

# Trigonometric Functions

- If in a circle of radius  $r$ , an arc of length  $l$  subtends an angle of  $\theta$  radians, then  $l = r\theta$ .
- Radian measure =  $\frac{\pi}{180} \times \text{Degree measure}$
- Degree measure =  $\frac{180}{\pi} \times \text{Radian measure}$
- A degree is divided into 60 minutes and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as 1', and one sixtieth of a minute is called a second, written as 1".  
Thus,  $1^\circ = 60'$  and  $1' = 60''$
- **Signs of trigonometric functions in different quadrants:**

Trigonometric function	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin x$	+ ve (Increases from 0 to 1)	+ ve (Decreases from 1 to 0)	-ve (Decreases from 0 to -1)	-ve (Increases from -1 to 0)
$\cos x$	+ ve (Decreases from 1 to 0)	-ve (Decreases from 0 to -1)	-ve (Increases from -1 to 0)	+ ve (Increases from 0 to 1)
$\tan x$	+ ve (Increases from 0 to $\infty$ )	-ve (Increases from $-\infty$ to 0)	+ ve (Increases from 0 to $\infty$ )	-ve (Increases from $-\infty$ to 0)
$\cot x$	+ ve (Decreases from $\infty$ to 0)	-ve (Decreases from 0 to $-\infty$ )	+ ve (Decreases from $\infty$ to 0)	-ve (Decreases from 0 to $-\infty$ )
$\sec x$	+ ve (Increases from 1 to $\infty$ )	-ve (Increases from $-\infty$ to -1)	-ve (Decreases from -1 to $-\infty$ )	+ ve (Decreases from $\infty$ to 1)
$\operatorname{cosec} x$	+ ve (Decreases from $\infty$ to 1)	+ ve (Increases from 1 to $\infty$ )	-ve (Increases from $-\infty$ to -1)	-ve (Decreases from -1 to $-\infty$ )

### Example 1:

If  $\sin \theta = -\frac{1}{\sqrt{3}}$ , where  $\pi < \theta < \frac{3\pi}{2}$ , then find the value of  $3 \tan \theta - \sqrt{3} \sec \theta$ .

### Solution:

Since  $\theta$  lies in the third quadrant, therefore  $\tan \theta$  is positive and  $\cos \theta$  (or  $\sec \theta$ ) is negative.

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\Rightarrow \cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

$$\Rightarrow \cos \theta = \pm \sqrt{1 - \left(-\frac{1}{\sqrt{3}}\right)^2} = \pm \sqrt{1 - \frac{1}{3}} = \pm \sqrt{\frac{2}{3}}$$

$$\therefore \cos \theta = -\sqrt{\frac{2}{3}}$$

$$\Rightarrow \sec \theta = -\sqrt{\frac{3}{2}}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-\frac{1}{\sqrt{3}}}{-\sqrt{\frac{2}{3}}} = \frac{1}{\sqrt{2}}$$

$$\therefore 3 \tan \theta - \sqrt{3} \sec \theta = 3 \times \frac{1}{\sqrt{2}} - \sqrt{3} \times \left(-\sqrt{\frac{3}{2}}\right) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 3\sqrt{2}$$

**Example 2:** Find the value of  $\cos 390^\circ \cos 510^\circ + \sin 390^\circ \cos (-660^\circ)$ .

**Solution:**

$$\cos 390^\circ = \cos (2 \times 180^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 510^\circ = \cos (3 \times 180^\circ - 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\sin 390^\circ = \sin (2 \times 180^\circ + 30^\circ) = \sin 30^\circ = \frac{1}{2}$$

$$\cos (-660^\circ) = \cos 660^\circ = \cos (4 \times 180^\circ - 60^\circ) = \cos 60^\circ = \frac{1}{2}$$

$$\therefore \cos 390^\circ \cos 510^\circ + \sin 390^\circ \cos (-660^\circ)$$

$$= \frac{\sqrt{3}}{2} \times \left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)$$

$$= -\frac{3}{4} + \frac{1}{4}$$

$$= -\frac{2}{4}$$

$$= -\frac{1}{2}$$

• **Domain and Range of trigonometric functions:**

Trigonometric function	Domain	Range
$\sin x$	$\mathbf{R}$	$[-1, 1]$
$\cos x$	$\mathbf{R}$	$[-1, 1]$
$\tan x$	$\mathbf{R} - \left\{x : x = \frac{(2n+1)\pi}{2}, n \in \mathbf{Z}\right\}$	$\mathbf{R}$
$\cot x$	$\mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\}$	$\mathbf{R}$
$\sec x$	$\mathbf{R} - \left\{x : x = \frac{(2n+1)\pi}{2}, n \in \mathbf{Z}\right\}$	$\mathbf{R} - [-1, 1]$
$\operatorname{cosec} x$	$\mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\}$	$\mathbf{R} - [-1, 1]$

• **Trigonometric identities and formulas:**

- $\operatorname{cosec} x = \frac{1}{\sin x}$
- $\sec x = \frac{1}{\cos x}$
- $\tan x = \frac{\sin x}{\cos x}$
- $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$

- $1 + \cot^2 x = \operatorname{cosec}^2 x$
- $\cos(2n\pi + x) = \cos x, n \in \mathbb{Z}$
- $\sin(2n\pi + x) = \sin x, n \in \mathbb{Z}$
- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
- $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
- $\cos(\pi - x) = -\cos x$
- $\sin(\pi - x) = \sin x$
- $\cos(\pi + x) = -\cos x$
- $\sin(\pi + x) = -\sin x$
- $\cos(2\pi - x) = \cos x$
- $\sin(2\pi - x) = -\sin x$

- If none of the angles  $x, y$  and  $(x \pm y)$  is an odd multiple of  $\frac{\pi}{2}$ , then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \text{ and } \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- If none of the angles  $x, y$  and  $(x \pm y)$  is a multiple of  $\pi$ , then

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}, \text{ and } \cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

- In particular,  $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = 2\cos^2 \frac{x}{2} - 1 = 1 - 2\sin^2 \frac{x}{2} = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

- $\sin 2x = 2\sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$

- In particular,  $\sin x = 2\sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

- $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

- In particular,

• **General solutions of some trigonometric equations:**

- $\sin x = 0 \Rightarrow x = n\pi$ , where  $n \in \mathbf{Z}$
- $\cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}$ , where  $n \in \mathbf{Z}$
- $\sin x = \sin y \Rightarrow x = n\pi + (-1)^n y$ , where  $n \in \mathbf{Z}$
- $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$ , where  $n \in \mathbf{Z}$
- $\tan x = \tan y \Rightarrow x = n\pi + y$ , where  $n \in \mathbf{Z}$

**Example 1:** Solve  $\cot x \cos^2 x = 2 \cot x$

**Solution:**

$$\cot x \cos^2 x = 2 \cot x$$

$$\Rightarrow \cot x \cos^2 x - 2 \cot x = 0$$

$$\Rightarrow \cot x (\cos^2 x - 2) = 0$$

$$\Rightarrow \cot x = 0 \text{ or } \cos^2 x = 2$$

$$\Rightarrow \frac{\cos x}{\sin x} = 0 \text{ or } \cos x = \pm \sqrt{2}$$

$$\Rightarrow \cos x = 0 \text{ or } \cos x = \pm \sqrt{2}$$

Now,  $\cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}$ , where  $n \in \mathbf{Z}$

$$\text{and } \cos x = \pm \sqrt{2}$$

But this is not possible as  $-1 \leq \cos x \leq 1$

Thus, the solution of the given trigonometric equation is  $x = (2n + 1)\frac{\pi}{2}$  where  $n \in \mathbf{Z}$ .

**Example 2:** Solve  $\sin 2x + \sin 4x + \sin 6x = 0$ .

**Solution:**

$$\sin 4x + (\sin 2x + \sin 6x) = 0$$

$$\Rightarrow \sin 4x + 2 \sin\left(\frac{2x+6x}{2}\right) \cos\left(\frac{2x-6x}{2}\right) = 0$$

$$\Rightarrow \sin 4x + 2 \sin 4x \cos 2x = 0$$

$$\Rightarrow \sin 4x(1 + 2 \cos 2x) = 0$$

$$\Rightarrow \sin 4x = 0 \text{ or } 1 + 2 \cos 2x = 0$$

$$\Rightarrow \sin 4x = 0 \text{ or } \cos 2x = -\frac{1}{2}$$

$$\sin 4x = 0$$

$$\Rightarrow 4x = n\pi, n \in \mathbf{Z}$$

$$\Rightarrow x = \frac{n\pi}{4}, n \in \mathbf{Z}$$

$$\cos 2x = -\frac{1}{2}$$

$$\Rightarrow \cos 2x = \cos \frac{2\pi}{3}$$

$$\Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3}, m \in \mathbf{Z}$$

$$\Rightarrow x = m\pi \pm \frac{\pi}{3}, m \in \mathbf{Z}$$

Thus,  $x = \frac{n\pi}{4}$  or  $x = m\pi \pm \frac{\pi}{3}$ , where  $m, n \in \mathbf{Z}$