

Principle of Mathematical Induction

- **Principle of Mathematical Induction**

- There are some mathematical statements or results that are formulated in terms of n , where n is a positive integer. To prove such statements, the well-suited principle that is used, based on the specific technique, is known as the principle of mathematical induction.
- To prove a given statement in terms of n , firstly, we assume the statement as $P(n)$.

Thereafter, we examine the correctness of the statement for $n = 1$, i.e., $P(1)$ is true.

Then, assuming that the statement is true for $n = k$, where k is a positive integer, we prove that the statement is true for $n = k + 1$, i.e., truth of $P(k)$ implies the truth of $P(k + 1)$. Then, we say $P(n)$ is true for all natural numbers n .

Example: For all $n \in \mathbb{N}$, prove that

$$\frac{4}{3} + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^n = 4 \left[\left(\frac{4}{3}\right)^n - 1 \right]$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n): \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^n = 4 \left[\left(\frac{4}{3}\right)^n - 1 \right]$$

For $n = 1$, $P(n): \frac{4}{3} = 4 \left[\frac{4}{3} - 1 \right] = 4 \times \frac{1}{3} = \frac{4}{3}$, which is true.

Now, assume that $P(x)$ is true for some positive integer k . This means

$$\frac{4}{3} + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^k = 4 \left[\left(\frac{4}{3}\right)^k - 1 \right] \quad - (1)$$

We shall now prove that $P(k + 1)$ is also true.

Now, we have

$$\left[\frac{4}{3} + \left(\frac{4}{3} \right)^2 + \dots + \left(\frac{4}{3} \right)^k \right] + \left(\frac{4}{3} \right)^{k+1}$$

$$= 4 \left[\left(\frac{4}{3} \right)^k - 1 \right] + \left(\frac{4}{3} \right)^{k+1}$$

$$= 4 \left(\frac{4}{3} \right)^k - 4 + \left(\frac{4}{3} \right)^k \times \frac{4}{3}$$

$$= \left(\frac{4}{3} \right)^k \times \left[4 + \frac{4}{3} \right] - 4$$

$$= \left(\frac{4}{3} \right)^k \times \frac{16}{3} - 4$$

$$= \left(\frac{4}{3} \right)^k \times \frac{4}{3} \times 4 - 4$$

$$= \left(\frac{4}{3} \right)^{k+1} \times 4 - 4 = 4 \left[\left(\frac{4}{3} \right)^{k+1} - 1 \right]$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true. Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n .