Matrices

- A **matrix** is an ordered rectangular array of numbers or functions. The numbers or functions are called the **elements** or the **entries** of the matrix.

  For example, $\begin{bmatrix} -10 & \sin x & \log x \\ e^x & 2 & -9 \end{bmatrix}$ is a matrix having 6 elements. In this matrix, number of rows = 2 and number of columns = 3

- A matrix having *m* rows and *n* columns is called a matrix of order $m \times n$. In such a matrix, there are *mn* numbers of elements.

  For example, the order of the matrix $\begin{bmatrix} \sin x & \cos x \\ -1 & 1+\sin x \\ 0 & \cos x \end{bmatrix}$ is 3×2 as the numbers of rows and columns of this matrix are 3 and 2 respectively.

- A matrix *A* is said to be a **row matrix**, if it has only one row. In general, $A = [a_{ij}]_{1\times n}$ is a row matrix of order $1 \times n$.

  For example, $[-9 \quad 6 \quad 5 \quad e \quad \sin x]$ is a row matrix of order 1×5.

- A matrix *B* is said to be a **column matrix**, if it has only one column. In general, $B = [b_{ij}]_{m\times 1}$ is a column matrix of order $m \times 1$.

  For example, $B = \begin{bmatrix} -6 \\ 19 \\ 13 \end{bmatrix}$ is a column matrix of order 3×1.

- A matrix *C* is said to be a **square matrix**, if the number of rows and columns of the matrix are equal. In general, $C = [b_{ij}]_{m\times n}$ is a square matrix, if $m = n$

  For example, $C = \begin{bmatrix} -1 & 9 \\ 5 & 1 \end{bmatrix}$ is a square matrix.

- A square matrix A is said to be a **diagonal matrix**, if all its non-diagonal elements are zero. In general, $A = [a_{ij}]_{m \times n}$ is a diagonal matrix, if $a_{ij} = 0$ for $i \neq j$

- A matrix is said to be a **rectangular matrix,** if the number of rows is not equal to the number of columns.

  For example: $\begin{bmatrix} 8 & 3 & 9 \\ 1 & 6 & 7 \end{bmatrix}$ is a rectangular matrix.

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal (denoted as $A = B$) if they are of the same order and each element of A is equal to the corresponding element of B i.e., $a_{ij} = b_{ij}$ for all i and j.

For example: $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ and $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ are equal but $\begin{bmatrix} 15 & 11 \\ 7 & 2 \end{bmatrix}$ and $\begin{bmatrix} 7 & 2 \\ 15 & 11 \end{bmatrix}$ are not equal.

Example: If $\begin{bmatrix} 7 & x-y \\ 13 & 3y+z \end{bmatrix} = \begin{bmatrix} 2x+y & 5 \\ 2x+y+z & 3 \end{bmatrix}$, then find the values of x, y and z.

Solution:

Since the corresponding elements of equal matrices are equal,

$2x + y = 7$… (1)

$x - y = 5$… (2)

$2x + y + z = 13$… (3)

$3y + z = 3$… (4)

On solving equations (1) and (2), we obtain $x = 4$ and $y = -1$.

On substituting the value of y in equation (4), we obtain $z = 6$.

Thus, the values of x, y and z are 4, −1 and 6 respectively.

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ can be added, if they are of the same order.

  The sum of two matrices A and B of same order $m \times n$ is defined as matrix $C = [c_{ij}]_{m\times n}$, where $c_{ij} = a_{ij} + b_{ij}$ for all possible values of i and j.

- The difference of two matrices A and B is defined, if and only if they are of same order. The difference of the matrices A and B is defined as $A - B = A + (-1)B$
- If A, B, and C are three matrices of same order, then they follow the following properties related to addition:
  - Commutative law: $A + B = B + A$
  - Associative law: $A + (B + C) = (A + B) + C$
  - Existence of additive identity: For every matrix A, there exists a matrix O such that $A + O = O + A = A$. In this case, O is called the additive identity for matrix addition.
  - Existence of additive inverse: For every matrix A, there exists a matrix $(-A)$ such that $A + (-A) = (-A) + A = O$. In this case, $(-A)$ is called the additive inverse or the negative of A.

Example: Find the value of x and y, if:

$$\begin{bmatrix} 2x+3y & 9 \\ -2 & 4x-7y \end{bmatrix} + 2\begin{bmatrix} 3x+\frac{5}{2}y & -11 \\ -13 & 3x-\frac{3}{2}y \end{bmatrix} = \begin{bmatrix} 56 & -13 \\ -28 & 30 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2x+3y & 9 \\ -2 & 4x-7y \end{bmatrix} + 2\begin{bmatrix} 3x+\frac{5}{2}y & -11 \\ -13 & 3x-\frac{3}{2}y \end{bmatrix} = \begin{bmatrix} 56 & -13 \\ -28 & 30 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3y & 9 \\ -2 & 4x-7y \end{bmatrix} + \begin{bmatrix} 6x+5y & -22 \\ -26 & 6x-3y \end{bmatrix} = \begin{bmatrix} 56 & -13 \\ -28 & 30 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8(x+y) & -13 \\ -28 & 10(x-y) \end{bmatrix} = \begin{bmatrix} 56 & -13 \\ -28 & 30 \end{bmatrix}$$

Therefore, we have

$8(x+y) = 56$ and $10(x-y) = 30$

$\Rightarrow x + y = 7$ … (1)

And

$x - y = 3$ … (2)

Solving equation (1) and (2), we obtain $x = 5$ and $y = 2$

- The multiplication of a matrix A of order $m \times n$ by a scalar k is defined as

  $kA = kA = k[a_{ij}]_{m\times n} = [k(a_{ij})]_{m\times n}$

- If A and B are matrices of same order and k and l are scalars, then
  - $k(A + B) = kA + kB$
  - $(k + l) A = kA + lA$
- The negative of a matrix B is denoted by –B and is defined as (–1)B.

- The product of two matrices A and B is defined, if the number of columns of A is equal to the number of rows of B.
- If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ are two matrices, then their product is defined as $AB = C = [c_{ik}]_{m\times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$

  For example, if $A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 1 & -9 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 9 \\ 7 & 2 \\ 0 & 1 \end{bmatrix}$, then

  $$AB = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 1 & -9 \end{bmatrix} \times \begin{bmatrix} -5 & 9 \\ 7 & 2 \\ 0 & 1 \end{bmatrix}$$

  $$= \begin{bmatrix} 2\times(-5) + (-3)\times7 + 7\times0 & 2\times9 + (-3)\times2 + 7\times1 \\ 0\times(-5) + 1\times7 + (-9)\times0 & 0\times9 + 1\times2 + (-9)\times1 \end{bmatrix}$$

  $$= \begin{bmatrix} -31 & 19 \\ 7 & -7 \end{bmatrix}$$

- If A, B, and C are any three matrices, then they follow the following properties related to multiplication:
  - Associative law: $(AB)\,C = A\,(BC)$
  - Distribution law: $A\,(B + C) = AB + AC$ and $(A + B)\,C = AC + BC$, if both sides of equality are defined.

- Existence of multiplicative identity: For every square matrix A, there exists an identity matrix I of same order such that $IA = AI = A$. In this case, I is called the multiplicative identity.

- Multiplication of two matrices is not commutative. There are many cases where the product AB of two matrices A and B is defined, but the product BA need not be defined.

  For example, if $A = \begin{bmatrix} -1 & 5 \end{bmatrix}_{1\times 2}$ and $B = \begin{bmatrix} 0 & 1 & -4 \\ 3 & 2 & -1 \end{bmatrix}_{2\times 3}$, then AB is defined where as BA is not defined.

- If A is a matrix of order $m \times n$, then the matrix obtained by interchanging the rows and columns of A is called the transpose of matrix A. The transpose of A is denoted by A' or A^T. In other words, if $A = [a_{ij}]_{m\times n}$, then $A' = [a_{ij}]_{n\times m}$

  For example, the transpose of the matrix $\begin{bmatrix} 2 & 8 & -3 \\ 1 & 11 & 9 \end{bmatrix}$ is $\begin{bmatrix} 2 & 1 \\ 8 & 11 \\ -3 & 9 \end{bmatrix}$.

- For any matrices A and B of suitable orders, the properties of transpose of matrices are given as:
  - $(A')' = A$
  - $(kA)' = kA'$, where k is a constant
  - $(A+B)' = A' + B'$
  - $(AB)' = B'A'$

- If A is square matrix such that $A' = A$, then A is called a symmetric matrix. I.e., square matrix $A = [a_{ij}]$ is symmetric if $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

  For example, let $A = \begin{bmatrix} 1 & 5 & -8 \\ 5 & -2 & 6 \\ -8 & 6 & 4 \end{bmatrix}$. Now, $A' = \begin{bmatrix} 1 & 5 & -8 \\ 5 & -2 & 6 \\ -8 & 6 & 4 \end{bmatrix} = A$.
  Thus, A is a symmetric matrix.

- If A is a square matrix such that $A' = -A$, then A is called a skew symmetric matrix. I.e., A square matrix $A = [a_{ij}]$ is skew symmetric if $a_{ij} = -a_{ji}$ for all possible values of i and j.

For $i = j$, $a_{ii} = -a_{ii}$ i.e. $a_{ii} = 0$. This means, all the diagonal elements of a skew symmetric matrix are 0.

For example, let $A = \begin{bmatrix} 0 & -5 & -8 \\ 5 & 0 & -6 \\ 8 & 6 & 0 \end{bmatrix}$.

Now, $A' = \begin{bmatrix} 0 & 5 & 8 \\ -5 & 0 & 6 \\ -8 & -6 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -5 & -8 \\ 5 & 0 & -6 \\ 8 & 6 & 0 \end{bmatrix} = -A$.

Thus, A is a skew symmetric matrix.

- For any square matrix A with entries as real numbers, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.
- Every square matrix can be expressed as the sum of a symmetric matrix and a skew symmetric matrix. In other words, if A is any square matrix, then A can be expressed as $P + Q$, where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$. Here, P is symmetric matrix and Q is a skew symmetric matrix.

Example: Express the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ -5 & -2 & 4 \\ -9 & 6 & -7 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution:

We have

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -5 & -2 & 4 \\ -9 & 6 & -7 \end{bmatrix}$$

$$\therefore A' = \begin{bmatrix} 1 & -5 & -9 \\ 2 & -2 & 6 \\ -3 & 4 & -7 \end{bmatrix}$$

Now, $P = \frac{1}{2}(A + A') = \frac{1}{2}\left[\begin{bmatrix} 1 & 2 & -3 \\ -5 & -2 & 4 \\ -9 & 6 & -7 \end{bmatrix} + \begin{bmatrix} 1 & -5 & -9 \\ 2 & -2 & 6 \\ -3 & 4 & -7 \end{bmatrix}\right] = \begin{bmatrix} 1 & -\frac{3}{2} & -6 \\ -\frac{3}{2} & -2 & 5 \\ -6 & 5 & -7 \end{bmatrix}$

$$\therefore P' = \begin{bmatrix} 1 & -\frac{3}{2} & -6 \\ -\frac{3}{2} & -2 & 5 \\ -6 & 5 & -7 \end{bmatrix} = P$$

Thus, P is a symmetric matrix.
Now,

- The various elementary operations or transformations on a matrix are as follows:
  - $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
  - $R_i \leftrightarrow kR_i$ *or* $C_i \leftrightarrow kC_j$, where k is a non-zero constant
  - $R_i \leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + kC_j$, where k is a constant.

For example, by applying $R_1 \rightarrow R_1 - 7R_3$ to the matrix $\begin{bmatrix} -9 & 5 & 8 \\ 5 & 6 & 11 \\ 2 & -1 & 0 \end{bmatrix}$, we obtain $\begin{bmatrix} -23 & 12 & 8 \\ 5 & 6 & 11 \\ 2 & -1 & 0 \end{bmatrix}$.

- If A and B are the square matrices of same order such that $AB = BA = I$, then B is called the inverse of A and A is called the inverse of B. i.e., $A^{-1} = B$ and $B^{-1} = A$
- If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$
- If the inverse of a square matrix exists, then it is unique.
- If the inverse of a matrix exists, then it can be calculated either by using elementary row operations or by using elementary column operations.

Example: Find the inverse of the matrix: $A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$

Solution:
We know that $A = IA$. Therefore, we have

$$\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to \sin\theta\, R_1$ and $R_2 \to \cos\theta\, R_2$, we have

$$\begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ -\cos^2\theta & \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 \\ 0 & \cos\theta \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$, we have

$$\begin{bmatrix} \sin^2\theta + \cos^2\theta & 0 \\ -\cos^2\theta & \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & -\cos\theta \\ 0 & \cos\theta \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -\cos^2\theta & \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & -\cos\theta \\ 0 & \cos\theta \end{bmatrix} A$$

Applying $R_2 \to R_2 + \cos^2\theta\, R_1$, we have

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \sin\theta\cos^2\theta & \cos\theta\left(1 - \cos^2\theta\right) \end{bmatrix} A$$