sqrt{1 + \cos x}$$

{using algebra of limits}

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{(e^x - 1)}{\sin x} \times \sqrt{1 + \cos 0} = \sqrt{2} \lim_{x \rightarrow 0} \frac{(e^x - 1)}{\sin x}$$

Dividing numerator and denominator by x-

$$Z = \sqrt{2} \lim_{x \rightarrow 0} \frac{\left(\frac{e^x - 1}{x}\right)}{\frac{\sin x}{x}}$$

$$\Rightarrow Z = \sqrt{2} \frac{\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x}\right)}{\lim_{x \rightarrow 0} \frac{\sin x}{x}}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = \sqrt{2} \frac{\log e}{1}$$

$$\{\because \log e = 1\}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}} = \sqrt{2}$$

30. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 5} \frac{e^x - e^5}{x - 5}$$

Answer

As we need to find $\lim_{x \rightarrow 5} \frac{e^x - e^5}{x - 5}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \frac{\lim_{x \rightarrow 5} (e^x - e^5)}{x - 5} = \frac{(e^5 - e^5)}{5 - 5} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

$$\text{As } Z = \lim_{x \rightarrow 5} \frac{e^x - e^5}{x - 5}$$

$$\Rightarrow Z = \lim_{x \rightarrow 5} \frac{e^5 \left(\frac{e^x}{e^5} - 1 \right)}{x - 5}$$

$$\Rightarrow Z = \lim_{x \rightarrow 5} \frac{e^5 (e^{x-5} - 1)}{x - 5}$$

{using properties of exponents}

$$\Rightarrow Z = e^5 \lim_{x \rightarrow 5} \frac{(e^{x-5} - 1)}{x - 5}$$

{using algebra of limits}

As, $x \rightarrow 5$

$$\therefore x - 5 \rightarrow 0$$

Let, $y = x - 5$

$$\therefore \text{if } x \rightarrow 5 \Rightarrow y \rightarrow 0$$

Hence, Z can be rewritten as-

$$Z = e^5 \lim_{y \rightarrow 0} \frac{(e^y - 1)}{y}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = e^5 \log e$$

$$\{\because \log e = 1\}$$

Hence,

$$\lim_{x \rightarrow 5} \frac{e^x - e^5}{x - 5} = e^5$$

31. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^{x+2} - e^2}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^{x+2} - e^2}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \frac{\lim_{x \rightarrow 0} (e^{x+2} - e^2)}{x} = \frac{(e^2 - e^2)}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^{x+2} - e^2}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^2(e^x - 1)}{x}$$

$$\Rightarrow Z = e^2 \lim_{x \rightarrow 0} \frac{(e^x - 1)}{x}$$

{using algebra of limits}

$$\text{Use the formula: } \lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$$

$$\therefore Z = e^2 \log e$$

$$\{\because \log e = 1\}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^{x+2} - e^2}{x} = e^2$$

32. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{e^{\cos x} - 1}{\cos x}$$

Answer

As we need to find $\lim_{x \rightarrow \frac{\pi}{2}} \frac{e^{\cos x} - 1}{\cos x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{e^{\cos x} - 1}{\cos x} = \frac{e^{\cos \frac{\pi}{2}} - 1}{\cos \frac{\pi}{2}} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

As $x \rightarrow \pi/2$

$\therefore \cos x \rightarrow 0$

Let, $y = \cos x$

$\therefore$ if $x \rightarrow \pi/2 \Rightarrow y \rightarrow 0$

Hence, Z can be rewritten as-

$$\lim_{y \rightarrow 0} \frac{(e^y - 1)}{y}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$\therefore Z = 1$

{ $\because \log e = 1$ }

Hence,

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{e^{\cos x} - 1}{\cos x} = 1$$

33. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^{3+x} - \sin x - e^3}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^{3+x} - \sin x - e^3}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^{3+x} - \sin x - e^3}{x} = \frac{e^{3+0} - \sin 0 - e^3}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^{3+x} - \sin x - e^3}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 5} \frac{e^3(e^x - 1) - \sin x}{x}$$

$$\Rightarrow Z = e^3 \lim_{x \rightarrow 5} \frac{(e^x - 1)}{x} - \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

{using algebra of limits}

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = e^3 \log e - 1 \quad \{\because \log e = 1\}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^{3+x} - \sin x - e^3}{x} = e^3 - 1$$

34. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{2}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{2}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^x - x - 1}{2} = \frac{e^0 - 0 - 1}{2} = \frac{1 - 1}{2} = 0 \text{ (not indeterminate)}$$

As we got a finite value, so no need to do any modifications.

Hence,

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{2} = 0$$

35. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x} = \frac{e^0 - e^0}{0} = \frac{1 - 1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^{3x} - 1 - e^{2x} + 1}{x}$$

{Adding and subtracting 1 in numerator}

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} - \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$$

{using algebra of limits}

To get the form as present in the formula we multiply and divide 3 and 2 into both terms respectively:

$$\Rightarrow Z = 3 \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{3x} - 2 \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{2x}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = 3 \log e - 2 \log e = 3 - 2 = 1$$

{using $\log e = 1$ }

Hence,

$$\lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{x} = 1$$

36. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x} = \frac{e^0 - 1}{\tan 0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

This question is a direct application of limits formula of exponential limits.

As, $x \rightarrow 0$

$$\therefore \tan x \rightarrow 0$$

Let, $y = \tan x$

$$\therefore \text{if } x \rightarrow 0 \Rightarrow y \rightarrow 0$$

Hence, Z can be rewritten as-

$$\lim_{y \rightarrow 0} \frac{(e^y - 1)}{y}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \log e = 1$$

$$\{\because \log e = 1\}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x} = 1$$

37. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^{bx} - e^{\sin x}}{x - \sin x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0} \frac{e^{bx} - e^{\sin x}}{bx - \sin x}$$

We can directly find the limiting value of a function by putting the value of variable at which the limiting value is asked, if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty$, .. etc)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^{bx} - e^{\sin x}}{bx - \sin x} = \frac{e^0 - e^{\sin 0}}{0 - \sin 0} = \frac{1-1}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential limits.

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^{bx} - e^{\sin x}}{bx - \sin x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^{\sin x} \left(\frac{e^{bx} - e^{\sin x}}{e^{\sin x}} - 1 \right)}{bx - \sin x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^{\sin x} (e^{bx - \sin x} - 1)}{bx - \sin x}$$

{using properties of exponents}

$$\Rightarrow Z = \lim_{x \rightarrow 0} e^{\sin x} \times \lim_{x \rightarrow 0} \frac{(e^{bx - \sin x} - 1)}{bx - \sin x}$$

{using algebra of limits}

$$\Rightarrow Z = e^{\sin 0} \times \lim_{x \rightarrow 0} \frac{(e^{bx - \sin x} - 1)}{bx - \sin x} = e^0 \times \lim_{x \rightarrow 0} \frac{(e^{bx - \sin x} - 1)}{bx - \sin x}$$

$$\therefore Z = \lim_{x \rightarrow 0} \frac{(e^{bx - \sin x} - 1)}{bx - \sin x}$$

As, $x \rightarrow 0$

$$\therefore bx - \sin x \rightarrow 0$$

Let, $y = bx - \sin x$

$$\therefore \text{if } x \rightarrow 0 \Rightarrow y \rightarrow 0$$

Hence, Z can be rewritten as-

$$Z = \lim_{y \rightarrow 0} \frac{(e^y - 1)}{y}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \log e = 1$$

$$\{\because \log e = 1\}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^{bx} - e^{\sin x}}{bx - \sin x} = 1$$

38. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty-\infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{x} = \frac{e^0 - 1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

This question is a direct application of limits formula of exponential limits.

$$\therefore Z = \lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{x}$$

To get the desired form, we proceed as follows-

Dividing numerator and denominator by $\tan x$ -

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{\frac{e^{\tan x} - 1}{\tan x}}{\frac{x}{\tan x}}$$

Using algebra of limits-

$$Z = \lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x} \times \lim_{x \rightarrow 0} \frac{\tan x}{x}$$

Use the formula - $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$ (sandwich theorem)

$$\therefore Z = \lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x} \times 1 = \lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{\tan x}$$

As, $x \rightarrow 0$

$$\therefore \tan x \rightarrow 0$$

Let, $y = \tan x$

$$\therefore \text{if } x \rightarrow 0 \Rightarrow y \rightarrow 0$$

Hence, Z can be rewritten as-

$$\lim_{y \rightarrow 0} \frac{(e^y - 1)}{y}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \log e = 1$$

$$\{\because \log e = 1\}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^{\tan x} - 1}{X} = 1$$

39. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \frac{e^0 - e^{\sin 0}}{0 - \sin 0} = \frac{1-1}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

This question is a direct application of limits formula of exponential limits.

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^{\sin x} (e^{\frac{x}{\sin x}} - 1)}{x - \sin x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{e^{\sin x} (e^{x - \sin x} - 1)}{x - \sin x}$$

{using properties of exponents}

$$\Rightarrow Z = \lim_{x \rightarrow 0} e^{\sin x} \times \lim_{x \rightarrow 0} \frac{(e^{x - \sin x} - 1)}{x - \sin x}$$

{using algebra of limits}

$$\Rightarrow Z = e^{\sin 0} \times \lim_{x \rightarrow 0} \frac{(e^{x - \sin x} - 1)}{x - \sin x} = e^0 \times \lim_{x \rightarrow 0} \frac{(e^{x - \sin x} - 1)}{x - \sin x}$$

$$\therefore Z = \lim_{x \rightarrow 0} \frac{(e^{x - \sin x} - 1)}{x - \sin x}$$

As, $x \rightarrow 0$

$$\therefore x - \sin x \rightarrow 0$$

Let, $y = x - \sin x$

$$\therefore \text{if } x \rightarrow 0 \Rightarrow y \rightarrow 0$$

Hence, Z can be rewritten as-

$$Z = \lim_{y \rightarrow 0} \frac{(e^y - 1)}{y}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = \log e = 1$$

$$\{\because \log e = 1\}$$

Hence,

$$\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} = 1$$

40. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{3^{2+x} - 9}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{3^{2+x} - 9}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{3^{x+2} - 3^2}{x} = \frac{3^2 - 3^2}{0} = \frac{0}{0} \text{ (indeterminate)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\text{and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

This question is a direct application of limits formula of exponential limits.

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{3^{x+2} - 3^2}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{3^2(3^x - 1)}{x}$$

$$\Rightarrow Z = 9 \lim_{x \rightarrow 0} \frac{(3^x - 1)}{x}$$

{using algebra of limits}

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = 9 \log 3$$

Hence,

$$\lim_{x \rightarrow 0} \frac{3^{x+2} - 9}{x} = 9 \log_e 3$$

41. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{a^x - a^{-x}}{x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{a^x - a^{-x}}{x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{a^x - a^{-x}}{x} = \lim_{x \rightarrow 0} \frac{a^0 - a^{-0}}{0} = \frac{1-1}{0} = \frac{0}{0} \text{ (indeterminate form)}$$

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

This question is a direct application of limits formula of exponential and logarithmic limits.

To get the desired forms, we need to include mx and nx as follows:

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{a^x - a^{-x}}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{a^{-x} \left(\frac{a^x}{a^{-x}} - 1 \right)}{x} = \lim_{x \rightarrow 0} \frac{a^{-x} (a^{2x} - 1)}{x}$$

{using law of exponents}

$$\Rightarrow Z = \lim_{x \rightarrow 0} a^{-x} \times \lim_{x \rightarrow 0} \frac{(a^{2x} - 1)}{x}$$

{using algebra of limits}

$$\Rightarrow Z = a^{-0} \times \lim_{x \rightarrow 0} \frac{(a^{2x} - 1)}{x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} \frac{(a^{2x} - 1)}{x}$$

To get the form as present in the formula we multiply and divide by 2

$$\therefore Z = \lim_{x \rightarrow 0} \frac{(a^{2x} - 1)}{2x} \times 2$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$

$$\therefore Z = 2 \log a$$

Hence,

$$\lim_{x \rightarrow 0} \frac{a^x - a^{-x}}{x} = 2 \log_e a$$

42. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, .. etc)

$$\text{Let } Z = \lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{0(e^0 - 1)}{1 - \cos 0} = \frac{0}{1 - 1} = \frac{0}{0} \text{ (indeterminate form)}$$

∴ We need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

It also involves a trigonometric term, so there is a possibility of application of Sandwich theorem- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x}$$

$$\text{As, } 1 - \cos x = 2\sin^2(x/2)$$

$$\therefore Z = \lim_{x \rightarrow 0} \frac{x(e^x - 1)}{2\sin^2\left(\frac{x}{2}\right)}$$

$$\Rightarrow Z = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x(e^x - 1)}{\sin^2\left(\frac{x}{2}\right)}$$

To get the desired form to apply the formula we need to divide numerator and denominator by x^2 .

$$\Rightarrow Z = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\frac{x(e^x - 1)}{x^2}}{\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2 \times 4}} = \frac{4}{2} \lim_{x \rightarrow 0} \frac{\frac{(e^x - 1)}{x}}{\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2}$$

Using algebra of limits, we have-

$$Z = 2 \frac{\lim_{x \rightarrow 0} \frac{(e^x - 1)}{x}}{\lim_{x \rightarrow 0} \left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = 2 \frac{\log e}{1^2}$$

$$\Rightarrow Z = 2 \log e = 2$$

Hence,

$$\lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x} = 2$$

43. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{x \left(x - \frac{\pi}{2} \right)}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{x \left(x - \frac{\pi}{2} \right)}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty - \infty$, .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{x \left(x - \frac{\pi}{2} \right)} = \frac{2^{-\cos \frac{\pi}{2}} - 1}{\frac{\pi}{2} \left(\frac{\pi}{2} - \frac{\pi}{2} \right)} = \frac{0}{0} \text{ (indeterminate form)}$$

∴ we need to take steps to remove this form so that we can get a finite value.

TIP: Most of the problems of logarithmic and exponential limits are solved using the formula $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\text{As } Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{x \left(x - \frac{\pi}{2} \right)}$$

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{\left(x - \frac{\pi}{2} \right)} \times \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{x} \quad \{\text{using algebra of limits}\}$$

$$\Rightarrow Z = \lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{\left(x - \frac{\pi}{2} \right)} \times \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{\left(x - \frac{\pi}{2} \right)}$$

$$\Rightarrow Z = \frac{2}{\pi} \lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{\sin(x - \frac{\pi}{2})} - 1}{\left(x - \frac{\pi}{2} \right)} \quad \{\because \sin(x - \pi/2) = -\cos x\}$$

As $x \rightarrow \pi/2$

∴ $x - \pi/2 \rightarrow 0$

Let $x - \pi/2 = y$ and $y \rightarrow 0$

Z can be rewritten as-

$$Z = \frac{2}{\pi} \lim_{y \rightarrow 0} \frac{2^{\sin(y)} - 1}{y}$$

Dividing numerator and denominator by $\sin y$ to get the form present in the formula

$$Z = \frac{2}{\pi} \lim_{y \rightarrow 0} \frac{\frac{2^{\sin(y)} - 1}{\sin y}}{\frac{y}{\sin y}}$$

Using algebra of limits:

$$Z = \frac{2}{\pi} \lim_{y \rightarrow 0} \frac{2^{\sin y} - 1}{\sin y} \times \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

Use the formula: $\lim_{x \rightarrow 0} \frac{(a^x - 1)}{x} = \log a$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\therefore Z = \frac{2}{\pi} \log_e 2$$

Hence,

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{2^{-\cos x} - 1}{x \left(x - \frac{\pi}{2} \right)} = \frac{2}{\pi} \log_e 2$$

Exercise 29.11

1. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \pi} \left(1 - \frac{x}{\pi} \right)^{\pi}$$

Answer

As we need to find $\lim_{x \rightarrow \pi} \left(1 - \frac{x}{\pi} \right)^{\pi}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty - \infty$, 0^∞ .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \pi} \left(1 - \frac{x}{\pi}\right)^{\pi} = \left(1 - \frac{\pi}{\pi}\right)^{\pi} = (1 - 1)^{\pi} = 0^{\pi} = 0$$

As it is not taking any indeterminate form.

$$\therefore Z = 0$$

Hence,

$$\lim_{x \rightarrow \pi} \left(1 - \frac{x}{\pi}\right)^{\pi} = 0$$

2. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{1/2x}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{1/2x}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty \cdot \infty$, 1^{∞} .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{1/2x} = \left\{1 + \tan^2 \sqrt{0}\right\}^{1/0} = (1)^{\infty} \text{ (indeterminate)}$$

As it is taking indeterminate form.

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

$$\text{As, } Z = \lim_{x \rightarrow 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{\frac{1}{2x}}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{\frac{1}{2x}}$$

Taking log both sides-

$$\Rightarrow \log Z = \lim_{x \rightarrow 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{\frac{1}{2x}}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow 0^+} \frac{\log(1 + \tan^2 \sqrt{x})}{2x}$$

$$\{\because \log a^m = m \log a\}$$

$$\text{Now it gives us a form that can be reduced to } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

Dividing numerator and denominator by $\tan^2 \sqrt{x}$ -

$$\log Z = \lim_{x \rightarrow 0^+} \frac{\frac{\log(1 + \tan^2 \sqrt{x})}{\tan^2 \sqrt{x}}}{\frac{2x}{\tan^2 \sqrt{x}}}$$

using algebra of limits -

$$\log Z = \frac{\lim_{x \rightarrow 0^+} \frac{\log(1 + \tan^2 \sqrt{x})}{\tan^2 \sqrt{x}}}{\lim_{x \rightarrow 0^+} \frac{2x}{\tan^2 \sqrt{x}}} = \frac{A}{B}$$

$$A = \lim_{x \rightarrow 0^+} \frac{\log(1 + \tan^2 \sqrt{x})}{\tan^2 \sqrt{x}}$$

Let, $\tan^2 \sqrt{x} = y$

As $x \rightarrow 0^+ \Rightarrow y \rightarrow 0^+$

$$\therefore A = \lim_{y \rightarrow 0} \frac{\log(1+y)}{y}$$

Use the formula - $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\therefore A = 1$$

$$\text{Now, } B = \lim_{x \rightarrow 0^+} \frac{2x}{\tan^2 \sqrt{x}}$$

$$\Rightarrow B = 2 \lim_{x \rightarrow 0^+} \left(\frac{\sqrt{x}}{\tan \sqrt{x}} \right)^2$$

Use the formula - $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

$$\therefore B = 2$$

Hence,

$$\log Z = \frac{A}{B} = \frac{1}{2}$$

$$\Rightarrow \log_e Z = 1/2$$

$$\therefore Z = e^{1/2}$$

Hence,

$$\lim_{x \rightarrow 0^+} \{1 + \tan^2 \sqrt{x}\}^{1/2x} = \sqrt{e}$$

3. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} (\cos x)^{1/\sin x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} (\cos x)^{1/\sin x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty \cdot \infty$, 1^∞ .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} (\cos x)^{1/\sin x} = \{\cos 0\}_{\sin 0}^1 = (1)^\infty \text{ (indeterminate)}$$

As it is taking indeterminate form-

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

$$\text{As, } Z = \lim_{x \rightarrow 0} (\cos x)^{1/\sin x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} (\cos x)^{1/\sin x}$$

Taking log both sides-

$$\Rightarrow \log Z = \lim_{x \rightarrow 0} \log(\cos x)^{1/\sin x}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow 0} \left\{ \frac{\log \cos x}{\sin x} \right\}$$

$$\{\because \log a^m = m \log a\}$$

Now it gives us a form that can be reduced to $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\log Z = \lim_{x \rightarrow 0} \left\{ \frac{\log(1+\cos x - 1)}{\sin x} \right\} \text{ {adding and subtracting 1 to cos x to get the form}}$$

Dividing numerator and denominator by $\cos x - 1$ to match with form in formula

$$\therefore \log Z = \lim_{x \rightarrow 0} \left\{ \frac{\frac{\log(1+\cos x - 1)}{\cos x - 1}}{\frac{\sin x}{\cos x - 1}} \right\}$$

using algebra of limits -

$$\log Z = \frac{\lim_{x \rightarrow 0} \frac{\log(1+\cos x - 1)}{\cos x - 1}}{\lim_{x \rightarrow 0} \frac{\sin x}{\cos x - 1}} = \frac{A}{B}$$

$$\therefore A = \lim_{x \rightarrow 0} \frac{\log(1+\cos x - 1)}{\cos x - 1}$$

Let, $\cos x - 1 = y$

As $x \rightarrow 0 \Rightarrow y \rightarrow 0$

$$\therefore A = \lim_{y \rightarrow 0} \frac{\log(1+y)}{y}$$

Use the formula - $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\therefore A = 1$$

$$\text{Now, } B = \lim_{x \rightarrow 0} \frac{\sin x}{\cos x - 1}$$

$$\because \cos x - 1 = -2\sin^2(x/2) \text{ and } \sin x = 2\sin(x/2)\cos(x/2)$$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{-2\sin^2\left(\frac{x}{2}\right)} = -\lim_{x \rightarrow 0} \cot \frac{x}{2}$$

$$\therefore B = -\cot 0 = \infty$$

$$\therefore B = \infty$$

Hence,

$$\log Z = \frac{A}{B} = \frac{1}{\infty} = 0$$

$$\Rightarrow \log_e Z = 0$$

$$\therefore Z = e^0 = 1$$

Hence,

$$\lim_{x \rightarrow 0} (\cos x)^{1/\sin x} = 1$$

4. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} (\cos x + \sin x)^{1/x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} (\cos x + \sin x)^{1/x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty \cdot \infty, 1^\infty$.. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} (\cos x + \sin x)^{\frac{1}{x}} = \{\cos 0 + \sin 0\}^{\frac{1}{0}} = (1)^{\infty} \text{ (indeterminate)}$$

As it is taking indeterminate form-

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

$$\text{As, } Z = \lim_{x \rightarrow 0} (\cos x + \sin x)^{\frac{1}{x}}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} (\cos x + \sin x)^{\frac{1}{x}}$$

Taking log both sides-

$$\Rightarrow \log Z = \lim_{x \rightarrow 0} \log (\cos x + \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow 0} \left\{ \frac{\log(\cos x + \sin x)}{x} \right\}$$

$$\{\because \log a^m = m \log a\}$$

$$\text{Now it gives us a form that can be reduced to } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\log Z = \lim_{x \rightarrow 0} \left\{ \frac{\log(1 + \cos x + \sin x - 1)}{x} \right\}$$

{adding and subtracting 1 to cos x to get the form}

Dividing numerator and denominator by cos x + sin x - 1 to match with form in formula

$$\therefore \log Z = \lim_{x \rightarrow 0} \left\{ \frac{\frac{\log(1 + \cos x + \sin x - 1)}{\cos x + \sin x - 1}}{\frac{x}{\cos x + \sin x - 1}} \right\}$$

using algebra of limits -

$$\log Z = \frac{\lim_{x \rightarrow 0} \frac{\log(1 + \cos x + \sin x - 1)}{\cos x + \sin x - 1}}{\lim_{x \rightarrow 0} \frac{x}{\cos x + \sin x - 1}} = \frac{A}{B}$$

$$\therefore A = \lim_{x \rightarrow 0} \frac{\log(1 + \cos x + \sin x - 1)}{\sin x + \cos x - 1}$$

$$\text{Let, } \cos x + \sin x - 1 = y$$

$$\text{As } x \rightarrow 0 \Rightarrow y \rightarrow 0$$

$$\therefore A = \lim_{y \rightarrow 0} \frac{\log(1+y)}{y}$$

$$\text{Use the formula - } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\therefore A = 1$$

$$\text{Now, } B = \lim_{x \rightarrow 0} \frac{x}{\cos x + \sin x - 1}$$

$$\because \cos x - 1 = -2\sin^2(x/2) \text{ and } \sin x = 2\sin(x/2)\cos(x/2)$$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{x}{-2\sin^2(\frac{x}{2}) + 2\sin(\frac{x}{2})\cos(\frac{x}{2})}$$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{x}{2\sin(\frac{x}{2})\{\cos(\frac{x}{2}) - \sin(\frac{x}{2})\}}$$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{\frac{x}{2}}{\sin(\frac{x}{2})} \times \lim_{x \rightarrow 0} \frac{1}{\cos(\frac{x}{2}) - \sin(\frac{x}{2})}$$

$$\text{Use the formula - } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{1}{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)} = \frac{1}{\cos 0 - \sin 0}$$

$$\therefore B = 1$$

Hence,

$$\log Z = \frac{A}{B} = \frac{1}{1} = 1$$

$$\Rightarrow \log_e Z = 1$$

$$\therefore Z = e^1 = e$$

Hence,

$$\lim_{x \rightarrow 0} (\cos x + \sin x)^{1/x} = e$$

5. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} (\cos x + a \sin x)^{1/x}$$

Answer

As we need to find $\lim_{x \rightarrow 0} (\cos x + a \sin x)^{1/x}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty \cdot \infty$, 1^∞ .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 0} (\cos x + a \sin x)^{1/x} = \{\cos 0 + a \sin 0\}^{1/0} = (1)^\infty \text{ (indeterminate)}$$

As it is taking indeterminate form-

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

$$\text{As, } Z = \lim_{x \rightarrow 0} (\cos x + a \sin x)^{1/x}$$

$$\Rightarrow Z = \lim_{x \rightarrow 0} (\cos x + a \sin x)^{1/x}$$

Taking log both sides-

$$\Rightarrow \log Z = \lim_{x \rightarrow 0} \log (\cos x + a \sin x)^{1/x}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow 0} \left\{ \frac{\log(\cos x + a \sin x)}{x} \right\}$$

$$\{\because \log a^m = m \log a\}$$

$$\text{Now it gives us a form that can be reduced to } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

Adding and subtracting 1 to $\cos x$ to get the form-

$$\log Z = \lim_{x \rightarrow 0} \left\{ \frac{\log(1 + \cos x + a \sin x - 1)}{x} \right\}$$

Dividing numerator and denominator by $\cos x + a \sin x - 1$ to match with form in formula

$$\therefore \log Z = \lim_{x \rightarrow 0} \left\{ \frac{\frac{\log(1 + \cos x + a \sin x - 1)}{\cos x + a \sin x - 1}}{\frac{x}{\cos x + a \sin x - 1}} \right\}$$

using algebra of limits -

$$\log Z = \frac{\lim_{x \rightarrow 0} \frac{\log(1 + \cos x + a \sin x - 1)}{a \sin x + \cos x - 1}}{\lim_{x \rightarrow 0} \frac{x}{\cos x + a \sin x - 1}} = \frac{A}{B}$$

$$\therefore A = \lim_{x \rightarrow 0} \frac{\log(1 + \cos x + a \sin x - 1)}{a \sin x + \cos x - 1}$$

Let, $\cos x + a \sin x - 1 = y$

As $x \rightarrow 0 \Rightarrow y \rightarrow 0$

$$\therefore A = \lim_{y \rightarrow 0} \frac{\log(1+y)}{y}$$

Use the formula - $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

$$\therefore A = 1$$

$$\text{Now, } B = \lim_{x \rightarrow 0} \frac{x}{\cos x + a \sin x - 1}$$

$$\because \cos x - 1 = -2\sin^2(x/2) \text{ and } \sin x = 2\sin(x/2)\cos(x/2)$$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{x}{-2\sin^2\left(\frac{x}{2}\right) + 2a\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{x}{2\sin\left(\frac{x}{2}\right)\{a\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\}}$$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{\frac{x}{2}}{\sin\left(\frac{x}{2}\right)} \times \lim_{x \rightarrow 0} \frac{1}{a\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}$$

Use the formula - $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\Rightarrow B = \lim_{x \rightarrow 0} \frac{1}{a\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)} = \frac{1}{a\cos 0 - \sin 0}$$

$$\therefore B = 1/a$$

Hence,

$$\log Z = \frac{A}{B} = \frac{1}{\frac{1}{a}} = a$$

$$\Rightarrow \log_e Z = a$$

$$\therefore Z = e^a = e^a$$

Hence,

$$\lim_{x \rightarrow 0} (\cos x + a \sin x)^{\frac{1}{x}} = e^a$$

6. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \left\{ \frac{x^2 + 2x + 3}{2x^2 + x + 5} \right\}^{\frac{3x-2}{3x+2}}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow \infty} \left\{ \frac{x^2 + 2x + 3}{2x^2 + x + 5} \right\}^{\frac{3x-2}{3x+2}}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty \cdot \infty$, 1^∞ .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow \infty} \left\{ \frac{x^2 + 2x + 3}{2x^2 + x + 5} \right\}^{\frac{3x-2}{3x+2}} = \left(\frac{\infty}{\infty} \right)^{\frac{\infty}{\infty}} \text{ (indeterminate)}$$

As it is taking indeterminate form-

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

$$Z = \lim_{x \rightarrow \infty} \left\{ \frac{x^2 + 2x + 3}{2x^2 + x + 5} \right\}^{\frac{3x-2}{3x+2}}$$

Take the log to bring the term in the product so that we can solve it more easily.

Taking log both sides-

$$\log Z = \lim_{x \rightarrow \infty} \log \left\{ \frac{x^2 + 2x + 3}{2x^2 + x + 5} \right\}^{\frac{3x-2}{3x+2}}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow \infty} \left(\frac{3x-2}{3x+2} \right) \log \left(\frac{x^2 + 2x + 3}{2x^2 + x + 5} \right)$$

$$\{\because \log a^m = m \log a\}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow \infty} \left(\frac{3x-2}{3x+2} \right) \times \lim_{x \rightarrow \infty} \log \left(\frac{x^2 + 2x + 3}{2x^2 + x + 5} \right)$$

{using algebra of limits}

Still, if we put $x = \infty$ we get an indeterminate form,

Take the highest power of x common and try to bring x in the denominator of a term so that if we put $x = \infty$ term reduces to 0.

$$\therefore \log Z = \lim_{x \rightarrow \infty} \left(\frac{x \left(3 - \frac{2}{x} \right)}{x \left(3 + \frac{2}{x} \right)} \right) \times \lim_{x \rightarrow \infty} \log \left(\frac{x^2 \left(1 + \frac{2x}{x^2} + \frac{3}{x^2} \right)}{x^2 \left(2 + \frac{x}{x^2} + \frac{5}{x^2} \right)} \right)$$

$$\Rightarrow \log Z = \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x}}{3 + \frac{2}{x}} \times \lim_{x \rightarrow \infty} \log \frac{1 + \frac{2}{x} + \frac{3}{x^2}}{2 + \frac{1}{x} + \frac{5}{x^2}}$$

$$\Rightarrow \log Z = \frac{3 - \frac{2}{\infty}}{3 + \frac{2}{\infty}} \times \log \frac{1 + \frac{2}{\infty} + \frac{3}{\infty^2}}{2 + \frac{1}{\infty} + \frac{5}{\infty^2}}$$

$$\Rightarrow \log Z = \frac{3}{3} \times \log \frac{1}{2} = \log \frac{1}{2}$$

$$\therefore \log_e Z = \log \frac{1}{2}$$

$$\Rightarrow Z = 1/2$$

Hence,

$$\lim_{x \rightarrow \infty} \left\{ \frac{x^2 + 2x + 3}{2x^2 + x + 5} \right\}^{\frac{3x-2}{3x+2}} = \frac{1}{2}$$

7. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}^{\frac{1 - \cos(x-1)}{(x-1)^2}}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow 1} \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}^{\frac{1 - \cos(x-1)}{(x-1)^2}}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty-\infty, 1^\infty$.. etc.)

$$\text{Let } Z = \lim_{x \rightarrow 1} \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}^{\frac{1 - \cos(x-1)}{(x-1)^2}} = \left(\frac{5}{6}\right)^0 \text{ (indeterminate)}$$

As it is taking indeterminate form-

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

$$Z = \lim_{x \rightarrow 1} \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}^{\frac{1 - \cos(x-1)}{(x-1)^2}}$$

Take the log to bring the power term in the product so that we can solve it more easily.

Taking log both sides-

$$\log Z = \lim_{x \rightarrow 1} \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}^{\frac{1 - \cos(x-1)}{(x-1)^2}}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow 1} \frac{1 - \cos(x-1)}{(x-1)^2} \log \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}$$

$$\{\because \log a^m = m \log a\}$$

using algebra of limits-

$$\Rightarrow \log Z = \lim_{x \rightarrow 1} \left(\frac{1 - \cos(x-1)}{(x-1)^2} \right) \times \lim_{x \rightarrow 1} \log \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow 1} \left(\frac{1 - \cos(x-1)}{(x-1)^2} \right) \times \log \left(\frac{1^3 + 2 \cdot 1^2 + 1 + 1}{1^2 + 2 \cdot 1 + 3} \right)$$

$$\Rightarrow \log Z = \log \frac{5}{6} \lim_{x \rightarrow 1} \left(\frac{1 - \cos(x-1)}{(x-1)^2} \right)$$

$$\text{As, } 1 - \cos x = 2\sin^2(x/2)$$

$$\therefore \log Z = \log \frac{5}{6} \lim_{x \rightarrow 1} \left(\frac{2\sin^2 \frac{x-1}{2}}{(x-1)^2} \right)$$

$$\text{Let } (x-1)/2 = y$$

$$\text{As } x \rightarrow 1 \Rightarrow y \rightarrow 0$$

$\therefore$ Z can be rewritten as

$$\log Z = \log \frac{5}{6} \lim_{y \rightarrow 0} \left(\frac{2\sin^2 y}{4y^2} \right)$$

$$\Rightarrow \log Z = \frac{1}{2} \log \frac{5}{6} \lim_{y \rightarrow 0} \left(\frac{\sin y}{y} \right)^2$$

$$\text{Use the formula - } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\therefore \log Z = \frac{1}{2} \log \frac{5}{6} \times 1 = \log \left(\frac{5}{6} \right)^{\frac{1}{2}}$$

$$\Rightarrow \log Z = \log \sqrt{\frac{5}{6}}$$

$$\therefore Z = \sqrt{\frac{5}{6}}$$

Hence,

$$\lim_{x \rightarrow 1} \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}^{\frac{1 - \cos(x-1)}{(x-1)^2}} = \sqrt{\frac{5}{6}}$$

8. Question

Evaluate the following limits:

$$\lim_{x \rightarrow 0} \left\{ \frac{e^x + e^{-x} - 2}{x^2} \right\}^{1/x^2}$$

Answer

$$\text{Let } y = \lim_{x \rightarrow 0} \left\{ \frac{e^x + e^{-x} - 2}{x^2} \right\}^{1/x^2}$$

Putting the limit, we get,

$$y = \left(\frac{0}{0} \right)^\infty$$

This is an indeterminate form, so we need to solve this limit. Taking log on both sides we get,

$$\log_e y = \log_e \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{x^2}$$

$$y = e^{\lim_{x \rightarrow 0} \left\{ \frac{e^x + e^{-x} - 2}{x^2} - 1 \right\}}$$

Now, applying L-Hospital's rule, we get,

$$y = e^{\lim_{x \rightarrow 0} \frac{x^2 \{e^x - e^{-x}\} - \{(e^x + e^{-x} - 2)/x^2 - 1\} 4x^3}{x^4}}$$

Applying L-hospital rule again we get,

$$y = e^{\lim_{x \rightarrow 0} \frac{1}{2} \{ (\lim_{x \rightarrow 0} (x+1)) / \lim_{x \rightarrow 0} (6+6x+x^2) \}}$$

$$y = e^{1/2}$$

9. Question

Evaluate the following limits:

$$\lim_{x \rightarrow a} \left\{ \frac{\sin x}{\sin a} \right\}^{\frac{1}{x-a}}$$

Answer

$$\text{As we need to find } \lim_{x \rightarrow a} \left\{ \frac{\sin x}{\sin a} \right\}^{\frac{1}{x-a}}$$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form (0/0 or ∞/∞ or $\infty \cdot \infty$, 1^∞ .. etc.)

$$\text{Let } Z = \lim_{x \rightarrow a} \left\{ \frac{\sin x}{\sin a} \right\}^{\frac{1}{x-a}} = \left(\frac{\sin a}{\sin a} \right)^\infty = 1^\infty \text{ (indeterminate)}$$

As it is taking indeterminate form-

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

$$Z = \lim_{x \rightarrow a} \left\{ \frac{\sin x}{\sin a} \right\}^{\frac{1}{x-a}}$$

Take the log to bring the power term in the product so that we can solve it more easily.

Taking log both sides-

$$\log Z = \lim_{x \rightarrow a} \left(\frac{1}{x-a} \right) \log \left\{ \frac{\sin x}{\sin a} \right\}$$

$$\{\because \log a^m = m \log a\}$$

$$\text{Now it gives us a form that can be reduced to } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\Rightarrow \log Z = \lim_{x \rightarrow a} \left(\frac{1}{x-a} \right) \log \left\{ 1 + \frac{\sin x - \sin a}{\sin a} \right\}$$

Dividing numerator and denominator by $\frac{\sin x - \sin a}{\sin a}$ to get the desired form and using algebra of limits we have-

$$\log Z = \lim_{x \rightarrow a} \frac{\log \left\{ 1 + \frac{\sin x - \sin a}{\sin a} \right\}}{\frac{\sin x - \sin a}{\sin a}} \times \lim_{x \rightarrow a} \frac{\sin x - \sin a}{\sin a (x-a)}$$

$$\text{if we assume } \frac{\sin x - \sin a}{\sin a} = y \text{ then as } x \rightarrow a \Rightarrow y \rightarrow 0$$

$$\Rightarrow \log Z = \lim_{y \rightarrow 0} \frac{\log(1+y)}{y} \times \lim_{x \rightarrow a} \frac{\sin x - \sin a}{\sin a (x-a)}$$

$$\text{Use the formula- } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\therefore \log Z = 1 \times \lim_{x \rightarrow a} \frac{\sin x - \sin a}{\sin a (x-a)}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow a} \frac{\sin x - \sin a}{\sin a (x-a)} = \frac{1}{\sin a} \lim_{x \rightarrow a} \frac{\sin x - \sin a}{(x-a)}$$

$$\text{Now it gives us a form that can be reduced to } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Try to use it. We are basically proceeding with a hit and trial attempt.

$$\Rightarrow \log Z = \frac{1}{\sin a} \lim_{x \rightarrow a} \frac{\sin(x-a+a) - \sin a}{(x-a)}$$

$$\because \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\Rightarrow \log Z = \frac{1}{\sin a} \lim_{x \rightarrow a} \frac{\sin(x-a) \cos a + \cos(x-a) \sin a - \sin a}{(x-a)}$$

$$\Rightarrow \log Z = \frac{1}{\sin a} \lim_{x \rightarrow a} \frac{\sin(x-a) \cos a}{(x-a)} + \frac{1}{\sin a} \lim_{x \rightarrow a} \frac{\cos(x-a) \sin a - \sin a}{x-a}$$

$$\Rightarrow \log Z = \frac{\cos a}{\sin a} \lim_{x \rightarrow a} \frac{\sin(x-a)}{(x-a)} + \frac{\sin a}{\sin a} \lim_{x \rightarrow a} \frac{\cos(x-a) - 1}{x-a}$$

$$\Rightarrow \log Z = \cot a \lim_{x \rightarrow a} \frac{\sin(x-a)}{(x-a)} - 1 \lim_{x \rightarrow a} \frac{2 \sin^2 \frac{x-a}{2}}{\left(\frac{x-a}{2} \right)^2} \times \frac{(x-a)}{4}$$

$$\text{Use the formula- } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\Rightarrow \log Z = \cot a - 0$$

$$\therefore \log Z = \cot a$$

$$\therefore Z = e^{\cot a}$$

Hence,

$$\lim_{x \rightarrow a} \left\{ \frac{\sin x}{\sin a} \right\}^{\frac{1}{x-a}} = e^{\cot a}$$

10. Question

Evaluate the following limits:

$$\lim_{x \rightarrow \infty} \left\{ \frac{3x^2 + 1}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}}$$

Answer

As we need to find $\lim_{x \rightarrow \infty} \left\{ \frac{3x^2 + 1}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}}$

We can directly find the limiting value of a function by putting the value of the variable at which the limiting value is asked if it does not take any indeterminate form ($0/0$ or ∞/∞ or $\infty \cdot \infty$, 1^∞ .. etc.)

Let $Z = \lim_{x \rightarrow \infty} \left\{ \frac{3x^2 + 1}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}} = \left(\frac{\infty}{\infty} \right)^{\frac{\infty}{\infty}}$ (indeterminate)

As it is taking indeterminate form-

$\therefore$ we need to take steps to remove this form so that we can get a finite value.

$$Z = \lim_{x \rightarrow \infty} \left\{ \frac{3x^2 + 1}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}}$$

Take the log to bring the term in the product so that we can solve it more easily.

Taking log both sides-

$$\log Z = \lim_{x \rightarrow \infty} \left\{ \frac{3x^2 + 1}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow \infty} \left(\frac{x^3}{1+x} \right) \log \left(\frac{3x^2 + 1}{4x^2 - 1} \right)$$

$$\{\because \log a^m = m \log a\}$$

$$\Rightarrow \log Z = \lim_{x \rightarrow \infty} \left(\frac{x^3}{1+x} \right) \times \lim_{x \rightarrow \infty} \log \left(\frac{3x^2 + 1}{4x^2 - 1} \right)$$

{using algebra of limits}

Still, if we put $x = \infty$ we get an indeterminate form,

Take highest power of x common and try to bring x in denominator of a term so that if we put $x = \infty$ term reduces to 0.

$$\therefore \log Z = \lim_{x \rightarrow \infty} \left(\frac{x^3}{x \left(1 + \frac{1}{x} \right)} \right) \times \lim_{x \rightarrow \infty} \log \left(\frac{x^2 \left(3 + \frac{1}{x^2} \right)}{x^2 \left(4 - \frac{1}{x^2} \right)} \right)$$

$$\Rightarrow \log Z = \lim_{x \rightarrow \infty} \frac{x^2}{1 + \frac{1}{x}} \times \lim_{x \rightarrow \infty} \log \frac{3 + \frac{1}{x^2}}{4 - \frac{1}{x^2}}$$

$$\Rightarrow \log Z = \frac{\infty}{1 + \frac{1}{\infty}} \times \log \frac{3 + \frac{1}{\infty^2}}{4 - \frac{1}{\infty^2}}$$

$$\Rightarrow \log Z = \log \frac{3}{4} \times \infty = -\infty$$

{ $\because \log (3/4)$ is a negative value as $3/4 < 1$ }

$$\therefore \text{Log}_e Z = -\infty$$

$$\Rightarrow Z = e^{-\infty} = 0$$

Hence,

$$\lim_{x \rightarrow \infty} \left\{ \frac{3x^2 + 1}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}} = 0$$