

Relations and Functions

- **Cartesian product of two sets:** Two non-empty sets P and Q are given. The Cartesian product $P \times Q$ is the set of all ordered pairs of elements from P and Q , i.e.,

$$P \times Q = \{(p, q) : p \in P \text{ and } q \in Q\}$$

Example: If $P = \{x, y\}$ and $Q = \{-1, 1, 0\}$, then $P \times Q = \{(x, -1), (x, 1), (x, 0), (y, -1), (y, 1), (y, 0)\}$

If either P or Q is a null set, then $P \times Q$ will also be a null set, i.e., $P \times Q = \emptyset$.

In general, if A is any set, then $A \times \emptyset = \emptyset$.

- **Property of Cartesian product of two sets:**
 - If $n(A) = p$, $n(B) = q$, then $n(A \times B) = pq$.
 - If A and B are non-empty sets and either A or B is an infinite set, then so is the case with $A \times B$
 - $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here, (a, b, c) is called an ordered triplet.
 - $A \times (B \cap C) = (A \times B) \cap (A \times C)$
 - $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- Two ordered pairs are equal if and only if the corresponding first elements are equal and the second elements are also equal. In other words, if $(a, b) = (x, y)$, then $a = x$ and $b = y$.

Example: Show that there does not exist $x, y \in \mathbb{R}$ if $(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2)$.

Solution: It is given that

$$(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2).$$

$$\Rightarrow x - y + 1 = y - x - 4 \text{ and } 4x - 2y - 6 = 7x - 5y - 2$$

$$\Rightarrow 2x - 2y + 5 = 0 \quad \dots (1)$$

And $-3x + 3y - 4 = 0 \quad \dots (2)$

Now,

$$\frac{2}{-3} = -\frac{2}{3}, \frac{-2}{3} = -\frac{2}{3} \text{ and } \frac{5}{-4} = -\frac{5}{4}$$

Since $\frac{2}{-3} = \frac{-2}{3} \neq \frac{5}{-4}$, equations (1) and (2) have no solutions. This shows that there does not exist $x, y \in \mathbb{R}$ if $(x - y + 1, 4x - 2y - 6) = (y - x - 4, 7x - 5y - 2)$.

In general, for any two sets A and B, $A \times B \neq B \times A$.

- **Relation:** A relation R from a set A to a set B is a subset of the Cartesian product $A \times B$, obtained by describing a relationship between the first element x and the second element y of the ordered pairs (x, y) in $A \times B$.
- The image of an element x under a relation R is y , where $(x, y) \in R$
- **Domain:** The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R .
- **Range and Co-domain:** The set of all the second elements in a relation R from a set A to a set B is called the range of the relation R . The whole set B is called the co-domain of the relation R . $\text{Range} \subseteq \text{Co-domain}$

Example: In the relation X from $\mathbf{W}$ to $\mathbf{R}$, given by $X = \{(x, y): y = 2x + 1; x \in \mathbf{W}, y \in \mathbf{R}\}$, we obtain $X = \{(0, 1), (1, 3), (2, 5), (3, 7) \dots\}$. In this relation X , domain is the set of all whole numbers, i.e., domain = $\{0, 1, 2, 3 \dots\}$; range is the set of all positive odd integers, i.e., range = $\{1, 3, 5, 7 \dots\}$; and the co-domain is the set of all real numbers. In this relation, 1, 3, 5 and 7 are called the images of 0, 1, 2 and 3 respectively.

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$.

If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq} .

- A relation R from a set A to a set B is said to be a **function** if for every a in A , there is a unique b in B such that $(a, b) \in R$.
- If R is a function from A to B and $(a, b) \in R$, then b is called the **image** of a under the relation R and a is called the **preimage** of b under R .
- For a function R from set A to set B , set A is the **domain** of the function; the images of the elements in set A or the second elements in the ordered pairs form the **range**, while the whole of set B is the **codomain** of the function.

For example, in relation $f = \{(-1, 3), (0, 2), (1, 3), (2, 6), (3, 11)\}$ since each element in A has a unique image, therefore f is a function.

Each image in B is 2 more than the square of pre-image.

Hence, the formula for f is $f(x) = x^2 + 2$ Or $f: x \rightarrow x^2 + 2$

Domain = $\{-1, 0, 1, 2, 3\}$

Co-domain = $\{2, 3, 6, 11, 13\}$

Range = $\{2, 6, 3, 11\}$

- **Real-valued Function:** A function having either $\mathbb{R}$ (real numbers) or one of its subsets as its range is called a real-valued function. Further, if its domain is also either $\mathbb{R}$ or a subset of $\mathbb{R}$, it is called a real function.

Types of functions:

- **Identity function:** The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $y = f(x) = x$, for each $x \in \mathbb{R}$, is called the identity function.

Here, $\mathbb{R}$ is the domain and range of f .

- **Constant function:** The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $y = f(x) = c$, for each $x \in \mathbb{R}$, where c is a constant, is a constant function.

Here, the domain of f is $\mathbb{R}$ and its range is $\{c\}$.

- **Polynomial function:** A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be a polynomial function if for each $x \in \mathbb{R}$, $y = f(x) = a_0 + a_1x + \dots + a_nx^n$ where n is a non-negative integer and $a_0, a_1, \dots, a_n \in \mathbb{R}$.

- **Rational function:** The functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$, are called rational functions.
- **Modulus function:** The function $f: \mathbb{R} \rightarrow \mathbb{R}^+$ defined by $f(x) = |x|$, for each $x \in \mathbb{R}$, is called the modulus function.

In other words,

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- **Signum function:** The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is called the signum function. Its domain is $\mathbb{R}$ and its range is the set $\{-1, 0, 1\}$.

- **Greatest Integer function:** The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = [x]$, $x \in \mathbb{R}$, assuming the value of the greatest integer less than or equal to x , is called the greatest integer function.

Example: $[-2.7] = -3$, $[2.7] = 2$, $[2] = 2$

- **Linear function:** The function f defined by $f(x) = mx + c$, for $x \in \mathbb{R}$, where m and c are constants, is called the linear function. Here, $\mathbb{R}$ is the domain and range of f .

- **Addition and Subtraction of functions:** For functions $f: X \rightarrow \mathbb{R}$ and $g: X \rightarrow \mathbb{R}$, we define

- Addition of Functions

$$(f + g): X \rightarrow \mathbb{R} \text{ by } (f + g)(x) = f(x) + g(x), \quad x \in X$$

- Subtraction of Functions

$$(f - g): X \rightarrow \mathbb{R} \text{ by } (f - g)(x) = f(x) - g(x), x \in X$$

Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ &= (2x - 3) + (x^2 + 3x + 2) \\ &= x^2 + 5x - 1\end{aligned}$$

$$(f - g) = f(x) - g(x)$$

- **Multiplication of real functions:** For functions $f: X \rightarrow \mathbb{R}$ and $g: X \rightarrow \mathbb{R}$, we define

$$\begin{aligned}&= (2x - 3) - (x^2 + 3x + 2) \\ &= -x^2 - x - 5\end{aligned}$$

- Multiplication of two real functions

$$(fg): X \rightarrow \mathbb{R} \text{ by } (fg)(x) = f(x) \cdot g(x) \quad x \in X$$

- Multiplication of a function by a scalar

$$(af): X \rightarrow \mathbb{R} \text{ by } (af)(x) = af(x) \quad x \in X \text{ and } a \text{ is a real number}$$

Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$$\begin{aligned}(fg)(x) &= f(x) \times g(x) \\ &= (2x - 3) \times (x^2 + 3x + 2) \\ &= 2x^3 + 3x^2 - 5x - 6\end{aligned}$$

- **Addition and Subtraction of functions:** For functions $f: X \rightarrow \mathbb{R}$ and $g: X \rightarrow \mathbb{R}$, we define

$$\begin{aligned}(2f)(x) &= 2f(x) \\ &= 2 \times (2x - 3) \\ &= 4x - 6\end{aligned}$$

Addition of Functions

$$(f + g): X \rightarrow \mathbb{R} \text{ by } (f + g)(x) = f(x) + g(x), x \in X$$

- Subtraction of Functions

$(f - g): X \rightarrow \mathbb{R}$ by $(f - g)(x) = f(x) - g(x), x \in X$

Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$$(f + g)(x) = f(x) + g(x)$$

$$= (2x - 3) + (x^2 + 3x + 2)$$

$$= x^2 + 5x - 1$$

$$(f - g) = f(x) - g(x)$$

$$= (2x - 3) - (x^2 + 3x + 2)$$

$$= -x^2 - x - 5$$