Three Dimensional Geometry

- **Direction cosines (d.c.'s) of a line:**
  - d.c.'s of a line are the cosines of angles made by the line with the positive direction of the coordinate axes.
  - If l, m, and n are the d.c.'s of a line, then $l^2 + m^2 + n^2 = 1$
  - d.c.'s of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$, where PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- **Direction ratios (d.r.'s) of a line:**
  - d.r.'s of a line are the numbers which are proportional to the d.c.'s of the line.
  - d.r.'s of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are given by $x_1 - x_2, y_1 - y_2, z_1 - z_2$ or $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

- If a, b, and c are the d.r.'s of a line and l, m, and n are its d.c.'s, then $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, \text{ and } n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Equation of a line through a given point and parallel to a given vector:**
  - **Vector form:** Equation of a line that passes through the given point whose position vector is $\vec{a}$ and which is parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$, where λ is a constant.

- **Cartesian form:**
  - Equation of a line that passes through a point (x_1, y_1, z_1) having d.r.'s as a, b, c is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
  - Equation of a line that passes through a point (x_1, y_1, z_1) having d.c.'s as l, m, n is given by $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

- **Equation of a line passing through two given points:**
  - **Vector form:** Equation of a line passing through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$, where $\lambda \in \mathbf{R}$
  - **Cartesian form:** Equation of a line that passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by, $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

- **Co-planarity of two lines**
  - **Vector form:** Two lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ are co-planar, if $\left(\vec{a_2} - \vec{a_1}\right).\left(\vec{b_1} \times \vec{b_2}\right) = 0$
  - **Cartesian form:** Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are co-planar, if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- **Angle between two Non-skew lines:**
  - **Cartesian form:**
    - If l_1, m_1, n_1, and l_2, m_2, n_2 are the d.c.'s of two lines and θ is the acute angle between them, then $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$
    - If a_1, b_1, c_1 and a_2, b_2, c_2 are the d.r.'s of two lines and θ is the acute angle between them, then $\cos\theta = \left|\dfrac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$
  - **Vector form:** If θ is the acute angle between the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_1}$, then $\cos\theta = \left|\dfrac{b_1 . b_2}{|b_1||b_2|}\right|$

- Two lines with d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are
  - perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$
  - parallel, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

- Two lines in space are said to be skew lines, if they are neither parallel nor intersecting. They lie in different planes.
- Angle between two skew lines is the angle between two intersecting lines drawn from any point (preferably from the origin) parallel to each of the skew lines.
- **Shortest Distance between two skew lines:** The shortest distance is the line segment perpendicular to both the lines.

- **Vector form:** Distance between two skew lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given by,

$$d = \left| \frac{\left(\vec{b_1} \times \vec{b_2}\right) \cdot \left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|} \right|$$

- **Cartesian form:** The shortest distance between two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is given by,

$$d = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(b_1 c_2 - b_2 c_1\right)^2 + \left(c_1 a_2 - c_2 a_1\right)^2 + \left(a_1 b_2 - a_2 b_1\right)^2}} \right|$$

- The shortest distance between two parallel lines $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu \vec{b}$ is given by,

$$d = \left| \frac{\vec{b} \times \left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b}\right|} \right|$$

- **Equation of a plane in normal form:**
  - **Vector form:** Equation of a plane which is at a distance of *d* from the origin, and the unit vector $\hat{n}$ normal to the plane through the origin is $\vec{r} \cdot \hat{n} = d$, where $\vec{r}$ is the position vector of a point in the plane

- **Cartesian form:** Equation of a plane which is at a distance *d* from the origin and the d.c.'s of the normal to the plane as *l*, *m*, *n* is $lx + my + nz = d$

- **Equation of a plane perpendicular to a given vector and passing through a given point:**
  - **Vector form:** Equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}).\vec{N} = 0$, where $\vec{r}$ is the position vector of a point in the plane
  - **Cartesian form:** Equation of plane passing through the point (x_1, y_1, z_1) and perpendicular to a given line whose d.r.'s are A, B, C is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

- **Equation of a plane passing through three non-collinear points:**
  - **Cartesian form:** Equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
  - **Vector form:** Equation of a plane that contains three non-collinear points having position vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$, where $\vec{r}$ is the position vector of a point in the plane

- **Planes passing through the intersection of two planes:**
  - **Vector form:** Equation of the plane passing through intersection of two planes $\vec{r}.\vec{n_1} = \vec{d_1}$ and $\vec{r}.\vec{n_2} = \vec{d_2}$ is given by, $\vec{r}.(\vec{n_1} + \lambda\vec{n_2}) = d_1 + \lambda d_2$, where λ is a non-zero constant

- **Cartesian form:** Equation of a plane passing through the intersection of two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$, is given by, $(A_1x + B_1y + C_1z + D_1) + \lambda (A_2x + B_2y + C_2z + D_2) = 0$, where λ is a non-zero constant

- **Angle between two planes:** The angle between two planes is defined as the angle between their normals.
  - **Vector form:** If θ is the angle between the two planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$, then $\cos\theta = \left| \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} \right|$

    Note that if two planes are perpendicular to each other, then $\vec{n_1} \cdot \vec{n_2} = 0$; and if they are parallel to each other, then $\vec{n_1}$ is parallel to $\vec{n_2}$.

  - **Cartesian form:** If θ is the angle between the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$, then $\cos\theta = \left| \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$

    Note that if two planes are perpendicular to each other, then $A_1A_2 + B_1B_2 + C_1C_2 = 0$; and if they are parallel to each other, then $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

- **Distance of a point from a plane:**

- **Vector form:** The distance of a point, whose position vector is $\vec{a}$, from the plane $\vec{r}.\hat{n} = d$ is $|d - \vec{a}.\hat{n}|$.

  Note:

  - If the equation of the plane is in the form of $\vec{r}.\vec{N} = d$, where $\vec{N}$ is the normal to the plane, then the perpendicular distance is $\frac{|\vec{a}.\vec{N} - d|}{|\vec{N}|}$.
  - Length of the perpendicular from origin to the plane $\vec{r}.\vec{N} = d$ is $\frac{|d|}{|\vec{N}|}$.

- **Cartesian form:** The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is $\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$.

- **Angle between a line and a plane:** The angle Φ between a line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}.\vec{n} = d$ is the complement of the angle between the line and the normal to the plane and is given by $\Phi = \sin^{-1}\left|\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right|$