

Limits and Derivatives

- **Derivatives**

- Suppose f is a real-valued function and a is a point in its domain of definition. The derivative of f at a [denoted by $f'(a)$] is defined as

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}, \text{ provided the limit exists.}$$

Derivative of $f(x)$ at a is denoted by $f'(a)$.

- Suppose f is a real-valued function. The derivative of f {denoted by $f'(x)$ or $\frac{d}{dx}[f(x)]$ } is defined as

$$\frac{d}{dx}[f(x)] = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \text{ provided the limit exists.}$$

This definition of derivative is called the first principle of derivative.

Example: Find the derivative of $f(x) = x^2 + 2x$ using first principle of derivative.

Solution: We know that $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h) + 2(x+h) - (x^2 + 2x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + h^2 + 2hx + 2x + 2h - x^2 - 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 + 2hx + 2h}{h} \\ &= \lim_{h \rightarrow 0} (h + 2x + 2) \\ &= 0 + 2x + 2 = 2x + 2 \\ f'(x) &= 2x + 2 \end{aligned}$$

- **Derivatives of Polynomial Functions**

For the functions u and v (provided u' and v' are defined in a common domain),

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- $(u \pm v)' = u' \pm v'$
- $(uv)' = u'v + uv'$ (Product rule)
- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ (Quotient rule)

- **Derivatives of Trigonometric Functions**

- $\frac{d}{dx}(x^n) = nx^{n-1}$ for any positive integer n
- $\frac{d}{dx}(a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0) = na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \dots + a_1$

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$

Example: Find the derivative of the function $f(x) = (3x^2 + 4x + 1) \cdot \tan x$

Solution: We have,

$f(x) = (3x^2 + 4x + 1) \cdot \tan x$
Differentiating both sides with respect to x ,
 $f'(x) = (3x^2 + 4x + 1) \cdot \frac{d}{dx} \tan x + \tan x \cdot \frac{d}{dx} (3x^2 + 4x + 1)$
 $f'(x) = (3x^2 + 4x + 1) \cdot \sec^2 x + \tan x \cdot (6x + 4)$