Application of Integrals

- Area of the region bounded by the curve $y = f(x)$, x-axis, and the lines $x = a$ and $x = b$ ($b > a$) is given by
  $A = \int_a^b y\,dx$ or $A = \int_a^b f(x)\,dx$
- The area of the region bounded by the curve $x = g(y)$, y-axis, and the lines $y = c$ and $y = d$ is given by
  $A = \int_a^b x\,dy$ or $A = \int_c^d g(y)\,dx$
- If a line $y = mx + p$ intersects a curve $y = f(x)$ at $x = a$ and $x = b$, ($b > a$), then the area (A) of region bounded by the curve $y = f(x)$ and the line $y = mx + p$ is

  $A = \int_a^b (y_1 - y_2)\,dx$, where $y_1 = mx + p$ and $y_2 = f(x)$

  $A = \int_a^b [(mx + p) - f(x)]\,dx$

- If a line $y = mx + p$ intersects a curve $x = g(y)$ at $y = c$ and $y = d$,($d > c$), then the area (A) of region bounded by the curve $x = g(y)$ and the line $y = mx + p$ is

$A = \int_{cy}^d (x_1 - x_2)\,dy$, where $x_1 = \frac{y - p}{m}$ and $x_2 = g(y)$

$A = \int_c^d \left[\left(\frac{y - p}{m}\right) - g(y)\right] dy$

Example 1: Find the area of the region in the first and third quadrant enclosed by the *x*-axis and the line $y = \sqrt{3}x$, and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: The given equations are

$$y = \sqrt{3}x \quad \text{... (1)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{... (2)}$$

Substituting $y = \sqrt{3}x$ in equation (2), we obtain

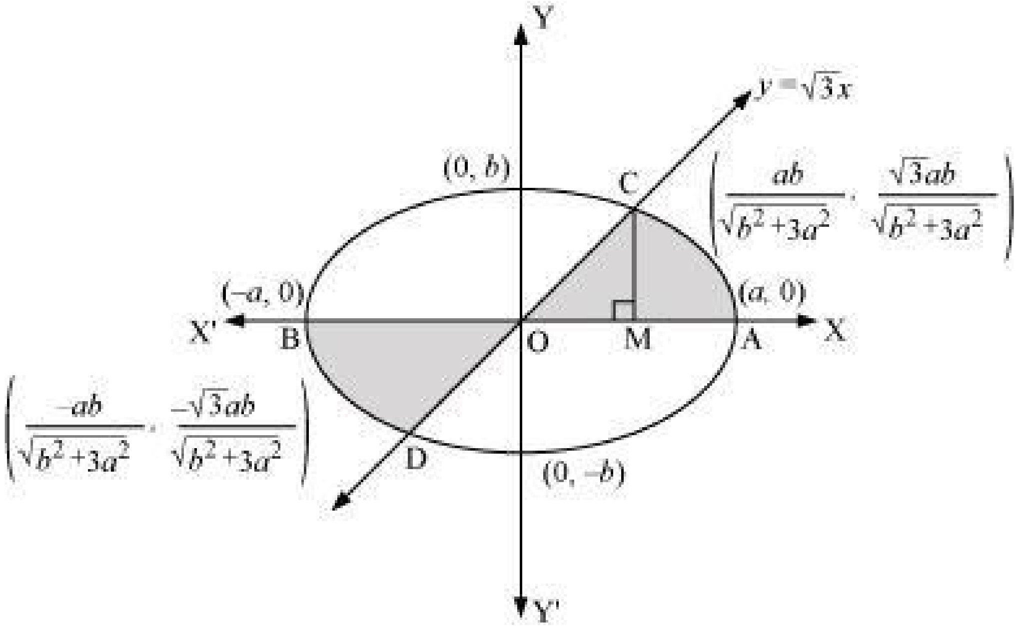
$$\frac{x^2}{a^2} + \frac{3x^2}{b^2} = 1$$

$$\Rightarrow x^2(b^2 + 3a^2) = a^2b^2$$

$$\Rightarrow x = \pm \frac{ab}{\sqrt{b^2+3a^2}}$$

$$\therefore y = \pm \frac{\sqrt{3}ab}{\sqrt{b^2+3a^2}}$$

Hence, the line meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at C $\left(\frac{ab}{\sqrt{b^2+3a^2}}, \frac{\sqrt{3}ab}{\sqrt{b^2+3a^2}}\right)$ and D $\left(\frac{-ab}{\sqrt{b^2+3a^2}}, \frac{-\sqrt{3}ab}{\sqrt{b^2+3a^2}}\right)$ in the first and third quadrant respectively.



In the figure, CM ⊥ XX′

Now, area OCMO = $\int_0^{\frac{ab}{\sqrt{b^2+3a^2}}} \sqrt{3}x\,dx = \frac{\sqrt{3}}{2}\left[x^2\right]_0^{\frac{ab}{\sqrt{b^2+3a^2}}} = \frac{\sqrt{3}a^2b^2}{2(b^2+3a^2)}$

Area ACMA

$$= \int_{\frac{ab}{\sqrt{b^2+3a^2}}}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\, dx = \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{ab}{\sqrt{b^2+3a^2}}}^{a}$$

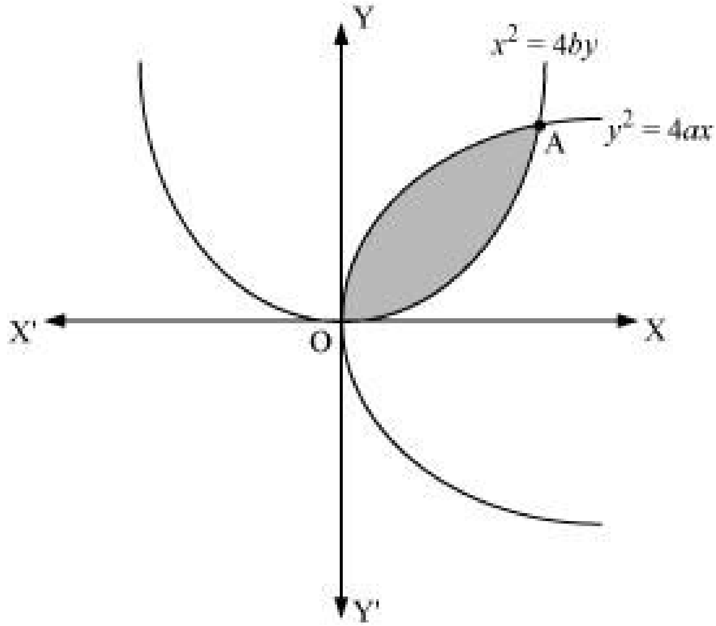
$$= \frac{b}{a}\left[\frac{a}{2}\times 0 + \frac{a^2}{2}\times\sin^{-1}1 - \left(\frac{ab}{2\sqrt{b^2+3a^2}}\times\frac{\sqrt{3}a^2}{\sqrt{b^2+3a^2}} + \frac{a^2}{2}\sin^{-1}\frac{b}{\sqrt{b^2+3a^2}}\right)\right]$$

$$= \frac{\pi}{4}ab - \frac{\sqrt{3}a^2b^2}{2\left(b^2+3a^2\right)} - \frac{ab}{2}\sin^{-1}\frac{b}{\sqrt{b^2+3a^2}}$$

a. The area of the region enclosed between two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is given by,

$$A = \begin{cases} \int_a^b [f(x) - g(x)]\,dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b] \\ \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx \\ \text{where } a < c < b \text{ and } f(x) \geq g(x) \text{ in } [a, c] \text{ and } f(x) \leq g(x) \text{ in } [c, b] \end{cases}$$

Example 2: Show that region bounded by two parabolas (shown in the figure) $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16}{3}ab$.



Solution:

The point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4by$ are 0 (0, 0) and A$4\sqrt[3]{ab^2}, 4\sqrt[3]{a^2b}$.

Here, $y^2 = 4ax \Rightarrow y = 2\sqrt{a}\sqrt{x} = f(x)$ and $x^2 = 4by \Rightarrow y = \frac{x^2}{4b} = g(x)$

It can be observed that $f(x)$ ³ $g(x)$ in $\left[0, 4\sqrt[3]{ab^2}\right]$.

Therefore, required area of the shaded region

$$= \int_0^{4\sqrt[3]{ab^2}} [f(x) - g(x)]\,dx$$

$$= \int_0^{4(ab^2)^{\frac{1}{2}}} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right) dx$$

$$= \left[\frac{2\sqrt{a}\cdot x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4b} \times \frac{x^3}{3}\right]_0^{4(ab^2)^{\frac{1}{3}}}$$

$$= \frac{4}{3}\sqrt{a}\left[8(ab^2)^{\frac{1}{2}}\right] - \frac{1}{12b} \cdot [64ab^2]$$

$$= \frac{32}{3}ab - \frac{16}{3}ab$$

$$= \frac{16}{3}ab$$