

Factorisation of polynomials

- **Remainder Theorem**

If $p(x)$ is a polynomial of degree greater than or equal to one and a is any real number then if $p(x)$ is divided by the linear polynomial $x - a$, the remainder is $p(a)$.

Example:

Find the remainder when $x^5 - x^2 + 5$ is divided by $x - 2$.

Solution:

$$p(x) = x^5 - x^2 + 5$$

The zero of $x - 2$ is 2.

$$p(2) = 2^5 - 2^2 + 5 = 32 - 4 + 5 = 33$$

Therefore, by remainder theorem, the remainder is 33.

- **Factor Theorem**

If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then

- $x - a$ is a factor of $p(x)$, if $p(a) = 0$.
- $p(a) = 0$, if $(x - a)$ is a factor of $p(x)$.

Example:

Determine whether $x + 3$ is a factor of $x^3 + 5x^2 + 5x - 3$.

Solution:

The zero of $x + 3$ is -3 .

$$\text{Let } p(x) = x^3 + 5x^2 + 5x - 3$$

$$\begin{aligned} p(-3) &= (-3)^3 + 5(-3)^2 + 5(-3) - 3 \\ &= -27 + 45 - 15 - 3 \\ &= -45 + 45 \\ &= 0 \end{aligned}$$

Therefore, by factor theorem, $x + 3$ is the factor of $p(x)$.

- Factorisation of quadratic polynomials of the form $ax^2 + bx + c$ can be done using Factor theorem and splitting the middle term.

Example 1:**Factorize $x^2 - 7x + 10$ using the factor theorem.****Solution:**Let $p(x) = x^2 - 7x + 10$ The constant term is 10 and its factors are $\pm 1, \pm 2, \pm 5$ and ± 10 .

Let us check the value of the polynomial for each of these factors of 10.

$$p(1) = 1^2 - 7 \cdot 1 + 10 = 1 - 7 + 10 = 4 \neq 0$$

Hence, $x - 1$ is not a factor of $p(x)$.

$$p(2) = 2^2 - 7 \cdot 2 + 10 = 4 - 14 + 10 = 0$$

Hence, $x - 2$ is a factor of $p(x)$.

$$p(5) = 5^2 - 7 \cdot 5 + 10 = 25 - 35 + 10 = 0$$

Hence, $x - 5$ is a factor of $p(x)$.We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $x - 2$ and $x - 5$.

Thus, we can write the given polynomial as:

$$p(x) = x^2 - 7x + 10 = (x - 2)(x - 5)$$

Example 2:**Factorize $2x^2 - 11x + 15$ by splitting the middle term.****Solution:**The given polynomial is $2x^2 - 11x + 15$.Here, $a \cdot c = 2 \times 15 = 30$. The middle term is -11 . Therefore, we have to split -11 into two numbers such that their product is 30 and their sum is -11 . These numbers are -5 and -6 [As $(-5) + (-6) = -11$ and $(-5) \times (-6) = 30$].

Thus, we have:

$$\begin{aligned} 2x^2 - 11x + 15 &= 2x^2 - 5x - 6x + 15 \\ &= x(2x - 5) - 3(2x - 5) \\ &= (2x - 5)(x - 3) \end{aligned}$$

Note: A quadratic polynomial can have a maximum of two factors.

- Factorisation of cubic polynomials of the form $ax^3 + bx^2 + cx + d$ can be done using factor theorem and hit and trial method.

A cubic polynomial can have a maximum of three linear factors. So, by knowing one of these factors, we can reduce it to a quadratic polynomial.

Thus, to factorize a cubic polynomial, we first find a factor by the hit and trial method or by using the factor theorem, and then reduce the cubic polynomial into a quadratic polynomial and it is then solved further.

Example:

Factorise $p(x) = x^3 - 7x + 6$

Solution:

The constant term is 6.

The factors of 6 are $\pm 1, \pm 2, \pm 3$ and ± 6 .

Let $x = 1$

$$\begin{aligned} p(x=1) &= (1)^3 - 7(1) + 6 \\ &= 1 - 7 + 6 \\ &= 0 \end{aligned}$$

Thus, $(x - 1)$ is a factor of $p(x)$.

Now we have to group the term of $p(x)$ such that we can take $(x - 1)$ as common.

$$\begin{aligned} \text{Therefore, } p(x) &= x^3 - 7x + 6 \\ &= x^3 - x^2 + x^2 - x - 6x + 6 \\ &= x^2(x - 1) + x(x - 1) - 6(x - 1) \\ &= (x - 1)(x^2 + x - 6) \quad \dots (1) \end{aligned}$$

Now, we factorize $(x^2 + x - 6)$ by splitting its middle term.

$$\begin{aligned} x^2 + x - 6 &= x^2 + 3x - 2x - 6 \\ &= x(x + 3) - 2(x + 3) \\ &= (x - 2)(x + 3) \end{aligned}$$

From equation (1), we get

$$p(x) = (x - 1) (x - 2) (x + 3)$$

Hence, factors of polynomial $p(x)$ are $(x - 1)$, $(x - 2)$ and $(x + 3)$.